ULTRAPRODUCTS AND MODEL THEORY

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Abstract. The first-order model-theoretic description of mathematical structures is unable to always uniquely characterize models up to isomorphism when the models are not finite. In this paper I look to ultraproducts of models to remedy this somewhat. By taking the ultraproduct construction over models, we form a new model out of many that preserves all of the first-order logical sentences of "most" of the original models. This construction will be useful for characterizing when models are equivalent according to their first-order model-theoretic description, and for describing the class of models that are equivalent in this way.

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1. Introduction

The ultraproduct is formed by taking the cartesian product of at least a countable number of sets and identifying those elements that agree on a "large" set of sets in the cartesian product. By carefully making constraints on the meaning of "large" we are able to change the construction of the ultraproduct. In this paper, I exhibit the use of this technique in model theory where an ultraproduct is itself a model that preserves first-order logical statements. I show how this construction is useful in characterizing when models are equivalent according to their first-order model-theoretic description, and to describe the class of models that are equivalent in this way.

2. Preliminary Definitions for Model Theory

We begin by defining a few basic terms intrinsic to model theory.

Definition 2.1. A *first-order language* $\mathscr{L}$ is a set of *relation* symbols, *function* symbols, and *constant* symbols as well as variables $v_1, v_2, ..., v_i, ...,$, logical symbols $\forall$, $\exists$, (,), $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftarrow$, $\leftrightarrow$ and the binary equality relation $\equiv$. In this paper we

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will only deal with languages that are first-order, and we will denote them using script letters.

Definition 2.2. Given a language $\mathscr{L}'$ that contains all the symbols of $\mathscr{L}$ and perhaps some additional symbols, we call $\mathscr{L}'$ an *expansion* of $\mathscr{L}$, $\mathscr{L}$ a *reduction* of $\mathscr{L}'$, and write $\mathscr{L}' \supseteq \mathscr{L}$.

Definition 2.3. To give our language some meaning we have to interpret what the elements of our language mean. We therefore define an *interpretation function* $\mathcal{I}$ to be a correspondence between $\mathscr{L}$ and a *universe set* A so that $\mathcal{I}$ maps n-ary relations of $\mathscr{L}$ to n-ary relations $R \subseteq A^n$ on A, m-ary functions of $\mathscr{L}$ to m-ary functions $G : A^m \to A$ on A, and each constant symbol to a constant $x \in A$.

Definition 2.4. A *model* $\mathfrak{M}$ is a pair $\langle M, \mathcal{I}\rangle$ of a universe set M and an interpretation function $\mathcal{I}$. We will denote models by gothic letters throughout this paper and their universe sets by their corresponding capital letter.

Definition 2.5. A model $\mathfrak{M}$ is a *reduction* of $\mathfrak{M}'$ if

(i) $M \subseteq M'$.
(ii) Each n-ary relation R of $\mathfrak{M}$ is the restriction to M of the corresponding relation R' of $\mathfrak{M}'$.
(iii) Each m-ary function G of $\mathfrak{M}$ is the restriction to M of the corresponding function G' of $\mathfrak{M}'$
(iv) Each constant of $\mathfrak{M}$ is the corresponding constant of $\mathfrak{M}'$.

We will write this as $\mathfrak{M} \preceq \mathfrak{M}'$ and call $\mathfrak{M}'$ an *expansion* of $\mathfrak{M}$.

We will now work our way to a definition of a first-order theory, starting first with the idea of a first-order formula. The concept of a formula of $\mathscr{L}$ is defined inductively as follows.

Definition 2.6. The *terms* of $\mathscr{L}$ are strings of symbols of $\mathscr{L}$ which come about through finite applications of the following rules:

(i) A variable is a term.
(ii) A constant symbol is a term.
(iii) Given an m-ary function F and terms $t_1, t_2, ..., t_m$, then $F(t_1, ..., t_m)$ is a term.

The *atomic formulas* of $\mathscr{L}$ are strings of the following form:

(i) Given terms t_1, t_2, $t_1 \equiv t_2$ is an atomic formula.
(ii) Given an m-ary relation P and terms $t_1, t_2, ..., t_m$, then $P(t_1, ..., t_m)$ is an atomic formula.

