COMPLETENESS OF THE RANDOM GRAPH: TWO PROOFS

EUGENIA FUCHS

Abstract. We take a countably infinite random graph, state its axioms as a theory in first-order logic, and prove its completeness in two distinct ways. For the first proof, we show that the random graph is $\aleph_0$-categorical, meaning that it has only one countably infinite model up to isomorphism. Completeness follows. For the second proof, we show that the theory of the random graph admits quantifier elimination and use a fact about models to show completeness without assuming $\aleph_0$-categoricity.

Contents

1. Introduction

We will take an infinite random graph - an object that at first glance does not seem to belong to formal logic - and write down a set of sentences in predicate logic that precisely describe it. We will precede this with some basic notions in model theory, which give us the tools necessary to analyze the sentences. Then we will return to the random graph and prove one of its properties - which, on the surface, will also appear to have little connection with the logical properties of the statements we wrote down, which together we call the *theory* of the random graph. As it turns out,the infinite random graph has the property of $\aleph_0$-categoricity (hence *the* random graph). This means that any two countably infinite mathematical objects satisfying the properties that constitute a random graph will be isomorphic. We will do this using a standard argument technique called the *back-and-forth argument.*

Of course, the random graph is not the only $\aleph_0$-categorical object.

Perhaps a better-known one is the dense linear order without endpoints (DLO), whose best-known countable model is $\mathbb{Q}$, the set of rational numbers. The text on model theory frequently referenced here, Chang and Kiesler's *Model Theory*[1], gives the proof of another quality that the theory of DLO, or (intuitively) the set

Date: DEADLINE 31 August 2010.

of conditions that together constitute DLO, also possesses. This is the property of *quantifier elimination.* In this paper we will give a similar proof for the random graph.

We will then show that, after proving these two distinct properties of the infinite random graph - to which we will hereafter most often refer to as just *the random graph*- we will draw the conclusion that the theory is complete, giving two entirely different proofs that follow from these two important, powerful, and therefore very infrequently occurring characteristics.

Remark 1.1. Note on references: for facts cited but not proven, we refer the reader to Enderton[3], a standard text in mathematical logic.

2. Basics of Model Theory

We begin with some basic notions of model theory that are absolutely crucial to understanding what we will have done after this. Although we primarily use Chang [1] as reference, Simpson [4] gives a more condensed system of organizing the definitions which better suit the purpose of this paper. We have made use of some of his structuring here.

Before discussing models, or theories, or any such notions, we must first establish a language in which to operate - call it $\mathcal{L}$. We want a language that will describe mathematical universes for us in a formal way. The language has the following form:

Definition 2.1. A formal language $\mathcal{L}$ is a set with three types of symbols: constant, relation, and function symbols. Constant symbols are precisely what their name suggests. Relation symbols have n places (at least one); function symbols have m-placed inputs, single-placed outputs. Every language is allowed to operate with accepted logical symbols from sentential (Boolean) logic and predicate (first-order) logic, as well as the symbol of equality and variables.

Having a language, we define now a *structure* on that language.

Definition 2.2. Take a language $\mathcal{L}$. An $\mathcal{L}$-structure is is a nonempty set, say A, called the *universe*, with an interpretation function $\mathscr{F}$ on it, otherwise written as the ordered pair $(A,\mathscr{F})$. The interpretation function associates the set with the language in the following way:
(i) It assigns a particular element of A to each constant symbol $c \in \mathcal{L}$;
(ii) It assigns a subset of A^n to each n-ary relation symbol; and finally
(iii) It assigns a function $f : A^m \to A$ to each function symbol.

We now want to make statements using the elements of the set and our relations. But of what do we construct these statements? Before cotninuing to the next set of definitions, remark on the method by which they are defined - they are defined inductively - that is, each built up from basic pieces by repeat steps. In this way we define terms and formulas.

As given in Chang [1], we have the following definitions.

Definition 2.3. A *term* satisfies the following conditions:
(i) It is a constant, or
(ii) It is a variable, or
(iii) It has the form $F(x_1, x_2, ..., x_n)$, where F is an n-placed function symbol and each x_i is a term as well, or

(iv) It can be constructed by a finite application of the preceding steps.

