PERCOLATION AND THE EXISTENCE OF THE INFINITE OPEN CLUSTER

RILEY HECKEL

ABSTRACT. Imagine an infinite graph, where each edge is deleted with a predetermined probability, resulting in a subgraph with a structure unique to the chosen probability. For some probability there will be a subgraph with an infinitely large connected collection of vertices. This paper investigates the point at which such a collection of vertices appears on the 2-dimensional lattice. Additionally, properties regarding these subgraphs on more general graphs are stated. Specifically, the connection between the amenability of a graph and the uniqueness of the infinite connected collection of vertices will be defined.

1. INTRODUCTION

Suppose you are given an infinite graph, where each edge is deleted with a particular probability resulting in a subgraph containing 'clusters' of edges. A number of questions arise. What do we know about the geometry of the resulting subgraph? What changes are made if a different probability is chosen? What properties of the graph determine the resulting behavior? These are the primary questions comprising percolation theory, which has been a major topic of research for probabilists over the past fifty years. Because we are mainly concerned with infinite graphs, particular attention has been given to determining the probability at which an infinitely large connected cluster of edges will exist. It is clear that when edges are deleted with probability 0 the graph will be an infinitely large connected cluster. Additionally, if each edge is deleted with a large enough probability the resulting subgraph will have only finite-sized clusters. Thus, there must be some minimum probability at which there will almost surely be an infinitely large connected cluster of edges. This is referred to as the "critical probability," and is one of the main areas of focus in percolation theory.

Percolation theory has applications in various fields such as: the geometry of chemical bonding in solids, measuring resistance in electrical networks and even predicting how infectious diseases will spread in various parts of the world. Therefore, attention has been given to determine the properties of a number of common graphs under this systematic "edge-deletion". In particular, it took twenty years of research to determine that the critical probability of the 2-dimensional square lattice is $\frac{1}{2}$. We will investigate this result further in section 2. This will make use of a concept known as the "dual lattice," which has shown to be quite useful in proving results about the square lattice. Additionally, a great deal of information is known concerning percolation on more general graph structures. In particular, attention has recently been drawn to the Cayley graphs of groups and how the underlying

Date: AUGUST 14, 2008.

group structure affects the percolation on the Cayley graph. Specifically, in sections 3 and 4 we will look at how for some graphs there can may exist infinitely many infinite clusters while for other graphs there will always be a unique infinite cluster if we are above the critical probability. This will lead us to a conjecture regarding the uniqueness of the infinite cluster for a class of underlying groups.

2. Critical Probability On the Lattice $\mathbb{L}_2$

This section is concerned with explicitly determining the critical probability on the 2-dimensional square lattice $\mathbb{L}_2$. This lattice is defined by a vertex set V and edge set E where

$$V = \{(x, y) \in \mathbb{R}^2 | x, y \in \mathbb{Z}\}$$

and an edge $e \in E$ between any two vertices $u, v \in V$ exists if $|u - v| = 1$.

Definition 2.1. For each edge we assign a random variable, $X(e)$, uniformly distributed on the unit interval. For a given probability $p \in [0, 1]$ we say that e is *open* if $X(e) < p$ and e is *closed* if $X(e) \geq p$. Thus, each edge will be open with probability p and closed with probability $1 - p$.

The collection of open edges will yield a subgraph of $\mathbb{L}_2$ consisting of connected 'clusters' of open edges. For any $p \in [0, 1]$, the resulting subgraph of open edges is called a *percolation* on $\mathbb{L}_2$.

If for some $p \in [0, 1]$ the resulting subgraph contains a connected cluster of open edges of infinite size (i.e. infinitely many vertices), we call such an object an *infinite open cluster* or sometimes just *infinite cluster*.

The existence of an infinite open cluster does not depend on the state of any finite set of edges. Removing any finite set of edges from $\mathbb{L}_2$ can only remove finitely many edges from an infinite cluster if one exists. Hence, the event that there exists an infinite cluster is a tail event of the sequence of indicator variables (1 if an edge is open, 0 if closed) on the edges remaining after any finite set of edges is removed. Thus, by Kolgomorov's Zero-One Law, the event of existence of an infinite cluster must have probability 0 or 1.

Now, we let

$$\theta(p) = \mathbb{P}(\text{the origin is contained in an infinite cluster})$$

For small enough p, $\theta(p)$ will almost surely be 0, but for larger p, $\theta(p)$ may be strictly greater than 0. The event that the origin is part of an infinite cluster is contained in the event that there exists an infinite cluster. Thus, whenever $\theta(p) > 0$, there will almost surely exist an infinite cluster somewhere on the lattice by the argument above. Additionally, whenever $\theta(p) = 0$, there will almost surely not exist an infinite cluster anywhere.

We define the *critical probability* as:

$$p_c = \sup_{p \in [0,1]} (\theta(p) = 0)$$

The critical probability is therefore the value at which the probability of existence of an infinite cluster of edges changes from 0 to 1.

