The Heat Equation and the Li-Yau Harnack Inequality

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Abstract

In this paper, we develop the necessary mathematics for understanding the Li-Yau Harnack inequality. We begin with a derivation of the heat equation in one dimension. A basic overview of the geometry of Riemannian manifolds is presented, followed by the generalization of the heat equation to Riemannian manifolds. A brief discussion of the strong maximum principle and the second fundamental form is presented, followed by a proof of the Li-Yau Harnack inequalities.

1 Introduction

The flow of heat is a process whose study has lead to a wide variety of developments in mathematics. Indeed, Fourier's analysis of an equation describing the flow of heat in a conducting rod has lead to the development Fourier series. More recently, the study of the heat equation has lead to the development of the Ricci Flow, a concept which was used by to prove the Poincaré conjecture. One of the more useful concepts that has developed along this line of research has been the Harnack inequalities, of which the Li-Yau Harnack inequality is of particular interest.

The Li-Yau Harnack inequality (LYHI) is an inequality which applies to any solution of the heat equation on a Riemannian manifold with nonnegative Ricci curvature. It provides a lower bound on the maximum possible deviation of the logarithm of a solution to the heat equation from being a supersolution to Laplace's equation; namely, if u is a solution to the heat equation, then

$$\Delta \ln u \geq -\frac{n}{2t},$$

where n is the dimension of the manifold. This inequality holds regardless of the curvature of the manifold and the initial conditions of the heat equation. From this inequality it is straightforward to derive several other important inequalities, including the differential Harnack inequality. In this paper, I attempt to give a derivation of the Li-Yau Harnack Inequality. The reader is presumed to have some familiarity with the concept of tensors, the covariant derivate, the Riemann tensor, and the Ricci curvature. The LYHI on a manifold with boundary requires a basic understanding of the second fundamental form, of which a brief introduction is given. The proof of the LYHI for closed manifolds is completed before it is extended to manifolds with boundary, so if the reader is only interested in closed manifolds, the introduction to the second fundamental form may be skipped.

2 Derivation of the Heat Equation in n Dimensions

We begin with a simple derivation of the heat equation, giving us a very basic intuition for the heat equation. Consider a n dimensional cube C with side lengths of h centered at $x = (x_1, ..., x_n)$, and assume that the cube has a uniform temperature u(x,t) at time t. Now, consider the 2n adjacent cubes $C_i^{\pm}$ centered at $x = (x_1, ..., x_i \pm h, ..., x_n)$, and assume the cube $C_i^{\pm}$ has a uniform temperature of $u(x_1, ..., x_i \pm h, ..., x_n, t)$ at time t. The rate of heat flow from cube $C_i^{\pm}$ to cube C is proportional to the temperature gradient between the two cubes and the area of contact between the two cubes:

$$\frac{dQ_{C_i^{\pm} \to C}}{dt} = A \frac{u(x_1, ..., x_i \pm h, ..., x_n, t) - u(x_1, ..., x_n, t)}{h} \cdot h^{n-1},$$

where $Q_{C_i^{\pm} \to C}$ is the amount of heat that flows from $C_i^{\pm}$ into C, the first term approximates the temperature gradient, and h^{n-1} is the area of the (n-1) dimensional area of contact between any two adjacent cubes. The rate at which the heat flows into C from every cube is proportional to the rate at which the temperature changes times the volume of the cube:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{h^n} \sum_{i=1}^{n} \left[\frac{dQ_{C_i^{+} \to C}}{dt} + \frac{dQ_{C_i^{-} \to C}}{dt} \right] \\ &\propto \sum_{i=1}^{n} \left[\frac{u(x_1, ..., x_i + h, ..., x_n) - 2u(x_1, ..., x_n) + u(x_1, ..., x_i - h, ..., x_n)}{h^2} \right]. \end{aligned}$$

Letting $h \to 0$ and scaling the axes so that the constant of proportionality is 1, we find that:

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial {x_i}^2}. \tag{1}$$

This is the equation that represents heat flow in flat n dimensional flat euclidean space, and is the equation that we will generalize to arbitrary Riemannian manifolds.