From terms we make formulas. The basic building block of the formula is the *atomic formula.* An atomic formula is intuitively the most basic formula, which cannot be further decomposed.

Definition 2.4. An atomic formula ϕ must satisfy one of the following conditions. Note that the x_i are terms. So ϕ must be either:
(i) $x_1 = x_2$, or
(ii) $R(x_1, x_2, \ldots, x_n)$, where R is an n-placed relation symbol.

Now we build all other formulas from these.

Definition 2.5. A *formula* satisfies the following conditions:
(i) It is an atomic formula, or
(ii) It has the form $\neg\psi$ or $\psi \wedge \phi$, where ϕ and ψ are formulas, or
(iii) It is of the form $(\exists x)\psi$, where ψ is a formula, or
(iv) It can be expressed as a finite sequence of applications of the previous three steps.

Note that we do not need to include the $\forall$ quantifier because, following the rules of predicate logic, every statement of the form $(\forall y)\theta$ is equivalent to a statement of the form $(\neg\exists y)(\phi)$.

Also, we call repeated but finite applications of step (ii) to formulas a *boolean combination* of formulas.

In a formula with quantifiers $\forall$ or $\exists$, the variable associated with the quantifier, like the x in condition (iii), is a *bound* variable. A *sentence* is a formula in which all variables are bound by a quantifier. In formula with no quantifiers, all the variables are unbound, or free. We shall refer to such formulas as *open.*

Now we take an $\mathcal{L}$-structure in our language $\mathcal{L}$, say $\mathscr{A} = (A, \mathscr{F})$. Take a formula $\phi(v_1, v_2, \ldots, v_n$ in n variables. For a formal inductive definition of what it means to assign a values to a term, see Chang [1]. Given a particular assignment of values $\{x_1, \ldots, x_n)$ to ϕ, we ask whether this assignment *satisfies* ϕ in $\mathscr{A}$.

Definition 2.6. We say that an assignment $\{x_1, \ldots, x_n)$ *satisfies* the formula ϕ in $\mathscr{A}$, abbreviated by $\mathscr{A} \models \phi[x_1, x_2, \ldots, x_n]$, when the following conditions are met:

(i) If ϕ is $t_1 = t_2$, where t_1 and t_2 are terms in n variables, then an assignment $\{x_1, \ldots, x_n\}$ satisfies ϕ if and only if $t_1(x_1, \ldots, x_n) = t_2(x_1, \ldots, x_n)$;

(ii) If R is a relation symbol in $\mathcal{L}$ and ϕ is $R(t_1, t_2)$, then an assignment $\{x_1, \ldots, x_n\}$ satisfies ϕ if and only if it is the case that $R(t_1(x_1, \ldots, x_n), t_2(x_1, \ldots, x_n)$.

(iii) If ϕ is a boolean combination of formulas, $\{x_1, \ldots, x_n\}$ satisfies ϕ iff the configuration in which $\{x_1, \ldots, x_n\}$ satisfies the component formulas follows the rules for truth assignment in Boolean logic. For example, if ϕ is $\psi_1 \wedge \psi_2$, where ψ_1 and ψ_2 are formulas, $\{x_1, \ldots, x_n\}$ satisfies ϕ iff it satisfies ϕ_1 and ϕ_2 in $\mathscr{A}$. Similarly, if ϕ is $\neg\psi$, then $\{x_1, \ldots, x_n\}$ satisfies ϕ iff it does *not* satisfy ψ in $\mathscr{A}$.

(iv) If ϕ is $(\forall v_1)\psi$, where ψ is a formula and v_i a variable, $\{x_1, \ldots, x_n\}$ satisfies ϕ iff $\{x_2, \ldots, x_n\}$ satisfies ψ for any x_1. Of course, the choice of v_1 and x_1 instead of another v_i and corresponding x_i is arbitrary.

We leave it to the reader to see what happens in the case when ϕ is a sentence. Then we say that ϕ *holds* in $\mathscr{A}$. If ϕ is any formula, it holds in $\mathscr{A}$ if any variable assignment $\{x_1, \ldots, x_n\}$ satisfies it in $\mathscr{A}$.