Finally, the *formulas* of $\mathscr{L}$ are defined inductively as follows

(i) An atomic formula is a formula.
(ii) If ϕ is a formula, then so is $\neg\phi$.
(iii) If ϕ and ψ are formulas, then so is $\phi \wedge \psi$.
(iv) Given a formula ϕ and a variable x, $(\forall x)\phi$ is a formula too.

We also define the other symbols we need in the usual way. Thus, $\exists x := \neg(\forall x)\neg\phi$ and $\phi \vee \psi := \neg((\neg\phi) \wedge (\neg\psi))$. Of course, we also need some logical axioms, but we will assume that the reader is already familiar with those.

Definition 2.7. The variables of a formula that have no quantifier are called *free variables.* We call a formula that contains no free variables a *sentence.*

We will sometimes write formulas in the form $\phi(x_1, x_2, ..., x_n)$ denoting that the free variables of ϕ are a subset of $\{x_1, ..., x_n\}$. We will usually do this only when we wish to call attention to the free variables. We will also write $\bar{x}$ to mean $x_1, ..., x_n$, thus $\phi(\bar{x}) = \phi(x_1, ..., x_n)$. We will use this notation as a shorthand especially when we wish to suppress the n.

In any given model of $\mathscr{L}$, all sentences of $\mathscr{L}$ are either true or false. For any sentence ϕ of $\mathscr{L}$, we use the notation $\mathfrak{M} \models \phi$ to mean that ϕ is true in $\mathfrak{M}$.

Definition 2.8. A *theory* T of a language $\mathscr{L}$ is a set of sentences of $\mathscr{L}$. The theory of $\mathfrak{M}$, denoted $\mathrm{Th}(\mathfrak{M})$, is the set of all the sentences of $\mathscr{L}$ that are true in $\mathfrak{M}$.

For any sentence ϕ of $\mathscr{L}$, we use the notation $T \models \phi$ to mean that for every model $\mathfrak{N}$ of T, $\mathfrak{N} \models \phi$. This leads us to the following equivalence relation:

Definition 2.9. Given models $\mathfrak{M}$ and $\mathfrak{N}$, we write

$$\mathfrak{M} \equiv \mathfrak{N} \text{ if and only if } \mathrm{Th}(\mathfrak{M}) = \mathrm{Th}(\mathfrak{N}),$$

i.e., every sentence that is true in $\mathfrak{M}$ is true in $\mathfrak{N}$. When this happens, we say that $\mathfrak{M}$ and $\mathfrak{N}$ are *elementarily equivalent.*

Although we already used the symbol $\equiv$, this is not an abuse of notation since $\equiv$ was previously only defined between terms of $\mathscr{L}$, and now we are using $\equiv$ between models of $\mathscr{L}$. Checking that $\equiv$ is an equivalence relation is trivial, as are the following properties of elementary equivalence that we leave the reader to prove.

Proposition 2.10. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be models.*

1. *Th($\mathfrak{M}$) is complete and consistent (i.e., for all sentences ϕ in $\mathscr{L}$, either $\mathfrak{M} \models \phi$ or $\mathfrak{M} \models \neg\phi$, but never both).*
2. *If $\mathfrak{M}$ and $\mathfrak{N}$ are isomorphic, then $\mathfrak{M} \equiv \mathfrak{N}$.*

Unless $\mathfrak{M}$ is finite, the converse to statement 2 of the above proposition is not true, and indeed we will see a counterexample in the next section. Therefore, the description of a model that this logic gives us is somewhat ambiguous. One of the principal goals of this paper will be answering the question of when a given class of models includes all of the models of a given theory.

3. Ultraproducts

Definition 3.1. For a set A, an *ultrafilter* U over A is a subset of the power set $\mathcal{P}(A)$ with the properties that

(i) $A \in U$.
(ii) If $V \in U$ and $W \in U$ then $V \cap W \in U$.
(iii) If $V \in U$ and $V \subseteq C \subseteq A$ then $C \in U$.
(iv) For any set $K \in \mathcal{P}(A)$, either $K \in U$ or $A \setminus K \in U$.