One of the greatest milestones in percolation history occurred in 1980 when Harry Kesten[2] determined the critical probability of $\mathbb{L}_2$ to be $\frac{1}{2}$. For higher dimensional lattices the precise critical probabilities are still unknown. Thus, as you will see, the calculation of p_c for the 2-dimensional lattice requires some unique properties of the lattice. The proof that $p_c = \frac{1}{2}$ uses the following definition.

Definition 2.2. The *dual lattice* of $\mathbb{L}_2$, denoted $\mathbb{L}_2^*$, is defined by the vertex set $\{(x + \frac{1}{2}, y + \frac{1}{2}) | (x, y) \in \mathbb{Z}^2\}$, so that every edge of $\mathbb{L}_2^*$ intersects a unique edge of $\mathbb{L}_2$. It is easy to see that $\mathbb{L}_2$ and $\mathbb{L}_2^*$ are isomorphic as graphs by translating $\mathbb{L}_2$ by the vector $\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$.

We can define percolation on $\mathbb{L}_2^*$ as follows. An edge in $\mathbb{L}_2^*$ is open if and only if the corresponding edge in $\mathbb{L}_2$ that it crosses is open. Thus, percolation on the dual lattice has the property that no closed edge in $\mathbb{L}_2^*$ can cross an open edge in $\mathbb{L}_2$. Therefore, all closed clusters in $\mathbb{L}_2^*$ must be non-intersecting with the open clusters in $\mathbb{L}_2$. An example of this can be seen in Figure 1 below, where the orange lines represent open edges in $\mathbb{L}_2$ and the blue lines represent closed edges in $\mathbb{L}_2^*$.

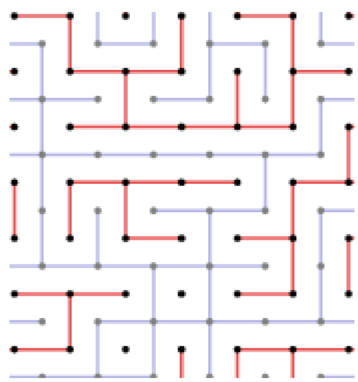

FIGURE 1. Percolation on $\mathbb{L}_2$ and $\mathbb{L}_2^*$

As a result, we will focus on the connected closed clusters in $\mathbb{L}_2^*$. Additionally, as we will see, the non-existence of an infinite open cluster in $\mathbb{L}_2$ implies the existence of an infinite closed cluster in $\mathbb{L}_2^*$ and vice versa. This will lead us to deduce that $p_c = \frac{1}{2}$.

Theorem 2.3. *On the 2-dimensional square lattice, $\mathbb{L}_2$, the critical probability $p_c \geq \frac{1}{2}$. In other words, $\theta(\frac{1}{2}) = 0$*

There are several methods used to prove that $p_c = \frac{1}{2}$. It was first proven by showing that $\theta(\frac{1}{2}) = 0$ so that $p_c \geq \frac{1}{2}$ and then by showing that $p_c \leq \frac{1}{2}$. The proof repeatedly uses the isomorphic nature of $\mathbb{L}_2$ and $\mathbb{L}_2^*$. An additional result concerning the lattice $\mathbb{L}_2$ is that any infinite cluster (if one exists) will almost surely be unique. This fact will be utilized in the following theorem, and will be justified in greater generality in the following section.

The proof of Theorem 2.2 will also require the following inequality, the proof of which can be found in Grimmett[1].

Lemma 2.4. *If A_p and B_p are increasing events in p on $\mathbb{L}_2$ such that $\mathbb{P}(A_{p_1}) \leq \mathbb{P}(A_{p_2})$ and $\mathbb{P}(B_{p_1}) \leq \mathbb{P}(B_{p_2})$ whenever $p_1 \leq p_2$, then*

$$\mathbb{P}(A_p \cap B_p) \geq \mathbb{P}(A_p) \Pr(B_p) \tag{2.5}$$

Examples of increasing events as mentioned in the Lemma include: the existence of an infinite cluster, the existence of an open path of edges connecting any two vertices, or the containment of a given vertex within an infinite cluster. We now have the tools we need in order to prove Theorem 2.2.

Proof. Assume for contradiction that $\theta(\frac{1}{2}) > 0$. This is equivalent to the event that there exists an infinite open cluster on the lattice when each edge is open with probability $\frac{1}{2}$. Let $B(n) = [0, n] \times [0, n]$ be the box in $\mathbb{L}_2$ of side length n with the bottom left corner at the origin. Define for each $n \in \mathbb{Z}^+$ the events $A_t(n), A_b(n), A_l(n), A_r(n)$ that some vertex on the top (respectively bottom, left right) of $B(n)$ is contained in an infinite open cluster on $\mathbb{L}_2 \setminus B(n)$. Now, since as n increases, the number of vertices on the top (and bottom, left, right) of $B(n)$ increases, it is clear that each of the $A_i(n)$ are increasing events in p and have the same probability (by symmetry) for $i = t, b, l$, or r.