Note that we assume that the formula is in sufficiently few variables and the assignment has sufficiently many elements for the above conditions to be well-defined. Also, when we use relation symbols, as in condition (iii), and then speak of it in the $\mathcal{L}$-structure $\mathscr{A}$, we assume that we have already applied the interpetation function $\mathscr{F}$. Finally, we again only choose to deal with one quantifier, this time $\forall$ because it is more convenient. As stated before, accounting for the other merely requires a negation.

Passing from formulas to sentences, we note that because sentences consist only of bound variables, we need not think of specific variable assignments; we ask immediately whether sentences do or do not hold, or whether that is indeterminate. We can also subject collections of sentences to analysis.

Definition 2.7. A *theory* T is a collection of sentences in the language $\mathcal{L}$.

Now we introduce the concept of a model to a theory. Informally, a model of a theory T is a world in which the collection of sentences are all true and can all make sense.

Definition 2.8. Take language $\mathcal{L}$ and theory T. The $\mathcal{L} - structure$ A is a *model* of T (or $\mathscr{A} \models T$) if, for every sentence $\sigma \in T$, σ holds in $\mathscr{A}$.

We use the same notation to say a theory models a sentence.

Definition 2.9. We say a *theory* $T \models \sigma$ (where σ is a sentence) if for any model $\mathscr{A}$, $\mathscr{A} \models T$ iff $\mathscr{A} \models \sigma$.

One important relation between formulas is equivalence in a given theory.

Definition 2.10. Let T be a theory in language $\mathcal{L}$. Two formulas ϕ and ψ are said to be T-equivalent if $T \models \phi \Leftrightarrow \psi$.

A theory can have multiple models; given a fixed model one can pick theories that will hold in the model. Taking all the theories we can construct within a model, one is special:

Definition 2.11. A theory T is *complete* in model $\mathscr{A}$ if for every sentence σ possible in $\mathscr{A}$, $T \models \sigma$ or $T \models \neg\sigma$, and for any two models $\mathscr{M}_1$ and $\mathscr{M}_2$ such that $\mathscr{M}_1 \models T$ and $\mathscr{M}_2 \models T$, $\mathscr{M}_1 \models \sigma \Leftrightarrow \mathscr{M}_2 \models \sigma$.

Make a note of the last definition, as it will be important to us later.

Now consider two models for the same theory.

Definition 2.12. Fix a language $\mathcal{L}$. two models, $\mathscr{A} = (A, \mathscr{F})$ and $\mathscr{B} = (B, \mathscr{G})$ are *isomorphic* if there exists a bijection $h : A \to B$ such that:

(i) For every constant term c in $\mathcal{L}$, $\mathscr{F}(c)$ corresponds to $\mathscr{G}(c)$;

(ii) For every relation symbol R in n places, where R_A, R_B denote $\mathscr{F}(R), \mathscr{G}(R)$, and for every n-tuple $(a_1, \ldots, a_n) \in A^n$, $R_A(a_1, \ldots, a_n)$ iff $R_B(h(a_1), \ldots, h(a_n))$.

(iii) For every function symbol F in m places, if we let F_A, F_B denote $\mathscr{F}(F), \mathscr{G}(F)$, it must be the case that for every m-tuple $(a_1, \ldots, a_n) \in A^m$, $F_A(a_1, \ldots, a_n) = F_B(h(a_1), \ldots, h(a_n))$.

Note that this definition easily applies to any $\mathcal{L}$-structures, not merely models.

Also,

Definition 2.13. A $\mathcal{L}$-structure $\mathscr{N}$ isomorphically embeds into another structure $\mathscr{M}$ if it is isomorphic to a subset of $\mathscr{M}$.

We distinguish isomorphism from another quality of models called elementary equivalence.

Definition 2.14. Two $\mathcal{L}$-structures $\mathscr{A}$ and $\mathscr{B}$ are *elemenarily equivalent* if, for any sentence σ_1 such that $\mathscr{A} \models \sigma_1$, $\mathscr{B} \models \sigma_1$, and for any sentence σ_2 such that $\mathscr{B} \models \sigma_2$, $\mathscr{A} \models \sigma_2$ as well.