Intuitively we think of the ultrafilter as containing the subsets of A that are "large." Not all ultrafilters are created equal however. In particular, we would like to separate the trivial ultrafilters from the non-trivial ones. The trivial ultrafilters we will call principal.

Definition 3.2. An ultrafilter U on A is *principal* if there is an element $d \in A$ that acts as a dictator of U. That is, if $J \subseteq \mathcal{P}(A)$ then $J \in U$ if and only if $d \in J$. Ultrafilters without dictators are called *non-principal*.

Definition 3.3. Given a family of sets A_i with $i \in I$, an index set I with an ultrafilter U on I, we consider the cartesian product $\prod_{i\in I} A_i$, and on this product we define the relation $\sim$:

$$\text{For } f, g \in \prod_{i\in I} A_i, f \sim g \text{ if and only if } \{i \in I \mid f_i = f_i\} \in U.$$

It is easy to see that this is indeed an equivalence relation, and we can therefore look at the equivalence classes $f_U = \{g \mid g \sim f\}$. We can then define the *ultraproduct* $\prod_U A_i = \prod_{i\in I} A_i \Big/ U$ as the set of $\sim$ equivalence classes:

$$\prod_{i\in I} A_i \Big/ U = \{f_U \mid f \in \prod_{i\in I} A_i\}.$$

Just like a cartesian product, we can think of the elements $f_U \in \prod_U A_i$ as the equivalence classes of functions that map each $i \in I$ to an element $f_U(i)$ of A_i.

We now connect this to the material in section 2.

Definition 3.4. Given an ultrafilter U over a set I and a family $\mathfrak{M}_i$ of models of $\mathscr{L}$ with universe set M_i for each $i \in I$, the *ultraproduct* $\prod_{i\in I} \mathfrak{M}_i \Big/ U$ is the unique model of $\mathscr{L}$ whose universe set is the ultraproduct $\prod_U M_i$ and with formulas from the $\mathfrak{M}_i$'s in the following way. For each atomic formula $\phi(x_1, ..., x_k)$ with at least one symbol from $\mathscr{L}$, and for every $f_1, ..., f_k \in \prod_{i\in I} M_i$,

$$\prod_{i\in I} \mathfrak{M}_i \Big/ U \models \phi(f_{1U}, f_{2U}, ..., f_{kU}) \text{ iff } \{i \mid \mathfrak{M}_i \models \phi(f_1(i), ..., f_2(i))\} \in U.$$

It is not difficult to show that the ultraproduct is well-defined, but it is tedious. We leave the details to the reader.

In the case that all of the $\mathfrak{M}_i$ are the same, i.e. that there is a model $\mathfrak{N}$ such that $\mathfrak{M}_i = \mathfrak{N} \; \forall \; i \in I$, we will call this ultraproduct the *ultrapower* of $\mathfrak{N}$, and denote it by $\mathfrak{N}^{|I|}/U$.

Whew. That was a lot of definitions! Let's move on to some of the results that ultraproducts give us. We will first give without proof some of the properties of ultraproducts that make them powerful. We would like to know when we can actually build ultrafilters. To this end, we have this

Theorem 3.5 (Finite Intersection Property)**.** *Given $X \subseteq \mathcal{P}(I)$, if the intersection over any finite subset of X is nonempty, then there exists an ultrafilter U over $\mathcal{P}(I)$ so that $X \subseteq U$.*

Much of the power of ultraproducts comes from the following two theorems.

Theorem 3.6 (Expansion Theorem)**.** *Given a language $\mathscr{L}$ and an expansion $\mathscr{L}' \supset \mathscr{L}$, let I be a non-empty set, U an ultrafilter over I, so that for each $i \in I$ we have a model $\mathfrak{M}_i$ for $\mathscr{L}$ and we have a model $\mathfrak{M}_i' \succeq \mathfrak{M}_i$ in $\mathscr{L}'$. Then $\prod_U \mathfrak{M}_i' \succeq \prod_U \mathfrak{M}_i$ in $\mathscr{L}'$.*

Theorem 3.7 (Łoś's Theorem)**.** *For all sentences ϕ of $\mathscr{L}$,*

$$\prod_U \mathfrak{M}_i \models \phi \text{ if and only if } \{i \in I \mid \mathfrak{M}_i \models \phi\} \in U.$$

In the case of an ultrapower, $\mathfrak{M} \equiv \prod_U \mathfrak{M}^I$

Ultraproducts give us a very natural proof of the following major theorem of model theory, the Compactness Theorem.