3 Geometry of Riemannian Manifolds

Before we may generalize equation (1) to manifolds, we review the necessary concepts of Riemannian geometry. Henceforth, we shall make use of the Einstein summation convention wherein an index that appears twice in the multiplication of matrices means that the index is summed over. Explicitly, we have:

$$S^{i_1 \ldots i \ldots i_q}_{j_1 \ldots j_r} T^{k_1 \ldots k_s}_{l_1 \ldots i \ldots l_t} = \sum_{i=1}^{n} S^{i_1 \ldots i \ldots i_q}_{j_1 \ldots j_r} T^{k_1 \ldots k_s}_{l_1 \ldots i \ldots l_t}.$$

Let M be an n dimensional Riemannian manifold. At each point of the manifold, there is a euclidean tangent space, allowing so to form at each point a set of basis vectors and basis co-vectors. We denote these vector fields as $\partial_i = \frac{\partial}{\partial x^i}$ and dx^i, respectively. We write v^i for the components of vectors and v_i for the components of co-vectors, where this is shorthand for:

$$v^i \to v^i \partial_i = v^i \frac{\partial}{\partial x^i},$$

$$v_i \to v_i dx^i,$$

respectively.

We define the **rank** of a tensor to be (N, M), where N and M are the number of raised and lowered indexes, respectively.

We define the **metric** on M to be a positive-definite tensor g_{ij} which allows us to define a notion of distance on the manifold. The metric carries with it all of the information about the manifold that we are interested in. We define the inverse metric g^{ij} by the equation:

$$\delta^i_j = g^{ik} g_{kj},$$

where δ^i_j is the **Kronecker delta**, defined by:

$$\delta^i_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

The metric gives a notion of distance as follows: if $\xi^i(t)$, $t \in [0, T]$ is a parametrized, continuously differentiable path in M, then we define the **length** of ξ by:

$$L(\xi) = \int_0^T \sqrt{g_{ij} \frac{d\xi^i}{dt} \frac{d\xi^j}{dt}} dt.$$

The metric allows us to define a notion of an inner product between two tensors. For two vectors $\xi = \xi^i, \eta = \eta^i$, we set:

$$\langle \xi, \eta \rangle = g_{ij} \xi^i \eta^j.$$

This notion easily generalizes to arbitrary tensors with the same number of vector/co-vector indexes. If $\xi = \xi^{i_1 \ldots i_r}_{j_1 \ldots j_s}, \eta = \eta^{i_1 \ldots i_r}_{j_1 \ldots j_s}$, then:

$$\langle \xi, \eta \rangle = g_{i_1 k_1} \cdots g_{i_r k_r} g^{j_1 l_1} \cdots g^{j_s l_s} \xi^{i_1 \ldots i_r}_{j_1 \ldots j_s} \eta^{k_1 \ldots k_r}_{l_1 \ldots l_s}.$$

We may use the metric and its inverse to raise and lower the indexes of arbitrary tensors. Given an arbitrary Vector $T = T^i \partial_i$ or co-vector $S = S_i dx^i$, we have

$$T_i = g_{ij} T^j,$$

$$S^i = g^{ij} S_j,$$

respectively. We define **Levi-Civitá connection** on M to be a matrix Γ^k_{ij} at each point of M by:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij} \right).$$

We henceforth refer to this as the **connection** on M. Given an arbitrary vector field $v = v^k \partial_k$, the connection allows us to define the **covariant derivative** of v:

$$\nabla_i v^k \equiv \partial_i v^k + \Gamma^k_{ij} v^j.$$

For a scalar u, on the other hand, the covariant derivative is simply the partial derivative:

$$\nabla_i u = \partial_i u.$$

Finally, the covariant derivative obeys the product rule: if X and Y are arbitrary tensor fields, then:

$$\nabla XY = (\nabla X)Y + X(\nabla Y).$$

These rules allow for the straightforward generalization of the covariant derivative to arbitrary tensors. Indeed, the covariant derivative of a co-vector follows from the product rule applied to the contraction of a vector with a co-vector, forming a scalar, and higher rank tensors may be contracted with vectors and co-vectors to form scalars, at which point the product rule may again be applied. With this generalization, we find that the covariant derivative is a map with (N, M) tensors to $(N, M+1)$ tensors.

We may define covariant derivatives in arbitrary directions as follows: if $X = X^i \partial_i$, we define the covariant derivative in the X direction to be:

$$\nabla_X = X^i \nabla_i.$$

While it is possible to define other connections on M, the **Levi-Cività connection** is uniquely defined by two important properties:

- Γ^k_{ij} is **torsion free**: $\Gamma^k_{ij} = \Gamma^k_{ji}$.
- Γ^k_{ij} is **compatible** with the metric: $\nabla_k g_{ij} = 0$.