It is left as an exercise to the reader to ascertain that any two isomorphic $\mathcal{L}$-structures are also elementarily equivalent.

These definitions are not an exhaustive base for model theory; they are, however, sufficient for us to continue our discussion.

3. The Random Graph

We switch now to a basic discussion of the random graph. We remind the reader of the following:

Definition 3.1. A *graph* G is a pair of sets (V, E) where $E \subseteq V \times V$. The elements of V we call *vertices*, and the elments of E we call *edges*. Two vertices $v_1, v_2 \in V$ are *adjacent* to each other if $(v_1, v_2) \in E$, in which case we shall write $v_1 \leftrightarrow v_2$, and $v_1 \not\leftrightarrow v_2$ otherwise. Note that this is not a standard symbol for adjacency. For our purposes, we will require that adjacency satisfies the following conditions:

(i) No vertex can be adjacent to itself.

(ii) For any two vertices $v_1, v_2 \in V$, if $(v_1) \leftrightarrow v_2$, then $v_2 \leftrightarrow v_1$.

A graph is usually visually represented with the vertices as points and the edges as line segments connecting the adjacent ones.

Now, let us contstruct a graph in the following way: We take a set V of vertices. In considering each pair $(v_i, v_j) \in V \times V$ - i.e., every potential edge - we toss a (possibly unfair) coin to decide whether (v_i, v_j) will be in the edge set E. Let $p \in [0, 1]$ denote the probability with which a given pair *is* included. We assume all the edges have the same probability of occurrence. We denote the set of graphs constructed in this manner by $\mathcal{G}(n, p)$, where n is the number of elements in the vertex set. Analogously, for countably infinite random graphs, we write $\mathcal{G}(\aleph_0, p)$, a set in which all the graphs have countably many vertices. From now on, when we refer to a random graph, we will mean an element of $\mathcal{G}(\aleph_0, p)$. See Diestel [2] for a more formal treatment of this construction, including the proof that it induces a valid probability measure on the set of graphs with n vertices.

Diestel also gives the following excercise about random graphs, which we include as a lemma. Before stating it, we define the following property:

Definition 3.2. A graph G has property $\mathcal{P}_{i,j}$, with $i, j = 0, 1, 2, 3, \ldots$ if, for any dijoint vertex sets V_1 and V_2 with $|V_1| \leq i$ and $|V_2| \leq j$, there exists a vertex $v \in G$ that satisifies three conditions:

(i) $v \notin V_1 \cup V_2$;

(ii) $v \leftrightarrow x$ for every $x \in V_1$; and

(iii) $v \not\leftrightarrow y$ for every $y \in V_2$.

Remark 3.3. We let i and j to range over zero as well in order to simplify the proof of $\aleph_0$-categoricity later.

And then the lemma:

Lemma 3.4. *An infinite graph $G \in \mathcal{G}(\aleph_0, p)$ has all the properties $\mathcal{P}_{i,j}$ with probability* 1.

Proof. The proof, which is an exercise in the probabalistic method, is left to the reader. □

By the above lemma it is always the case that every $\mathcal{P}_{i,j}$ holds for any infinite randomly-generated graph in $\mathcal{G}$. It is also clear that for this property to hold, the graph *must* be infinite. The exercise gives us an alternative way to construct a random graph, which is precisely what we will use to write down the first-order theory of the random graph.

First, we define a language $\mathcal{L}_{RG} = \{\leftrightarrow\}$. Note that any language assumes first-order logic symbols and equality.

We shall have a countably infinite number of axioms. They are as follows:

(i) $(\exists x, y)[\neg(x = y)]$ - stating that the graph is nonempty, with at least two distinct points.

(ii) $(\forall x, y)[(x \leftrightarrow y) \Rightarrow \neg(x = y)]$ - stating that adjacency is not reflexive.

(iii) $(\forall x, y)[(x \leftrightarrow y) \Rightarrow (y \leftrightarrow x)$ - stating that adjacency is symmetric.