Since we assumed that $\theta(\frac{1}{2}) > 0$, there exists an infinite cluster on the lattice almost surely

$$\mathbb{P}(A_t(n) \cup A_b(n) \cup A_l(n) \cup A_r(n)) \to 1 \quad \text{as} \quad n \to \infty$$

Now notice that by de Morgan's Law and (2.5)

$$\begin{aligned} 1 - \mathbb{P}(A_t(n) \cup A_b(n) \cup A_l(n) \cup A_r(n)) &= \Pr(A_t^c(n) \cap A_b^c(n) \cap A_l^c(n) \cap A_r^c(n)) \\ &\geq \mathbb{P}(A_t^c(n))\mathbb{P}(A_b^c(n))\mathbb{P}(A_l^c(n))\mathbb{P}(A_r^c(n)) \\ &= (1 - \mathbb{P}(A_t(n)))^4 \end{aligned}$$

since each of the $A_i(n)$ have the same probability of occurring. This tells us

$$\mathbb{P}(A_i(n)) \geq 1 - [1 - \mathbb{P}(A_t(n) \cup A_b(n) \cup A_l(n) \cup A_r(n))]^{\frac{1}{4}} \quad \text{for} \ \ i = t, b, l, \ \text{or} \ r \tag{2.6}$$

From this, we deduce that $\mathbb{P}(A_i(n)) \to 1$ as $n \to \infty$. Now, choose N large enough such that:

$$\mathbb{P}A_i(N)) > \frac{7}{8} \tag{2.7}$$

Define the dual box $B^*(n) \subset \mathbb{L}_2^*$ to be:

$$B^*(n) = \left\{ (x, y) + \left(\frac{1}{2}, \frac{1}{2} \right) \middle| 0 \leq x, y \leq n \right\}$$

In other words, $B^*(n)$ is the box in the dual lattice obtained by translating $B(n)$ by the vector $\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$. We define a percolation on $\mathbb{L}_2^*$ by setting each edge of $\mathbb{L}_2^*$ open if and only if the corresponding edge of $\mathbb{L}_2$ is open.

Now we define $A_t^*(n)$ (respectively $A_b^*(n), A_l^*(n), A_r^*(n)$ to be the event that some vertex on the top (resp. bottom, left, right) of $B^*(n)$ is contained in an infinite closed cluster of $\mathbb{L}_2^* \setminus T^*(n)$. Therefore, since each edge of $\mathbb{L}_2^*$ is closed with probability $\frac{1}{2}$, we have

$$\mathbb{P}(A_i^*(N)) = \mathbb{P}(A_i(N)) > \frac{7}{8} \quad \text{for } i = t, b, l, \ \text{or } r \tag{2.8}$$

Consider the event A that there exists infinite open clusters intersecting the left and right sides of $B(N)$ and infinite closed clusters intersecting the top and bottom of $B^*(N)$

$$A = A_l(N) \cap A_r(N) \cap A_t^*(N) \cap A_b^*(N)$$

From (2.7) and (2.8) we see

$$\begin{aligned}\mathbb{P}(A^c) &\leq \mathbb{P}(A_l^c(N)) + \mathbb{P}(A_r^c(N)) + \mathbb{P}(A_t^{*c}(N)) + \mathbb{P}(A_b^{*c}(N)) \\ &< \frac{1}{2}\end{aligned}$$

which tells us that $\mathbb{P}(A) \geq \frac{1}{2}$. If A occurs then we have an infinite open cluster on $\mathbb{L}_2 \setminus B(n)$ intersecting the left-hand side of $B(N)$ and an infinite open cluster on $\mathbb{L}_2 \setminus B(n)$ intersecting the right-hand side of $B(N)$. By the uniqueness of the infinite open cluster on $\mathbb{L}_2$ (to be proved in the next section), there must exist an open path either through $B(N)$ or an open path traveling around the top or bottom of $B(N)$. Using the construction of the dual lattice, no open edge in $\mathbb{L}_2$ can cross a closed edge in $\mathbb{L}_2^*$. Thus, the existence of infinite closed clusters on $\mathbb{L}_2^* \setminus B(n)$ intersecting the top and bottom of $B^*(N)$ prevents any open path from traveling around $B(N)$ to connect the left and right infinite open clusters on $\mathbb{L}_2$

By the uniqueness of the infinite open cluster, there must be a path traversing $B(N)$ connecting the infinite open clusters intersecting the left and right sides of $B(N)$. However, this creates disjoint infinite closed clusters intersecting the top and bottom of $B^*(N)$, which cannot occur (again by uniqueness). Therefore, A cannot occur. In other words,

$$\mathbb{P}(A) = 0$$

which contradicts our calculation that $\mathbb{P}(A) \geq \frac{1}{2}$. Thus, our initial assumption that $\theta(\frac{1}{2}) > 0$ must be false, indicating that $\theta(\frac{1}{2}) = 0$

□

Next we will prove that $p_c \leq \frac{1}{2}$, which, combined with $\theta(\frac{1}{2}) = 0$ implies that $p_c = \frac{1}{2}$ on $\mathbb{L}_2$. In order to do this, a few new definitions are required.