Now, if $X = X^i \partial_i$, $Y = Y^i \partial_i$, and $Z = Z^i \partial_i$ are vector fields on M, then we may define the **Riemann curvature tensor** by:

$$Rm(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where $[X, Y]$ is the Lie bracket of X and Y, and will often be zero. It is a straightforward, albeit tedious, matter to show that, under the Levi-Civit connection, the Riemann curvature tensor has the form:

$$R^k_{lij} = \partial_i \Gamma^k_{lj} - \partial_l \Gamma^k_{ij} + \Gamma^k_{mi} \Gamma^m_{lj} - \Gamma^k_{mj} \Gamma^m_{li}.$$

By lowering the upper index of the Riemann tensor, we obtain the commonly used form of the Riemann tensor:

$$R_{klij} = g_{km} R^m_{lij}.$$

By contracting the first and third indexes of the Riemann tensor, we obtain the **Ricci tensor**, also known as the **Ricci curvature**:

$$R_{ij} = R^k_{ikj}.$$

By contracting the Ricci tensor with the inverse metric on both indexes, we obtain the Ricci scalar, also known as the Ricci curvature:

$$R = g^{ij} R_{ij}.$$

The Second Fundamental Form

The proof of the LYHI on manifolds with boundary requires a generalization of the concept of a convex surface. In order to state this definition, we must first define the second fundamental form.

Suppose we have an n dimensional manifold M, and the submanifold ∂M with co-dimension 1 (i.e. ∂M is (n-1) dimensional). We define a unit normal vector field ν as follows: $|\nu(p)| = 1$ and $\nu(p) \in T_pN^{\perp} \equiv \{\mu \in T_pM | \langle \mu, v \rangle = 0 \text{ for all } v \in T_pN\}$, where T_pR is the tangent space of the manifold R at the point p.

Now, ν can be either an outward pointing on inward pointing vector field; we choose to define it as a the outward pointing vector field. With an explicit ν in hand, we are now able to define the second fundamental form:

$$II_p(v, w) \equiv -\langle \nabla_v \nu, w \rangle. \tag{2}$$

The second fundamental form gives us the following definition of convexity:

Definition 1. *A manifold M is* **convex** *iff, for all $p \in \partial M$ and $v \in T_pM$, $II_p(v, v) \geq 0$.*

4 Generalization of the Heat Equation to Riemannian Manifolds

Before we may generalize the heat equation to arbitrary Riemannian manifolds, we need to first write the heat equation as a tensor equation. This may be easily accomplished by noting that, in euclidean space with the standard structure:

$$g_{ij} = g^{ij} = \delta^i_j,$$

$$\Gamma^k_{ij} \equiv 0,$$

and

$$\nabla_i \equiv \partial_i.$$

Now, we rewrite the derivatives in the heat equation as covariant derivatives:

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial t^2} = \sum_{i=1}^{n} \nabla_i \nabla_i u = g^{ij} \nabla_i \nabla_j u \equiv \Delta u,$$

$$\partial_t u = \Delta u, \tag{3}$$

where Δ is the generalization of the Laplacian operator to curved manifolds. Both sides of this equation are scalars, so that we have generalized the heat equation to a tensor equation.

This definition of the Laplacian requires a minor clarification. When we take the covariant derivative of a scalar, a (0,0) tensor, we obtain a co-vector, a (0,1) tensor. Applying the covariant derivative again, we obtain a (0,2) tensor. In order to obtain the Laplacian, a scalar, we contract this tensor with both indexes of the inverse metric, giving us a (0,0) tensor (a scalar):

$$\Delta u = g^{ij}(\nabla_i(\nabla_j u)). \tag{4}$$

With the fully general equation in our hands, it now seems natural for us to consider questions of the existence and uniqueness of solutions to this equation. While it is true that the existence of solutions to this equation may be proven under appropriate assumptions of the initial conditions and boundary conditions (i.g. Neumann boundary conditions and smooth initial conditions), the proof of this would take us astray from our goal of proving the LYHI. The proof of uniqueness also requires appropriate initial conditions and boundary conditions assumptions, but follows as a natural consequence of the maximum principle, one of the necessary tools in our proof of the LYHI. We thus opt to prove uniqueness as an application of the maximum principle before we move on to the proof of the LYHI.