Let $n, m \geq 0$ be natural numbers. For ease of notation, let i and j range over the following integer values: $i = 0, 1, 2, \ldots, n$ and $j = 0, 1, 2, \ldots, m$. Then the $(n, m)^{th}$ axiom will be:

(n, m) $(\forall x_0, x_1, x_2, \ldots, x_n, y_0, y_1, y_2, \ldots, y_m)[(\bigwedge x_i \neq y_j) \Rightarrow (\exists z)(\bigwedge z \neq x_i \wedge \bigwedge z \neq y_j \wedge \bigwedge z \leftrightarrow x_i \wedge \bigwedge z \not\leftrightarrow y_j)]$

In English, this last is precisely the statement of the property $\mathcal{P}_{n,m}$.

4. The $\aleph_0$-categoricity of the random graph

The random graph, we shall see, is $\aleph_0$-categorical, which means that up to isomorphism, there is only one countably infinite model for its theory. To show this, we suppose that there exist two countable models in which this theory holds. We call them, say, $\mathscr{M}$ and $\mathscr{N}$. We now want to construct an isomorphism between them. And so:

Theorem 4.1. *There theory of the random graph, which we shall call RG for short, has only one countably infinite model up to isomorphism.*

Proof. To construct an isomorphism, which we will call f, we use a standard technique called a *back-and-forth argument.* Choose a vertex $x_1 \in \mathcal{M}$. The first axiom we listed assures us that such an x_1 exists. Because RG also holds in the model $\mathscr{N}$, it must contain some point y_1. Let $f(x_1) = y_1$. Also, by the first axiom, we have there exists at least one other point in $\mathscr{N}$ distinct from y_1.

Now, to choose a corresponding point in $\mathscr{M}$, we need to make sure that all the corresponding relations hold. In this case, the only relation we have to worry about is the edge relation. Suppose that $y_1 \leftrightarrow y_2$. Then we must choose a point x_2 in $\mathscr{M}$ that is analogously related to x_1. We know that some point $x \neq x_1$ exists, because by axiom (i) there are at least two elements in $\mathscr{M}$. Now suppose we define the sets $V_1 = \{x_1\}$ and $V_2 = \{x\}$. Then by one of the countably many other axioms, we can find a third point $x_2 \in \mathcal{M}$, $x_2 \neq x$ or x_1, such that x_2 is adjacent x_1 but not x. If we let $f(x_2) = y_2$, the desired edge relation is preserved. In the contrary case, if $y_2 \not\leftrightarrow y_1$, we simply choose x_2 adjacent to x and not to x_1. Now in set $\mathcal{M}$, we consider the aforementioned point x, which we shall rename as x_3. We consider its adjacency relations to the other two points, x_1 and x_2. We want to choose an image point $y_3 \in \mathcal{N}$ such that the same relations hold (pairwise) among y_3, y_1 and y_2 as among x_3, x_1 and x_2.

To do this, we use our remaining axioms, which give us the properties $\mathcal{P}_{m,n}$ for all natural numbers m, n. We verify that there will be a third point. Because in stating the $\mathcal{P}_{m,n}$ property we allowed m, n to be zero, which permits one of the sets V_1 and V_2 to be empty, we can easily find the third point in $\mathcal{N}$ the way we found the second in set $\mathcal{M}$ - that is, depending on the adjacency relation between y_1 and y_2, we set V_1 and V_2 to be, respectively, $\{y_1\}$ and $\{y_2\}$, $\{y_2\}$ and $\{y_1\}$,$\{y_1, y_2\}$ and $\emptyset$, or $\emptyset$ and $\{y_1, y_2\}$. And again, the only way all axioms can be satisfied is if our graph has infinitely many vertices. We define $y_3 = f(x_3)$. In a similar way, we find some arbitrary point $y_4 \in \mathcal{N}$ and pick a corresponing point $x_4 \in \mathcal{M}$ respecting the same adjacency relations as y_4 has with y_1, y_2, and y_3. Since this can be continued indefinitely, we will eventually have a full isomorphism in the model-theoretic definition of the term, respecting the relations in the language. We will then be done. □

Having proved that any countably infinite random graph is isomorphic to another, we can justify calling it *the* random graph.