Theorem 3.8 (Compactness Theorem)**.** *A set of sentences Σ of a language $\mathscr{L}$ has a model if and only if every finite subset of Σ has a model.*

Proof. The forward direction is trivial since it is clear that if Σ has a model $\mathfrak{M}$, then $\mathfrak{M}$ is also a model for any finite subset of Σ.

To prove the backwards direction, take I to be the set of all finite subsets of Σ. For each $i \in I$ by assumption there exists a model $\mathfrak{M}_i$, and we can define the set $S_i = \{j \in I \mid j \supseteq i\}$. Then the set $S = \{S_i \mid \forall i \in I\}$ satisfies the finite intersection property. Therefore by Theorem 3.5 there exists an ultrafilter U over I containing all the sets of the form S_i, and we can take the ultraproduct $\prod_U \mathfrak{M}_i$.

This ultraproduct is a model of Σ. This is because for any $\sigma \in \Sigma$, $\{i \in I \mid \mathfrak{M}_i \models \sigma\} \supseteq S_{\{\sigma\}}$ and since $S_{\{\sigma\}} \in U$, we see that $\{i \in I \mid \mathfrak{M}_i \models \sigma\} \in U$, by property (iii) of Definition 3.1. By Łoś's theorem then, $\prod_U \mathfrak{M}_i \models \sigma$, hence $\prod_U \mathfrak{M}_i$ is a model of Σ. □

Corollary 3.9. *There exists a model $\mathfrak{V}$ satisfying all of the sentences in* Th*($\mathbb{N}$) such that $\mathfrak{V} \ncong \mathbb{N}$.*

Proof. It is well known that the natural numbers are a model of the Peano Arithmetic axioms PA. We wish to define a language PA′ in the language of Peano Arithmetic along with a new constant symbol x. PA′ contains all of the axioms of PA as well as an axiom for each $n \in \mathbb{N}$, namely that $x > n$. Any finite subset of PA′ is satisfied by a model $\mathfrak{N}$ that is the standard model of arithmetic along with a constant corresponding to x which becomes the largest number of $\mathfrak{N}$. Thus by the compactness theorem there is a model $\mathfrak{V}$ that satisfies all of the axioms of PA′. Then $\mathfrak{V}$ satisfies all of the sentences in P, the Th($\mathbb{N}$), but $\mathfrak{V}$ contains an element corresponding to x, a "largest" element of $\mathfrak{V}$. Since no such element exists in $\mathbb{N}$, we see that $\mathfrak{V} \ncong \mathbb{N}$. □

This proves the existence of such a model, but the result of the compactness theorem was not merely existential; we actually constructed the model that we were looking for. Using the ultraproduct construction here, we can get the sort of construction we are looking for.

Construction 3.10. We will look at the ultrapower of the natural numbers. We get this ultrapower from considering $S_k = \{n \in \mathbb{N} \mid n \geq k\}$ for each $k \in \mathbb{N}$. Then it is clear that the S_k's satisfy the finite intersection property, so by Theorem 3.5 we can build a non-principal ultrafilter U on $\mathbb{N}$ so that $\forall k, S_k \subseteq U$. It is clear from the way that we constructed U that there can be no dictator of U and therefore that U is non-principal.