5 The Strong Maximum Principle

Let M be as above. Let $U = \overline{M} \backslash \partial M$, and let $U_T = U \times (0, T]$. We define the heat operator as: $(\partial_t - \Delta) \equiv \Box$. The heat equation may then be stated as:

$$\Box u = (\partial_t - \Delta)u = 0 \tag{5}$$

Strong Maximum Principle. *Assume $u \in C^\infty(U_T)$ and $q > 0$ in U_T, U connected. Then:*

i) If u is a subsolution of (4), i.e. $\Box u \leq 0$ in U_T, and if u attains a non-negative absolute maximum over U_T^- at an interior point $(x_0, t_0) \in U_T$, then u is constant on U_{t_0}.

ii) If u is a supersolution of (4), i.e. $\Box u \geq 0$ in U_T, and if u attains a non-positive absolute minimum over U_T^- at an interior point $(x_0, t_0) \in U_T$, then u is constant on U_{t_0}.

Proof. See [2] $\Box$

We also amend an important lemma, known as the Hopf boundary point lemma:

Hopf Boundary Point Lemma. *Suppose $\Box u \leq 0$ in U_T, and that u is not constant. By the strong maximum principle, u attains a maximum at a point p on the boundary. Then:*

$$\frac{\partial u}{\partial n}(p) > 0,$$

Where n is an outwardly oriented normal coordinate.

Proof. See [3]. $\Box$

Before moving on to the proof of the uniqueness of solutions to the heat equation, we first prove a useful corollary:

Corollary 1. *Suppose u is such that $\Box u = 0$ on U_T, and if attains an absolute extremum over $\overline{U}_T$ at an interior point (x_0, t_0). Then u is constant on U_{t_0}.*

Proof. We begin by noting that u is both a supersolution and a subsolution of (4). Suppose u attains an absolute maximum over $\overline{U}_T$ at an interior point (x_0, t_0). We may assume that the maximum is non-negative, since $\Box(u + C) = 0$ for C constant. Thus, we apply (i) from the strong maximum principle, and find that u is constant. Similarly, suppose u attains an absolute minimum over $\overline{U}_T$ at an interior point (x_0, t_0). We may assume that the minimum is non-positive, since $\Box(u + C) = 0$ for C constant. Thus, applying (ii), u is constant on U_{t_0}. □

5.1 Uniqueness of Solutions to the Heat Equation

Theorem 1. *Suppose u solves the heat equation ($\Box u = 0$) with the following boundary conditions:*

1. *Neumann boundary conditions on $\partial M \times [0, T]$; $\frac{\partial u}{\partial v} = 0$; physically, this states that the manifold is thermally insulated from the outside world*
2. *Initial conditions on $M \times [0]$: $u(x, 0) \equiv u_0(x)$, with $u_0(x)$ is smooth.*

u is unique.

Proof. Suppose there exist two solutions u, u' satisfying boundary conditions 1 and 2. Then $w = u - u'$ satisfies the boundary condition $w(x, 0) = 0$ as well as the Neumann boundary condition. Now, suppose w attains an absolute extremum at an interior point $(x_0, t_0) \in U_T$. Then w is constant by corollary 1. However, $w(x, 0) = 0 \Rightarrow w \equiv 0$. Thus, w attains its maximum and minimum on the boundary. If this isn't the case, then w attains its maximum minimum on the boundary. By the Hopf boundary point lemma,

$$\frac{\partial u}{\partial n}(p) > 0,$$

which contradicts the Neumann boundary condition. Thus, $u = u'$, proving the uniqueness. □

6 The Li-Yau Harnack Inequality

We begin by stating the LYHI:

Li-Yau Harnack Inequality. *Let M be an n dimensional, compact, Riemannian manifold with non-negative Ricci curvature and a convex boundary ∂M if $\partial M \neq \emptyset$. Let $u(x, t)$ satisfy the heat equation $\Box u = 0$ with Neumann boundary conditions ($\frac{\partial u}{\partial n} = 0$) on $\partial M \times (0, T]$, and assume that $u(x, t) > 0$ $\forall (x, t) \in M \times [0, T]$. Then:*

$$H = \frac{\partial_t u}{u} - \frac{|\nabla u|^2}{u^2} + \frac{n}{2t} \geq 0 \; \forall (x, t) \in M \times (0, T]. \tag{6}$$

Before beginning the proof, let us see how equation (6) is equivalent to the inequality presented in the introduction:

$$\Delta \ln u = g^{ij}\nabla_i(\nabla_j \ln u) = g^{ij}\nabla_i\left(\frac{\nabla_j u}{u}\right) = g^{ij}\left(\frac{\nabla_i\nabla_j u}{u} + (\nabla_j u)\nabla_i\left(\frac{1}{u}\right)\right)$$
$$= \frac{\Delta u}{u} - g^{ij}\frac{\nabla_i u\nabla_j u}{u^2} = \frac{\partial_t u}{u} - \frac{|\nabla u|^2}{u^2}.$$

Substituting this into equation (6), we find that:

$$H = \Delta \ln u + \frac{n}{2t} \geq 0,$$

$$\Delta \ln u \geq -\frac{n}{2t}.$$

We may now begin the proof of the LYHI. We follow the proof given in the original paper by Li and Yau, [1]. We first prove a technical result which may seem arbitrary at this point, but is pivotal to the proof of the LYHI.