An interesting thing to note is that, because we did not distinguish the points we chose in any way, we can build this isomorphism up from any point we happen to select. Even more interestingly and significantly, $\aleph_0$-categoricity also implies that the theory of the random graph is complete.

Recall that a theory T is complete when, for any sentence σ, $T \models \sigma$ or $T \models \neg\sigma$, and for any two models $\mathscr{M}_1$ and $\mathscr{M}_2$ such that $\mathscr{M}_1 \models T$ and $\mathscr{M}_2 \models T$, $\mathscr{M}_1 \models \sigma \Leftrightarrow \mathscr{M}_2 \models \sigma$. The two parts are in fact equivalent, because if there were some σ without a definte truth value given T, we could construct two models in which T would hold but one of which also modeled σ and the other $\neg\sigma$. Now we state and prove the completenes of the random graph.

But first, we state a fact we will require, in the following form:

Fact 4.2. *(The Downward Löwenheim-Skolem Theorem) Let $\mathcal{L}$ be a countable language and T be a countable collection of sentences. If there exists an infinite model $\mathscr{A}$ such that $\mathscr{A} \models T$, then $\mathscr{A}$ has a coutably infinite submodel $\mathscr{B}$ such that $\mathscr{B} \models T$. Note here that the cardinality of a model is simply the cardinality of the universe set.*

Having stated that, we can continue with the corollary.

Corollary 4.3. *The theory of the random graph is complete.*

Proof. Suppose the theory of the random graph, RG, were not complete. Any model of RG must be infinite, so there is an infinite model modeling RG. If RG is not complete, there exists a sentence σ such that neither it nor its negation is provable. This means, for our purposes, that there exist models $\mathscr{M}_1$ and $\mathscr{M}_2$ such that $\mathscr{M}_1 \models \{\sigma\} \cup RG$ and $\mathscr{M}_2 \models \{\neg\sigma\} \cup RG$. Both $\mathscr{M}_1$ and $\mathscr{M}_2$ are infinite. By the The Downward Löwenheim-Skolem Theorem, $\mathscr{M}_1$ and $\mathscr{M}_2$ must have countable submodels $\mathscr{N}_1$ and $\mathscr{N}_2$ such that $\mathscr{N}_1 \models \sigma$ and $\mathscr{N}_2 \models \neg\sigma$. But this cannot be the case, because by the $\aleph_0$-categoricity of RG, $\mathscr{N}_1$ and $\mathscr{N}_2$ must be isomorphic. This is a contradiction, so we are done. □

Now observe: we have constructed an isomorphism that preserves all basic edge relations - so to state that $x_1 \in \mathcal{M} \leftrightarrow x_2 \in \mathcal{M}$ is equivalent to making the same statement for $y_1, y_2 \in \mathcal{N}$. We know that any sentence made about the other. Now, suppose we take a formula with at least one open variable. If this formula is simply a statement about the adjacency of two variables - say, $v_1 \leftrightarrow v_2$ - we know that the variable assignments satisfying this formula in set $\mathcal{M}$ will also satisfy it in $\mathcal{N}$. This is also true negations and conjunctions of the basic statements, as well as repeated applications thereof. This follows from sentential logic. Now suppose that we add quantifiers. If we have a formula, say, in m free variables and n variables bound by quantifiers, what guarantee do we have that the valid variable assignments for that formula will hold up under our isomorphism? Well, instead of checking every formula individually, we rely on the fact that the theory of the random graph admits the *elimination of quantifiers.*

5. Quantifier Elimination in the Random Graph

We can now show that the random graph admits quantifier elimination - that is, that every formula is *RG*-equivalent to an open formula. More generally, we offer the following definition:

Definition 5.1. A theory T *admits quantifier elimination* if, for every formula ϕ, there exists an open formula ψ such that $T \models \phi \Leftrightarrow \psi$.

As we said before, it is clearly not feasible to check this statement for *each* formula in our language. It would therefore be helpful to divide all possible formulas into a few groups with which to work. The following lemma, slightly modified from that given in Chang [1], provides just such a simplification. Let J be the set of open formulas (using our specified language), and J^* be the set of all formulas RG-equivalent to some $\theta \in J$.