Now we have our ultrafilter U, let's boldly step ahead and build an ultrapower $\mathbb{N}^\lambda/U$, given some cardinal $\lambda \geq \aleph_0$. Now by Łoś's theorem, $\mathbb{N} \equiv \mathbb{N}^\lambda/U$, however, as we will show, they are not isomorphic. To do this we will consider the sentences $\phi_k(x) = \exists x(x \in S_k)$ and take them all together:

$$\psi_i(x) = \bigwedge_{k<i} \phi_k(x).$$

Then by the way we constructed U, $\mathbb{N}^\lambda/U \models \psi_i(x)$ for each $i < \aleph_0$, hence

$$\{t < \aleph_0 \mid \mathbb{N} \models \exists x \psi_i(x)\} \in U.$$

Observe that the S_k's form a descending chain in the ultrafilter,

$$\mathbb{N} = S_1 \supset S_2 \supset S_3 \supset \dots$$

but that $\bigcap_{i<\aleph_0} S_i = \emptyset$. So for each $i < \aleph_0$ we define an $l \in \mathbb{N}^\lambda/U$ so that $l(i) \in \mathbb{N}$ so that $\mathbb{N}_i \models \psi_i[l(i)]$. This ensures that for each i, $\mathbb{N}_i \models \phi_i[l(i)]$. Thus by Łoś's Theorem, $\mathbb{N}^\lambda/U \models \phi_i[l_U]$ for all $i > 0$, which means that l_U satisfies $\psi_i(x)$ in $\mathbb{N}^\lambda/U$ for each i, exactly what we wanted.

4. Saturation

Definition 4.1. Take a model $\mathfrak{M}$ of a language $\mathscr{L}$ and $A \subseteq M$, the universe set of $\mathfrak{M}$. Let $\mathscr{L}_A = \mathscr{L} \cup \{c_a \mid a \in A\}$, obtained by adding a constant c_a for each $a \in A$ to the original language $\mathscr{L}$. Let $\mathrm{Th}_A(\mathfrak{M})$ be the set of all sentences of $\mathscr{L}_A$ that are true in $\mathfrak{M}$.

Let p be the set of $\mathscr{L}_A$-formulas in free variables $v_1, v_2, ..., v_n$. Then p is called an *n-type* if $p \cup \mathrm{Th}_A(\mathfrak{M})$ is satisfiable. We call p a *complete n-type* if $\phi \in p$ or $\neg\phi \in p$ for every $\mathscr{L}_A$-formulas ϕ with the free variables $v_1, v_2, ..., v_n$. We denote by $S_n^{\mathfrak{M}}(A)$ the set of all complete n-types.

If p is a complete n-type over A, we say that $\mathfrak{M}$ *realizes* p if there exists an $\bar{a} = (a_1, a_2, ..., a_n) \in M^n$ so that $\mathfrak{M} \models \phi(\bar{a})$ for all $\phi \in p$.

Definition 4.2. For any cardinal κ, a model $\mathfrak{M}$ is *κ-saturated* if for all subsets $A \subseteq M$ with $|A| < \kappa$, $\mathfrak{M}$ realizes all complete types over A. In the case that $\kappa = |M|$, we say that $\mathfrak{M}$ is *saturated.*

We would like to know when models are saturated. Specifically, when are ultraproducts saturated? We answered this question already to some extent in Construction 3.10, but we will generalize this result somewhat. That construction required an ultrafilter that contained S_k's, but not their countable intersection. We therefore make the following definition.

Definition 4.3. A ultrafilter U is *countably incomplete* if U has a countable subset V with the property that $\bigcap V = \emptyset$.

Proposition 4.4. *Every countably incomplete ultraproduct over a countable language $\mathscr{L}$ is $\aleph_1$-saturated.*

Proof. We need to show that given a language $|\mathscr{L}| \leq \aleph_0$, a cardinal $\kappa \geq \aleph_1$ so that there exists a non-principal ultrafilter U over κ, as well as an ultraproduct $\mathfrak{N} := \prod_U \mathfrak{M}_\mathrm{i}$ of a family of models $\mathfrak{M}_i$ then $\mathfrak{N}$ realizes all the complete types over any $A \subseteq N$ with $|A| < \aleph_0$.