Notation 2.9. For any two vertices $(x_1, y_1), (x_2, y_2) \in \mathbb{L}_2$, we denote the event that there exists an open path from (x_1, y_1) to (x_2, y_2) by:

$$(x_1, y_1) \leftrightarrow (x_2, y_2)$$

Definition 2.10. The "mean open cluster size" is defined to be the expected value of the size of the open cluster containing the origin. If we denote $|C_o|$ as the number of vertices in the connected open cluster containing the origin, then the mean cluster size is:

$$\chi(p) = \sum_{n=1}^{\infty} n\mathbb{P}(|C_o| = n) = \sum_{n=1}^{\infty} \mathbb{P}(|C_o| \geq n)$$

Notice that we could have defined $|C_o|$ to be the size of the open cluster around any vertex in the lattice since the lattice is invariant under integral horizontal and vertical translations. We just choose the origin for convenience.

The following theorem requires a result concerning the mean cluster size whenever $p < p_c$.

Lemma 2.11. *For $p < p_c$, the mean size of the open cluster containing the origin is finite.*

I will not prove this here. The reader is referred to Menshikov[5], who was the first to prove this result in 1986. He showed that whenever $p < p_c$, the tail of the greatest distance (i.e. radius) between the origin and any vertex in the open cluster containing the origin decays exponentially, from which the proposition follows. In other words, if C_o denotes the open cluster containing the origin, we let A_n be the event that there exists a path of length n between the origin and some $(x, y) \in C_o$ but no path of length $n+1$ between the origin and any vertex in C_o, then:

$$\mathbb{P}(A_n) \approx \exp^{-\alpha(p)n} \text{ as } n \to \infty$$

for some positive function $\alpha(p)$. From this we can deduce that the expected cluster size $\chi(p)$ will be finite whenever $p < p_c$.

Definition 2.12. For any connected cluster C, we define the *boundary* of C to be:

$$\partial C = \{e \in E \,|\, \text{one endpoint of } e \text{ is in } C, \text{ while the other endpoint is not}\}$$

Theorem 2.13. *On the 2-dimensional square lattice $\mathbb{L}_2$, the critical probability $p_c \leq \frac{1}{2}$*

In order to prove this result, we will show that for $p < p_c$, the probability that the origin of the dual lattice is contained in an infinite closed cluster is strictly positive. In other words, for all $p < p_c$, $1 - p \geq p_c$. This shows us

$$1 - p_c \geq p_c$$

or

$$p_c \leq \frac{1}{2}$$

Proof. Since the mean cluster size is finite whenever $p < p_c$

$$\chi(p) = \sum_{n=1}^{\infty} \mathbb{P}(|C_o| \geq n) < \infty$$

Let $n > 0$. We let A_n be the event that a path π exists between vertices $(k, 0)$ and $(l, 0)$ for some $k < 0$ and some $l \geq n$ such that each vertex of π (other than $(k, 0)$ and $(l, 0)$) is strictly above the horizontal axis. If $(k, 0) \leftrightarrow (l, 0)$, then the vertex $(l, 0)$ is contained in an open cluster of size at least l. Therefore, conditioned on this path, the mean cluster size must be at least l. Using this observation, we notice that:

$$\begin{aligned} \mathbb{P}(A_n) &\leq \mathbb{P}\left(\bigcup_{l=n}^{\infty} (k,0) \leftrightarrow (l,0) \text{ for some } k < 0\right) \\ &\leq \sum_{l=n}^{\infty} \mathbb{P}(|C_o| \geq l) \end{aligned} \tag{2.14}$$

Since

$$\sum_{i=1}^{\infty} \mathbb{P}(|C_o| \geq i) < \infty$$

(2.14) tells us that we can choose N large enough such that $\mathbb{P}(A_N) \leq \frac{1}{2}$

Let D denote the set of vertices $(i + \frac{1}{2}, \frac{1}{2})$ for $0 \leq i \leq N$ in the dual lattice. Additionally, let $C(D)$ denote the set of vertices in the dual lattice that are connected to some vertex in D by a closed path. As before, we define an edge in $\mathbb{L}_2^*$ to be open if and only if the corresponding edge in $\mathbb{L}_2$ is open. Using this, the event $|C(D)| < \infty$ implies the event that $C(D)$ is surrounded above by an open path in the original lattice $\mathbb{L}_2$. This is because, for $|C(D)|$ finite, $\partial C(D)$ above the horizontal axis must be finite, and since any edge in $\partial C(D)$ is open, there exists an open path in $\mathbb{L}_2$ surrounding $C(D)$ above the horizontal. Thus, the occurrence of $|C(D)| < \infty$ implies that A_N occurs as well, yielding the containment $C(D) \subset A_N$. Using this, we see:

$$\mathbb{P}(|C(D)| < \infty) \leq \mathbb{P}(A_N) \leq \frac{1}{2}$$

which implies that

$$\mathbb{P}(|C(D)| = \infty) = 1 - \mathbb{P}(|C(D)| < \infty) > \frac{1}{2}$$

Thus, since there is a strictly positive probability that there exists an infinite closed cluster in the dual lattice, we have from earlier results that $\theta(1-p) > 0$. This implies that $1 - p \geq p_c$ whenever $p < p_c$. Thus, $p_c \leq \frac{1}{2}$, and we are done. $\square$