Lemma 5.2. *To show that the random graph admits quantifier elimination, it is sufficient to show that any formula of the form* $(\exists x)\psi$*, with* $\psi \in J$*, belongs to* J^**.*

Proof. Take a formula ϕ. If ϕ is an open formula, it is in J, and therefore in J^*, since it is certainly equivalent to itself. Now, consider all elements of J^*. For any $\phi \in J^*$, if ϕ is equivalent to some $\theta \in J$, $\neg\phi$ is equivalent to $\neg\theta$. $\neg\theta$ is in J, so $\neg\phi$ is in J^*. Take ϕ_1 and ϕ_2 are in J^*, so they are equivalent to θ_1 and θ_2 in J. Then $\phi_1 \wedge \phi_2$ is equivalent to $\theta_1 \wedge \theta_2$. Since J^* is closed under negation and conjunction, it is also closed on disjunction, conditional, and biconditional operators (from sentential logic). Now suppose we have ϕ in the form $(\exists x)\psi$, with $\psi \in J^*$. By the hypothesis of the lemma, ϕ is also an element of J^*. If ϕ has the form $(\forall x)\psi$, with $\psi \in J^*$, we simply write it equivalently as $\neg[(\exists x)(\neg\psi)]$. Since J^* is closed under negation, in this case ϕ is also in J^*. Any other formula can be built up by a finite sequence of these three operators - conjunction, negation, and the existential quantifier. Therefore given the lemma's hypothesis, all formulas are equivalent to quantifier-free ones. We are done. □

Chang gives another lemma that will simplify things for us:

Lemma 5.3. *Every open formula* ψ *is equivalent to a finite disjunction of terms* θ_j *of the form* $\bigwedge \sigma_i$*, where every* σ *is an atomic formula or the negation of one, and there are finitely many* σ *terms.*

Proof. Observe that we have two atomic formulas: $x = y$ and $x \leftrightarrow y$. Consider any statement, however complicated, that combines these two types of formula using standard symbols from sentential logic - negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$), implication ($\Rightarrow$), and biconditional ($\Leftrightarrow$). The last three are just shorthand notation, used for simplicity; they are, in fact, redundant, because they can be restated equivalently using only conjunctions and negations. Thus:

- $x \vee y$ is equivalent to $\neg(\neg x \wedge \neg y)$;
- $x \Rightarrow y$ is equivalent to $\neg x \vee y$; and
- $x \Leftrightarrow y$ is equivalent to $(x \wedge y) \vee (\neg x \wedge \neg y)$.

Thus any statement without existential or universal quantifiers can be made using combinatons of conjunctions and negations of atomic formulas - i.e., a boolean combination of atomic formulas. So the open formulas are precisely the Boolean combinations of basic formulas.

The reader is referred to Chang [1] for a more detailed proof. □

Now we turn once again to the theory of the random graph. This proof follows the structure given in Chang [1] of the same property in the theory of Dense Linear Order, for which the rational numbers are the only countably infinite model, up to isomorphism.

So we can now prove the following theorem:

Theorem 5.4. *The theory of the random graph admits elimination of quantifiers.*

Proof. By the firat lemma stated in this section, to prove that RG admits quantifier elimination we only need to show that any formula ϕ of the form $(\exists v)\psi$, with ψ an open formula, is equivalent to some open formula as well. We know that the formula must be in a finite number of variables - at least one. Suppose it has only one variable, v_1. If v - the bound variable - is not the same as v_1, we are done, because then ϕ is just equivalent to ψ. If they are the same, then ψ is a Boolean combination of one of the following formulas: $v = v$ and $v \leftrightarrow v$. It follows from our axioms that the first of these is true, and the second is false (adjacency is not reflexive) - for any value of the variable we choose to substitute. On the basis of this, we can find the truth value of any series of conjunctions and negations of these statements using standard sentential logic - a conjunction is only true if all of its conjuncts are true - otherwise false; a negation gives the truth value opposite of the sentence being negated. Or, to state this otherwise: for any boolean combination of the atomic formulas in one variable, given any model for which RG holds, either the combination itself or its negation will hold as well. And, given a specific such combination, we can determine which it will be. So if a boolean combination of atomic formulas, such as ψ, will either always hold in a model where RG holds or always fail, then surely $(\exists v)\psi$ must also always hold or never - always, if ψ does, never if it does not. Then $RG \models (\psi \Leftrightarrow \phi)$. Since ψ is open, we have ϕ equivalent to an open formula.