Lemma 1. *(Li-Yau, Lemma 1.1 of [1]) Let $f(x,t)$ be a smooth function on $M \times [0,T]$ such that $\partial_t f = \Delta f + |\nabla f|^2$, and set $F := t(|\nabla f|^2 - \partial_t f) = -t\Delta f$. If F has nonnegative Ricci curvature, then F satisfies the inequality*

$$(\Delta - \partial_t)F \geq -2\langle \nabla F, \nabla f\rangle - \frac{1}{t}F + \frac{2}{nt}F^2. \tag{7}$$

Proof. We begin with three straightforward calculations:

$$\begin{aligned}
\Delta|\nabla f|^2 &= g^{kl}\nabla_k\nabla_l(g^{ij}\nabla_i f\nabla_j f) = g^{ij}g^{kl}\nabla_k(\nabla_l\nabla_i f\nabla_j f + \nabla_i f\nabla_l\nabla_j f)\\
&= g^{ij}g^{kl}(\nabla_k\nabla_l\nabla_i f\nabla_j f + \nabla_l\nabla_i f\nabla_k\nabla_j f + \nabla_k\nabla_i f\nabla_l\nabla_j f + \nabla_i f\nabla_k\nabla_l\nabla_j f)\\
&= 2\langle \Delta\nabla f, \nabla f\rangle + 2\langle \nabla^2 f, \nabla^2 f\rangle = 2\langle \Delta\nabla f, \nabla f\rangle + 2|\mathrm{Hess}(f)|^2,
\end{aligned} \tag{8}$$

$$\begin{aligned}
\langle \Delta\nabla f, \nabla f\rangle &= g^{ij}g^{kl}\nabla_k\nabla_l\nabla_i f\nabla_j f\\
&= g^{ij}g^{kl}\nabla_k\nabla_i\nabla_l f\nabla_j f\\
&= g^{ij}g^{kl}(\nabla_i\nabla_k\nabla_l f\nabla_j f + R^p_{lki}\nabla_p f\nabla_j f)\\
&= \langle \Delta\nabla f, \nabla f\rangle + g^{ij}g^{kl}g^{pq}R_{qlki}\nabla_p f\nabla_j f\\
&= \langle \Delta\nabla f, \nabla f\rangle + g^{ij}g^{kl}g^{pq}R_{kilq}\nabla_p f\nabla_j f\\
&= \langle \Delta\nabla f, \nabla f\rangle + g^{ij}g^{pq}R_{iq}\nabla_p f\nabla_j f\\
&= \langle \Delta\nabla f, \nabla f\rangle + \mathrm{Ric}(\nabla f, \nabla f),
\end{aligned} \tag{9}$$

and

$$(\Delta f)^2 = |g^{ij}\nabla_i\nabla_j f|^2 \leq |g^{ij}|^2|\nabla_i\nabla_j f|^2 = n|\mathrm{Hess}(f)|^2. \tag{10}$$

Where the inequality is simply the Cauchy-Schwartz inequality, and $|g^{ij}|^2 = n$ is a property of the metric. We may now put (8), (9), and (10) together to find that:

$$\begin{aligned}\Delta|\nabla f|^2 &= 2\langle \Delta\nabla f, \nabla f\rangle + 2|\mathrm{Hess}(f)|^2 \\ &= 2\langle \nabla\Delta f, \nabla f\rangle + 2\mathrm{Ric}(\nabla f, \nabla f) + 2|\mathrm{Hess}(f)|^2 \\ &\geq 2\langle \nabla\Delta f, \nabla f\rangle + \frac{2}{n}(\Delta f)^2. \end{aligned} \tag{11}$$

Finally, we may find our result by noting that:

$$(\Delta - \partial_t)F = t(\Delta|\nabla f|^2 - \Delta(\partial_t f)) - \partial_t F \tag{12}$$

$$\begin{aligned} &\geq t(2\langle \nabla\Delta f, \nabla f\rangle + \frac{2}{n}(\Delta f)^2 + \partial_t \Delta f) - \partial_t F \\ &= 2t\left\langle \nabla\left(\frac{-F}{t}\right), \nabla f\right\rangle + \frac{2t}{n}\left(\frac{-F}{t}\right)^2 - t\partial_t\left(\frac{-F}{t}\right) - \partial_t F \\ &= -2\langle \nabla F, \nabla f\rangle - \frac{1}{t}F + \frac{2}{nt}F^2, \end{aligned} \tag{13}$$

as was to be shown. □

We are now ready to proceed with the proof of the LYHI.