In fact, we can even determine a lower bound on $\theta(1-p)$. we see that if $|C(D)| = \infty$, then at least one of the N vertices in the dual must lie in an infinite closed cluster. Thus, the probability that this vertex is the origin in dual is

$$\begin{aligned} \mathbb{P}\left[\left(\frac{1}{2}, \frac{1}{2}\right) \text{ is contained in an infinite closed cluster}\right] &\geq \frac{1}{N}\mathbb{P}(|C(D)| = \infty) \\ &> \frac{1}{2N} \end{aligned}$$

Combining the two results that $p_c \geq \frac{1}{2}$ and $p_c \leq \frac{1}{2}$ implies immediately that $p_c = \frac{1}{2}$ on the 2-dimensional square lattice $\mathbb{L}_2$. This is one of the most well-known results in percolation theory as it took more than twenty years of research to prove. It is an open problem to determine the critical probability on the higher dimensional square lattices (i.e. on $\mathbb{L}_d$ for $d \geq 3$).

We now turn our attention to percolation on more general graphs.

3. The Number of Possible Infinite Clusters

Percolation theory has recently been extended to considering percolation on more general graphs. Specifically, percolation on Cayley graphs for various groups has been examined. This greater generality allows for results that can be applied to a larger class of graphs, as opposed to considering graphs on a case by case basis. For example, we will see how the infinite cluster is unique for all amenable graphs, from which we deduce the uniqueness of the infinite cluster on $\mathbb{L}_2$, since $\mathbb{L}_2$ is seen to be amenable.

Definition 3.1. A *Cayley Graph*, $\Gamma(G, S)$, for a group G and a finite generating set $S \subset G$ is a graph where each vertex represents an element of G and an edge between g_1 and g_2 exists if and only if there is an $s \in S$ such that $g_1 = sg_2$.

Definition 3.2. A graph Γ is said to be *transitive* if for any vertex $v \in \Gamma$ there exists a graph automorphism $f \in \mathrm{Aut}(\Gamma)$ such that

$$v = f(v)$$

Intuitively, this definition merely implies that no vertex in a transitive graph is inherently unique, so the graph 'looks the same' everywhere. We will only consider percolation on transitive graphs so that separate attention does not need to be given to any particular vertex in the graph.

When considering percolation on Cayley graphs, it is natural to wonder how properties of G affect percolation on Γ. In particular, for which groups will the corresponding Cayley graph yield unique infinite clusters when above the critical probability? For some graphs - such as the d-dimensional lattice - the infinite cluster will always be unique, but for other graphs, there will be multiple infinite clusters (in fact, infinitely many of them) for certain probabilities.

A question naturally arises: if there is not necessarily a unique infinite cluster for some percolation, can we determine the number of infinite clusters that do in fact arise the subgraph? The answer to this is actually rather well-known, and may seem surprising.

Theorem 3.3. *If we let Γ be an infinite, connected, transitive graph and we determine a percolation on Γ with probability $p \in [0,1]$, then the number of infinite connected open clusters must equal $0, 1,$ or ∞.*

Before we prove this, a few preliminary definitions are needed.

Definition 3.4. For a given infinite Cayley graph $\Gamma(G,S)$ and a given $p \in [0,1]$, we allow each edge of Γ to be open with probability p and closed with probability $1-p$. The resulting subgraph of Γ consisting of the open edges is a *percolation of* Γ with parameter p. This subgraph will be denoted Γ_p.

For any subgraph $H \subset \Gamma$, we define *percolation on* H to be the subgraph of Γ_p restricted to the edges in H. We denote percolation on a subgraph H by $X(H)$.

Definition 3.5. For a given Cayley graph $\Gamma(G,S)$, the *ball of radius* k *centered at* v is the set

$$B(v,k) = \{w \in \Gamma : |v \leftrightarrow w| \leq k\}$$

where $|\cdot|$ denotes the shortest path length. Thus, $B(v,k)$ is the set of vertices within k edges from v. Similarly, we define the *edge-set of* $B(v,k)$ to be

$$B_E(v,k) = \{e \in E : \text{ both endpoints of } e \text{ are contained in } B(v,k)\}$$

Notice that this definition implies immediately

$$\lim_{k\to\infty} B_E(v,k) = \Gamma$$

Proof. Let N denote the number of infinite clusters on the subgraph $\Gamma_p \subset \Gamma$. For $n = 0,1,2,\ldots\infty$, let D_n be the event that there exists exactly n infinite open clusters. We first show that D_n occurs with probability either 0 or 1, so that N must be constant.