Since A is countable, $S(A)$ is too, and we can take $p \in S(A)$ and enumerate

$$p(x) = \{\phi_i(x; a_i^1, a_i^2, ..., a_i^{m_i}) \mid i \in \mathbb{N}\}.$$

Taking them all together:

$$\psi_i(x) = \bigwedge_{j \leq i} \phi_i(x; a_i^1, a_i^2, ..., a_i^{m_i})$$

Then by Łoś's theorem, since $\exists x \psi_i(x; c_i^1, c_i^1, ..., c_i^{m_i})$ is true in $\mathfrak{N}$ there is a "large" set of index models on which this is true, i.e.

$$\{t < \omega \mid \mathfrak{M}_t \models \exists x \ \psi_i(x; c_i^1[t], ..., c_i^{m_i}[t])\} \in U$$

Now since U is countably incomplete there is a descending chain

$$I = I_0 \supset I_1 \supset \dots$$

with each $I_n \in U$ but $\bigcap_{t<\omega} I_t = 0$. So we can define $X_0 := I$ and for each $0 < n < \omega$:

$$X_n = I_n \cap \{t < \omega \mid \mathfrak{M}_t \models \exists x \ \psi_i(x; c_i^1[t], ..., c_i^{m_i}[t])\}$$

Then each $X_n \in U$, and they form a descending chain $X_n \supset X_{n+1}$ with $\bigcap_{t<\omega} X_n = 0$

So for each $i \in I$ there exist a greatest $n(i) < \omega$ with $i \in X_{n(i)}$. We now choose an element $l \in \prod_{i \in I} M_i$ so that if $n(i) = 0$, $l(i)$ is allowed to be some arbitrary element of M_i. But if $n(i) > 0$, we make $l(i) \in A_i$ so that $\mathfrak{M}_i \models \psi_{n(i)}[l(i)]$. This ensures that whenever $0 < n$ and $i \in X_n$, we have that $n \leq n(i)$, so $\mathfrak{M}_i \models \phi_n[l(i)]$. Thus by Łoś's Theorem, $\prod_U \mathfrak{M}_i \models \phi_n[l_U]$ for all $n > 0$, which means that l_U satisfies A in $\prod_U \mathfrak{M}_i$ which suffices to prove the theorem. □

The last construction relied on the fact that we could find a countably incomplete ultrafilter. This turns out to be very doable in most cases. It is not hard to see that every principal ultrafilter is not countably incomplete (i.e. *countably complete*). But the existence of cardinals that admit a non-principal countably complete ultrafilter (called measurable cardinals) is not provable in ZFC [2].

We state without proof the following

Theorem 4.5 (Uniqueness of Saturated Models)**.** *Given elementarily equivalent saturated models $\mathfrak{M}, \mathfrak{N}$ of the same cardinality, $\mathfrak{M} \cong \mathfrak{N}$.*

A proof can be found in, say, [1], but it is tedious and we will only use the result here. We give a quick but nice corollary to Proposition 4.4 that can be extended to higher cardinals, but we will need an important assumption, the generalized continuum hypothesis (or GCH) which states that for any α, $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$.

Corollary 4.6. *Assuming GCH, given models $\mathfrak{M}, \mathfrak{N}$ of a countable language $\mathscr{L}$ with $|M|, |N| \leq \aleph_1$ we have $\mathfrak{M} \equiv \mathfrak{N}$ if and only if there exist ultrafilters U, D so that $\prod_U \mathfrak{M} \cong \prod_D \mathfrak{N}$.*

Proof. For the forward direction, assume $\mathfrak{M} \equiv \mathfrak{N}$ and take U, D to be non-principal ultrafilters over $\aleph_0$. Then by Proposition 4.4, $\prod_U \mathfrak{M}$, $\prod_D \mathfrak{N}$ are $\aleph_1$ saturated. Now

$$|M|, |N| \leq \aleph_1 = 2^{\aleph_0}$$

by GCH, so the ultrapowers $\prod_U \mathfrak{M}, \prod_D \mathfrak{N}$ have cardinality at most $\aleph_1$. By Łoś's Theorem they are elementarily equivalent, and so by the Uniqueness Theorem for Saturated Models, they are isomorphic.

The backwards direction follows immediately from Łoś's Theorem. □

Both Proposition 4.4 and Corollary 4.6 can be generalized to all cardinalities, but not in a straightforward way [1]. Though the full treatment of that is beyond the scope of this paper, we do give the following important theorem that in its full form proves that any theory with an infinite model has a model of each infinite cardinality. The half of this theorem which can be proved using ultraproducts follows.