Now, suppose that $\psi = \psi(v_1, ...v_n), n \geq 2$. Again, if $v \neq v_i$ for any $i = 1, 2, 3, \ldots, n$, then ϕ is RG-equivalent to ψ. If v is the same as some v_i is the most important case. To solve it, we use the second helper lemma.Now, if ψ is equivalent to a finite disjunction, when we write $(\exists v)\psi$ we can “distribute” the existential quantifier and write an equivalent statement of the form $\bigvee[(\exists v)\theta_j]$. Now suppose each θ_j uses variables v_1 through v_n, one of which is the same as v. Since θ is of the form $(\exists v_k)(\bigwedge \sigma_i)$, this is equivalent to writing the same sequence of σ_i that do not include v_k. Each θ_j, thus, is equivalent to some open formula not including v, so the disjunction of θs can also be written not including v. In that case, the existential quantifier is not necessary, and $(\exists v)\psi$ is equivalent to some formula in one fewer variables, but a certifiably open one. □

We have already proven that the theory of the random graph is complete by using the isomorphism constructed in a previous section. Suppose we had not done that - so for all we know, the random graphs may not be isomorphic. Let us take only the knowledge of qantifier elimination as foundation and attempt to prove completeness merely from quantifier elimination. To do this, we will use the following fact:

Fact 5.5. *Take the theory RG. Suppose $\mathscr{N}$ is an $\mathcal{L}$-structure that embeds isomorphically into any model $\mathscr{U}$ of RG. Then, if RG admits quantifier elimination, RG is complete.*

Theorem 5.6. *The theory of the random graph is complete. This can be proven without making use of the $\aleph_0$-categoricity of the random graph.*

Proof. Using the notation from the lemma, let $\mathscr{N} = (\{x\}, \mathscr{F}$, where $\{x\}$ is a set with one element and $\mathscr{F}$ is the interpretation function mapping the relation symbol $\leftrightarrow$ to $\emptyset$, the empty set. This clearly embeds into every model of RG. Since we have quantifier elimination for RG, we apply the lemma and are done. □

6. Conclusion

Thus we have proven completeness using two different properties of the random graph. Note that these two properties, $\aleph_0$-categoricity and elimination of quantifiers, are not related by any implication. There are examples of theories with one property and not the other. It is also worth noting that one of the earlier uses of quantifier elimination was precisely to prove completeness of a theory. With that, we leave the reader to explore the concept more on his own, and to attempt similar proofs for the Dense Linear Order.

7. Acknowledgments

Many, many effusive thanks to Maryanthe Malliaris whose advice gave my paper a direction, spine and skeleton, and who introduced me to a whole new field of mathematics to explore.

Likewise I would like to thank my mentors,my mentors, Damir Dzhafarov and Eric Astor, for patient, insightful, exhaustive comments (osobenno spasibo Damiru), hints, editing, and guidance (as well as some great leisure-time reading, Eric).

And of course to Peter May, without whom this program may not even have been in existence.

References

[1] C.C.Chang and H. Jerome Kiesler. *Model Theory, 2nd ed.* North Holland Publishing Company. 1977 (reprinted 1978). Logical definitions: Formulas -. 20-24 formulas, 25-29 satisfaction, 36-40 theories. Quantifier elimination pp. 49-53

[2] Reinhard Diestel. *Graph Theory.* Springer. 1999. Definition of Random Graph, pp. 228-230 Property $\mathcal{P}_{m,n}$: pp. 236, 246.

[3] Herbert Enderton and Herbert B. Enderton. *A Mathematical Introduction to Logic, 2nd ed.* Academic Press. 2001.

[4] Stephen G. Simpson. "Math 563: Model Theory", pp. 7-15. May 2, 1998. Pennsylvania State Math Dept Website. http://www.math.psu.edu/simpson/courses/math563/master.pdf