Assume for contradiction that for some n we have:

$$0 < \mathbb{P}(D_n) < 1 \tag{3.6}$$

If we just look at the subgraph of $X(B_E(v,k)) \subset \Gamma_p$ we can "guess" on the occurrence of D_n by defining an indicator variable as follows:

$$I_{n,v.k} = \begin{cases} 0 & \text{if } \mathbb{P}(D_n | X(B_E(v,k))) \leq \frac{1}{2}; \\ 1 & \text{if } \mathbb{P}(D_n | X(B_E(v,k))) > \frac{1}{2}. \end{cases}$$

If we let I_{D_n} denote the indicator variable for D_n as well, then for fixed v, it is an immediate consequence that

$$\lim_{k\to\infty} I_{n,v,k} = I_{D_n}$$

since

$$\lim_{k\to\infty} X(B_E(v,k)) = \Gamma_p$$

Next, define a sequence of vertices $\{u_k\}_{k=1}^{\infty}$ such that for each k, w_k is at least $2k$ vertices away from v. We have

$$B_E(v,k) \cap B_E(w_k,k) = \emptyset$$

Now, because of the transitivity of the graph, the variables $I_{n,v,k}$ and $I_{n,w_k,k}$ have the same distribution, and thus $I_{n,w_k,k}$ converges to I_{D_n} as $k \to \infty$. Hence,

$$\lim_{k\to\infty} \mathbb{P}(I_{n,w_k,k} = I_{n,v,k} = I_{D_n}) = 1 \tag{3.7}$$

However, since we saw that $B_E(v,k)$ and $B_E(w_k,k)$ are disjoint, they are independent (since the state of each edge in Γ is independent of other edges). Thus, from (3.6)

$$\begin{aligned}\lim_{k\to\infty} \mathbb{P}(I_{n,w_k,k} = 1 - I_{n,v,k} = 1) &= \lim_{k\to\infty} \mathbb{P}(I_{n,w_k,k} = 1)\mathbb{P}(1 - I_{n,v,k} = 1) \\ &= \mathbb{P}(D_n)(1 - \mathbb{P}(D_n)) > 0\end{aligned}$$

However, this contradicts the equality (3.7). Therefore, our initial assumption (3.6) must be false, so that D_n occurs with probability either 0 or 1.

Now, it remains to show that for $2 \le n < \infty$, $\mathbb{P}(D_n) = 0$. Assume for contradiction that $\mathbb{P}(D_n) = 1$ for some $2 \le n < \infty$. Since Γ is connected, there exists some $k \in \mathbb{Z}^+$ and some vertex $v \in \Gamma$ such that with positive probability $B(v,k)$ intersects each of the n infinite clusters. Since $|B(v,k)|$ is finite, there is a positive probability that there exist open paths in $B(v,k)$ connecting the infinite clusters (conditional on $B(v,k)$ intersecting each n infinite clusters). The occurrence of such open paths would result in a unique infinite cluster. Therefore, it follows that

$$\mathbb{P}(N = 1) > 0$$

which is a contradiction to the assumption that $\mathbb{P}(D_n) = 1$ for $2 \le n < \infty$. Thus, we have shown that the number of infinite open clusters must be constant for a percolation on any transitive graph, and that the number of infinite clusters must be: 0, 1, or ∞. □

This result is quite useful when studying the uniqueness of infinite clusters. If the event of infinitely many infinite clusters can be shown to have probability 0 on a given graph, then it follows that any infinite cluster must be unique on that graph.

The free group on 2 elements, F_2, is an example of a graph in which infinitely many infinite clusters arise. Removing any edge from this graph clearly separates the graph into disjoint connected components, so there will be infinitely many infinite clusters for any $p \in (p_c, 1)$, and uniqueness of the infinite cluster only occurs when $p = 1$ (see Figure 2).

It is not entirely known whether or not a given graph will always have a unique infinite cluster for $p \ge p_c$. However, there is an important result which determines this for the Cayley graphs of a large class of groups.

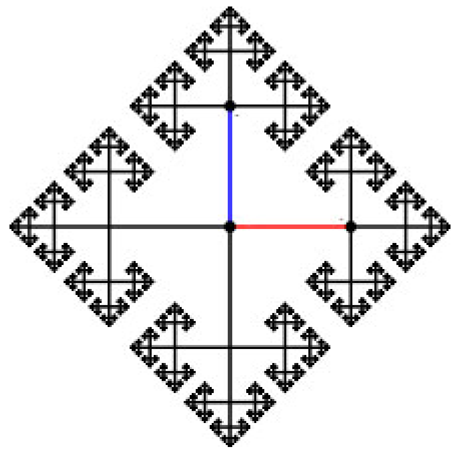

FIGURE 2. Cayley Graph of F_2

4. Amenable Graphs and Uniqueness of the Infinite Cluster

In the attempt to classify groups for which an infinite cluster will always be unique, the notion of amenability has great importance.