Theorem 4.7 (Upward Löwenheim-Skolem Theorem)**.** *Given a theory T of a language $\mathscr{L}$, if $\exists\ \mathfrak{M} \models T$ and $|M| \geq \aleph_0$, then there exist models of arbitrarily large cardinality that are elementarily equivalent to $\mathfrak{M}$.*

We will prove this by considering the ultraproduct $\mathfrak{N} := \mathfrak{M}^\lambda \big/ U$ and we will see that we can make $\mathfrak{N}$ as large as we want. This will prove the above theorem since by Łoś's Theorem $\mathfrak{M} \equiv \mathfrak{M}^\lambda \big/ U$. The difficult part of the proof lies in determining the cardinality of the ultraproduct. In general, this question is open, but certain restraints on the ultrafilter can enable us to bound the cardinality of the ensuing ultrapower. In particular, we will need the notion of regular ultrafilters.

Definition 4.8. Given a cardinal λ, an ultrafilter over I is λ-*regular* if and only if $\exists\ V \in U$ with $|V| = \lambda$ so that each $i \in I$ is a member of only finitely many elements of V.

These regular ultrafilters will enable us to prove the theorem, but we first need to prove that they exist.

Lemma 4.9. *For any set I with $|I| = \lambda \geq \aleph_0$, there exists a λ-regular ultrafilter U over I.*

Proof. We need to find a J of cardinality λ that has a λ-regular ultrafilter over it. To do this, we take $J = S_\omega(\lambda)$, the set of finite subsets of λ. Then for each $l \in \lambda$, $V_j := \{j \in J \mid l \in j\}$ and we take $V := \{V_j \mid l \in \lambda\}$. It is immediate that $|V| = \lambda$ and that each $j \in J$ is in only finitely many $V_j \in V$ since $j \in V_j$ means $l \in j$ and j is finite. Moreover, V satisfies the finite intersection property since given $V_{j_1}, \dots, V_{j_n} \in V$ we can see that

$$l_1, \dots, l_n \in V_{j_1} \cap \dots \cap V_{j_n}.$$

Therefore by Theorem 3.5 there is an ultrafilter U over λ that contains V. And we saw that V contains all of the properties needed to ensure that U be regular. $\square$

Now the theorem follows directly from the next lemma.

Lemma 4.10. *For a λ-regular ultrafilter U over I, if A is infinite, then $|\prod_U A| = |A^\lambda|$.*

Proof. Of course $|\prod_U A| \leq |A^\lambda|$ follows immediately from the construction of the ultraproduct. The tricky part will be to show that

$$|\prod_U A| \geq |A^\lambda|.$$

Let J be the set of finite sequences of elements of A. Then since $|A| \geq \aleph_0$, we have $|A| = |J|$. Now, since U is regular it must have a subset V with $|V| = |U|$ and each

$i \in I$ belongs to only finitely many elements of V. It therefore suffices for us to exhibit an injective map

$$g : V^A \to \prod_u J.$$

where V^A denotes the set of functions from V to A.

We define an order $\leq$ on V and define a function $h : I \to B$. We define $h(i)$ so that given $(v_1, \ldots, v_n)$ the finite sequence of all $v_j \in V$ so that $i \in v_j$ and we arrange them in their order according to $\leq$. Then taking an $h' : V \to A$ will enable us to write

$$h(i) = (h'(v_1), \ldots, h'(v_n)).$$

Taking the corresponding element in the ultraproduct we define $g(h') = h_U$. We need to show that g is injective. Take $s', t' \in V^A$ with $s' \neq t'$. This means that there is some $v \in V$ where $s'(v) \neq t'(v)$. Taking any $i \in v$, we know that v appears in the sequence of v_j's that contain i, say at the k^{th} spot, so $v = v_k$. Then by construction

$$s(i) = (\ldots, s'(e_k), \ldots) \neq (\ldots, h'(e_k), \ldots) = t'(i).$$

Therefore, $s'(i) \neq t'(i)$ for all $i \in v$ and we know that $v \in U$, so they are not equal on a "large" set of I, hence