Proof. Set $f = \ln(u)$. We begin by noting that, from above:

$$\Delta \ln u = \frac{\partial_t u}{u} - \left|\frac{\nabla u}{u}\right|^2 = \partial_t \ln(u) - |\nabla \ln(u)|^2 = \Delta f, \tag{14}$$

so that f satisfies the assumption of Lemma 1. We thus define F as in Lemma 1. Comparing inequality (6) to equation (14), we find that the LYHI is equivalent to:

$$\Delta f \geq -\frac{n}{2t},$$

or

$$F \leq \frac{n}{2}. \tag{15}$$

We may prove this by means of the result of Lemma 1, namely:

$$(\Delta - \partial_t)F \geq -2\langle \nabla F, \nabla f\rangle + \frac{2}{nt}F\left(F - \frac{n}{2}\right). \tag{16}$$

If (15) were not true, then there would be a maximum point of F on $M \times [0, T]$ such that $F(x_0, t_0) > n/2$. Since $F(x, 0) = 0$ by definition, $t_0 > 0$. We first consider the case in which M is a closed manifold. Since M lacks a boundary, x_0 is necessarily an interior point. Since (x_0, t_0) is a maximum for F, it is clear that:

$$\nabla F(x_0, t_0) = 0, \quad \Delta F(x_0, t_0) \leq 0, \quad \partial_t F(x_0, t_0) \geq 0.$$

Substituting this into (16), it we find that:

$$0 \geq (\Delta - \partial_t)F \geq 0 + \frac{2}{nt_0}F(x_0,t_0)\left(F(x_0,t_0) - \frac{n}{2}\right) > 0,$$

Which is clearly a contradiction. Thus, the theorem is has been demonstrated for manifolds without boundary.

Now, suppose $\partial M \neq \emptyset$.The above argument shows that $(x_0, t_0) \in \partial M \times (0, T]$. Now, by applying the Hopf boundary point lemma, we see that $\frac{\partial F}{\partial n} > 0$, where n is the normally oriented, outward pointing coordinate. Now, let us set up a basis $\{x_i\}$for the tangent space of M at (x_0, t_0) such that the coordinate x_n is normally oriented and outward pointing. Thus, in this coordinate system, we may express the Neumann boundary condition as $\nabla_n f = 0$. Now, we have:

$$\begin{aligned}\frac{\partial F}{\partial n}(x_0,t_0) &= t\nabla_n(|\nabla f|^2 - \partial_t f) = 2t\sum_{i=1}^{n}\nabla_n\nabla_i f\nabla_i f - \partial_t(\nabla f)\\ &= 2t\sum_{i=1}^{n}\nabla_n\nabla_i f\nabla_i f = \langle\nabla_n\nabla f, \nabla f\rangle = -2t_0 II_{x_0}(\nabla f, \nabla f) < 0,\end{aligned}$$

since the boundary is convex and $t_0 > 0$. This contradicts the Hopf boundary point lemma, and proves our theorem. □

Having proven the LYHI, we conclude the paper with a demonstration that the inequality is not strict; it may be saturated by some solutions to the heat equation. Consider the following solution to the heat equation in n dimensional Euclidean space with the usual structure:

$$u(x,t) = \frac{1}{\sqrt{t}}e^{-\frac{|\mathbf{x}|^2}{4t}},$$

$$\ln(u) = -\frac{1}{2}\ln(t) - \frac{|\mathbf{x}|^2}{4t}.$$

Substituting this into the LYHI, we find that:

$$\frac{\partial^2 \ln(u)}{\partial x_i^2} = -\frac{1}{2t},$$

so that

$$\Delta \ln(u) = \frac{\partial^2 \ln(u)}{\partial x_i^2}\sum_{i=1}^{n} = -\frac{n}{2t},$$

Which both satisfies and saturates the LYHI.

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