Definition 4.1. Let $\Gamma(G, S)$ be a given Cayley graph, and let $H \subset \Gamma$ be any subgraph. We determine the {boundary of H to be the edge-set

$$\partial H = \{(u, v) | u \in H, v \in \Gamma \setminus H\}$$

so that $|\partial H|$ represents the number of edges leaving H.

Definition 4.2. A Cayley graph $\Gamma(G, S)$ is *amenable* if and only if

$$h(G) = \inf_{H \subset \Gamma} \frac{|\partial H|}{|H|} = 0$$

where $|\cdot|$ denotes the number of vertices in the graph. If $h(G) > 0$, then $\Gamma(G, S)$ is considered *non-amenable*

The constant $h(G)$ is called the *edge iso-perimetric* constant. Amenability describes a great deal about the geometry of a graph. For an amenable graph, we must be able to select vertices to form a connected subgraph such that the number of boundary edges that are added is less than the number of vertices that are added. An example of this can be seen by the box $[-n, n] \times [-n, n] \subset \mathbb{L}_2$ which has $4n^2$ vertices and $4(2n - 1)$ boundary edges. Expanding this box by 1 vertex in each direction (i.e. taking the box $[-(n + 1), n + 1] \times [-(n + 1), n + 1]$) adds $4(2n - 1)$ vertices but only 8 boundary edges. It follows from taking n arbitrarily large that $\mathbb{L}_2$ is amenable.

On the other hand, non-amenable graphs have the property that selecting vertices to form a connected subgraph will always produce the same amount or more boundary edges as vertices. An example of this is F_2, in which any additional vertex being added to a connected subgraph will produce 2 additional boundary edges.

This definition of amenability concerns only properties of the Cayley graph, but amenability is also related to the structure of the group itself. Defining amenability for a group requires the ability to determine a probability measure on G that is left invariant on subsets of G. Since this definition concerns only the group structure itself, an amenable Cayley graph $\Gamma(G, S)$ for some generating set S will be amenable for any generating set.

The following Theorem illustrates the connection between amenability and uniqueness of the infinite cluster.

Theorem 4.3. *Let G be a connected, transitive, amenable group with critical probability p_c. Then for any $p \in (p_c, 1]$, the Cayley graph, $\Gamma(G, S)$ contains a unique infinite cluster. In other words, whenever there is positive probability of an infinite cluster existing, such an infinite cluster will almost surely be unique.*

Proof. First, assume that our percolation has probability $p \in (p_c, 1]$ so that there exists an infinite cluster in Γ. A vertex $v \in \Gamma$ is called a *trifurcation* if the following conditions hold:

- v is part of an infinite cluster
- v has only three neighboring vertices
- Removing v from Γ splits up the corresponding infinite cluster into exactly three disjoint infinite clusters (and therefore no finite clusters).

Assume Γ contains at least one trifurcation. If not, then the probability that any $v \in \Gamma$ is a trifurcation is 0, and the reader can skip to the paragraph following (4.4).

Now, let $A \subset \Gamma$ be a finite set of trifurcations which are all contained in the same infinite cluster, C. A trifurcation $v \in A$ is called an *outer member* if at least two of the disjoint infinite clusters resulting from the removal of v from Γ contain no other elements of A (i.e. $C \setminus \{v\}$ is comprised of three non-intersecting infinite clusters, at most one of which contains any trifurcations). We claim that A contains some outer member.

To see this, take $v_1 \in A$ and assume v_1 is not an outer member. Thus, $C \setminus \{v_1\}$ is the union of three disjoint infinite clusters such that at least two of the clusters contain elements of A. Let v_2 and v_3 be such elements. Assume v_3 is not outer, so that $C \setminus \{v_3\}$ results in three disjoint infinite clusters such that one cluster contains v_1 and v_2, and at least one of the other two clusters contains some $v_4 \in A$. Now if v_4 is not outer, then its removal results in one cluster containing v_1, v_2 and v_3 and at least one other cluster containing some v_5. Since A was taken to be finite, this process must terminate with some $v_n \in A$. Thus, $C \setminus \{v_n\}$ results in three disjoint infinite clusters, one of which contains $v_1, \ldots, v_{n-1}$ and the other two containing no other members of A. Therefore, A contains an outer member.

Given a connected infinite cluster, C, we claim that removing j trifurcations from C results in $j + 2$ disjoint infinite clusters. This is seen by induction. Removing 1 trifurcation clearly results in 3 disjoint infinite clusters. Therefore, assume the result holds for j trifurcations and let $T \subset C$ be a set of $j + 1$ trifurcations. Let v be an outer member of T. By assumption, $C \setminus \{T \setminus \{v\}\}$ results in $j + 2$ infinite clusters. Since v is an outer member, $C \setminus \{v\}$ consists of 3 disjoint infinite clusters, at most one of which contains elements of T. Thus, removing v from $C \setminus \{T \setminus \{v\}\}$ produces an additional disjoint infinite cluster, resulting in $j + 3$ clusters total, proving the claim.