$$g(s') = s_U \neq t_U = g(t').$$

Thus g is the injective function we are looking for. $\square$

This theorem gets its full power when paired which its counterpart, the Downward Löwenheim-Skolem Theorem which states that given a theory T of a language $\mathscr{L}$, if $\exists\, \mathfrak{M} \models T$ with $|M| = \kappa \geq \aleph_0$, then for any $\aleph_0 \leq \lambda \leq \kappa$, there exists an $\mathfrak{N} \equiv \mathfrak{M}$ with $|N| = \lambda$. We will not prove this result, but confine ourselves to pointing out that this result combined with Theorem 4.7 implies the following powerful result.

Theorem 4.11 (Löwenheim-Skolem)**.** *Given a theory T of a language $\mathscr{L}$ and a model $\mathfrak{M} \models T$, if $|M| = \kappa \geq \aleph_0$ then for all $\lambda \geq \aleph_0$ there exists a model $\mathfrak{N} \equiv \mathfrak{M}$ with $|N| = \lambda$.*

5. Elementary Classes

Definition 5.1. A class of models K for $\mathscr{L}$ is called an *elementary class* if and only if there exists a theory T such that K is exactly the class of all models of T.

Definition 5.2. A class of models K is *closed under elementary equivalence* if and only if given a model $\mathfrak{M} \in K$, if a model $\mathfrak{N} \equiv \mathfrak{M}$ then $\mathfrak{N} \in K$.

Definition 5.3. A class of models K is *closed under ultraproducts* if and only if every ultraproduct $\prod_U \mathfrak{V}_i \in K$ provided that each $\mathfrak{V}_i \in K$.

Theorem 5.4. *Take K to be a class of models. K is an elementary class if and only if K is closed under ultraproducts and elementary equivalence.*

Proof. For the forward direction, given a class K of models of a theory T, if a model $\mathfrak{M} \in K$ and a model $\mathfrak{N} \equiv \mathfrak{M}$ then it is clear that $\mathfrak{N}$ is also a model of T, and so is an element of K. Thus, K is trivially closed under elementary equivalence. Given an ultraproduct $\prod_U \mathfrak{M}_i$ where $\mathfrak{M}_i$ is a family of models in K, for all sentences

ϕ, if $\mathfrak{M}_i \models \phi$, then by Łoś's theorem $\prod_U \mathfrak{M}_i \models \phi$. Therefore K is closed under ultraproducts.

For the backwards direction, let K be a class of models that is closed under elementary equivalence and under ultraproducts. We wish to consider T the set of all sentences in $\mathscr{L}$ that are true in every $\mathfrak{V} \in K$. Hence T is a theory in $\mathscr{L}$ and every $\mathfrak{V} \in K$ is a model of T. Given $\mathfrak{U}$ a model of T, take Σ to be the set of all sentences of $\mathscr{L}$ that are true in $\mathfrak{U}$. We call I the set of finite subsets of Σ. Given $i = \{\sigma_1, \sigma_2, ..., \sigma_n\} \in I$, there exists a model $\mathfrak{V}_i \in K$ that is a model of i since otherwise the sentence $\neg(\sigma_1 \wedge \ldots \wedge \sigma_n)$ would be in T even though it is false in $\mathfrak{U}$, a model of T. We can therefore pick a model $\mathfrak{V}_i$ for each $i \in I$.

By the Compactness Theorem, there exists an ultraproduct $\prod_U \mathfrak{V}_i$ which is a model of Σ. Then $\prod_U \mathfrak{V}_i \in K$ since K is closed under ultraproducts. And since $\prod_U \mathfrak{V}_i$ is a model of Σ, which we defined as the set of sentences true in $\mathfrak{U}$, it follows that every model of Σ is elementarily equivalent to $\mathfrak{U}$. Therefore $\prod_U \mathfrak{V}_i \equiv \mathfrak{U}$, so $\mathfrak{U} \in K$. Therefore K is the class of all models of T, so K is an elementary class. $\square$

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