Let $W \subset \Gamma$ be a finite subgraph containing j trifurcations of some infinite cluster, C. Then C must intersect ∂W in at least $j + 2$ vertices. Consequently, W cannot contain more than $|\partial W| - 2$ trifurcations. Let p_t be the probability of a vertex being a trifurcation. Since Γ was assumed to be transitive, p_t does not depend on the particular vertex. Additionally, if $T(W) = |\{w \in W | \text{w is a trifurcation}\}|$, then as we just saw, $T(W) \leq |\partial W| - 2$ and $\mathbb{E}[T(W)] = p_t|W|$. Because of this

$$p_t = \frac{\mathbb{E}[T(W)]}{|W|} \leq \frac{|\partial W| - 2}{|W|} \tag{4.4}$$

By the assumption that Γ is amenable, the right side of the above equation can be as small as we want by choice of W. This implies that $p_t = 0$.

Now we show that the number of infinite clusters must be 1. Assume for contradiction that there are infinitely many infinite clusters. We can find a connected finite set W such that at least three disjoint infinite clusters intersect W. Since $|W| < \infty$, there are only finitely many possible percolations on the edge-set of W, so any particular percolation has positive probability.

Thus, since at least three disjoint infinite clusters intersect W, the event that there exists open paths in W from the three infinite clusters intersecting at only one vertex has positive probability. This event creates a trifurcation, so that the probability of a trifurcation on a vertex is positive. In other words $p_t > 0$ on Γ, which is a contradiction as was seen above.

Therefore, there cannot be infinitely many infinite clusters, and since we have $p \in (p_c, 1]$, there must only be one infinite cluster for any $p > p_c$ □

Corollary 4.5. *Given $p > p_c$, the infinite cluster on $\mathbb{L}_2$ is almost surely unique.*

Proof. Let $\epsilon > 0$ be given, and let $H = \{-n, n\} \times \{-n, n\}$ be the box of side length n with center at the origin. Then, $|H| = 4n^2$, and $|\partial H| = 4(2n - 1)$. If we set $n > \frac{1}{\epsilon} \pm \sqrt{\frac{1}{\epsilon^2} - \frac{1}{\epsilon}}$, then it can be checked that

$$\frac{|\partial H|}{|H|} = \frac{2n - 1}{n^2} < \epsilon$$

which shows us that we can make $\frac{|\partial H|}{|H|}$ as small as we want. Hence, $h(\mathbb{L}_2) = 0$, so $\mathbb{L}_2$ is amenable, and the result follows from Theorem 4.2. □

Given some non-amenable graph such that infinitely many infinite clusters may exist, we define the *uniqueness probability* to be

$$p_u = \inf \{p \,|\, \mathbb{P}(\text{there exists a unique infinite cluster} = 1)\}$$

Hence, for amenable graphs $p_c = p_u$ since whenever there is an infinite cluster it will be unique. However, for non-amenable graphs, $p_c \leq p_u$ since there may be some interval (p_c, p_u) where the infinite cluster will not be unique.An example of such a graph was given at the end of the previous section with F_2 in which $p_u = 1$.

It is still under investigation whether or not for all non-amenable graphs there exists a non-trivial interval $U = (p_c, p_u)$ such that for all $p \in U$ there will not be uniqueness of the infinite cluster. Thus, we have the following conjecture, which was first stated by Benjamini and Schramm[6] in 1996.

Conjecture 4.6. *If G is a non-amenable group and $S \subset G$ is a finite generating set, then*

$$p_c < p_u$$

on the Cayley graph, $\Gamma(G, S)$

Many results have been found determining conditions on G and S such that $p_c < p_u$. For example, it is known that for any non-amenable group and transitive Cayley graph $\Gamma(G, S)$ that there exists a finite generating set S such that $p_c < p_u$, but it is not known in general for any such S. It has also been shown that certain slight modifications can be made to the generating set which in the conjecture holds.

References

[1] Grimmett, Geoffrey. *Percolation* second edition. Springer, 1999

[2] Kesten, Harry. *The Critical Probability of Bond Percolation on the Square Lattice Equals* $\frac{1}{2}$. Communications in Mathematical Physics 74. 1980

[3] Haggstrom, Olle and Jonasson, Johan. *Uniqueness and Non-Uniqueness in Percolation Theory*. Probability Surveys, Volume 3. 2006

[4] Burton, R. M. and Keane, M. *Density and Uniqueness in Percolation*. Communications in Mathematical Physics 121. 1989.

[5] Menshikov M. V. *Coincidence of Critical Points In Percolation Problems*. Soviet Mathematics Doklady 33. 1986

[6] Benjamini, Itai and Schramm, Oded. *Percolation Beyond* $\mathbb{Z}^d$*, Many Questions and A Few Answers*