ALGEBRAIC TOPOLOGY

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Abstract. The focus of this paper is a proof of the Nielsen-Schreier Theorem, stating that every subgroup of a free group is free, using tools from algebraic topology.

Contents

1. Preliminaries

Notations 1.1.
I [0, 1] the unit interval
S^n the unit sphere in $\mathbb{R}^{n+1}$
$\times$ standard cartesian product
$\approx$ isomorphic to
$\vee$ the wedge sum
$A - B$ the space $\{x \in A | x \notin B\}$
A/B the quotient space of A by B.

In this paper we assume basic knowledge of set theory. We also assume previous knowledge of standard group theory, including the notions of homomorphisms and quotient groups.

Let us begin with a few reminders from algebra.

Definition 1.2. A **group** G is a set combined with a binary operator $\star$ satisfying:

- For all $a, b \in G$, $a \star b \in G$.
- For all $a, b, c \in G$, $(a \star b) \star c = a \star (b \star c)$.
- There exists an identity element $e \in G$ such that $e \star a = a \star e = a$.
- For all $a \in G$, there exists an inverse element $a^{-1} \in G$ such that $a \star a^{-1} = e$.

A convenient way to describe a particular group is to use a **presentation**, which consists of a set S of **generators** such that each element of the group can be written

Date: DEADLINE AUGUST 22, 2008.

as a product of elements in S, and a set R of **relations** which define under which conditions we are able to simplify our 'word' of product of elements in S.

Definition 1.3. A group is called a **free group** if there are no relations on its generators other than that of an element with its inverse.

Definition 1.4. A subset H of a group G is called a **subgroup** of G if H is also a group under the same binary operator as G. We write this as $H \leq G$.

2. The Fundamental Group

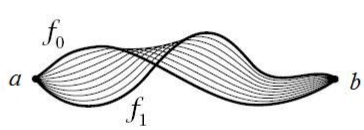
While algebra and topology seem at first to be very different branches of mathematics, they are related in surprising ways. Notions of algebra can be used to study properties from topological spaces under various maps. Our first object of study will be the fundamental group, which is, loosely put, the set of loops in a topological space. As its name suggests, it is indeed a group in the algebraic sense. In order to prove this fact, we shall first need to formalize the notion of 'loops in space'.

Definition 2.1. Let X be a topological space and $a, b \in X$. A **path** from a to b is a continuous function $f : I \to X$ such that $f(0) = a$ and $f(1) = b$.

We call a and b **endpoints**. When we look at the above definition, it is quite clear that even for the simplest of spaces the amount of different paths betweens two endpoints is colossal. In order to reduce that number we define the notion of homotopy between paths. Intuitively, two functions are homotopic if we can 'continuously deform' one into the other.

Definition 2.2. Given $a, b \in X$, we say two paths f_0 and f_1 are **homotopic** as paths if there exists a family of paths such that for all $t \in I$, f_t satisfies the following properties:

- $f_t(0) = a, f_t(1) = b$.
- The map $F : I \times I \to X$ defined by $F(s,t) = f_t(s)$ is continuous.



A family of paths satisfying the above conditions is called a **homotopy**. We also define a special type of homotopy which will be useful later on. As its name suggests, a deformation retract is a continuous map that deforms a space into one of its subspaces.

Definition 2.3. A **deformation retract** from a space X to a subspace A is a homotopy satisfying the additional properties that:

- $f_0 = id$.
- $f_1(X) \subset A$.
- $f_t|A = id$ for all t.

Proposition 2.4. *Given two fixed endpoints a and b, path homotopy is an equivalence relation on the set of all paths from a to b.*

Proof. Fix two points a and b in the space which will be the endpoints of all paths considered. Let us use the symbol '$\cong$' to mean 'is path homotopic to'. We must now prove that $\cong$ is reflexive, symmetric and transitive.

Reflexivity is trivial from the definition, since $f \cong f$ by the constant homotopy $f_t = f$; and so is symmetry since if $f_0 \cong f_1$ by the homotopy f_t, then $f_1 \cong f_0$ by the inverse homotopy f_{1-t}.

For transitivity, suppose $f_0 \cong f_1 = g_0$ via f_t and $g_0 \cong g_1$ via g_t. Then $f_0 \cong g_1$ via the homotopy h_t that equals f_{2t} on $[0, \frac{1}{2}]$ and equals g_{2t-1} on $[\frac{1}{2}, 1]$. The associated map $H(s,t)$ is indeed continuous since it is continuous when restricted to each of the intervals, and it agrees at $t = \frac{1}{2}$ since $f_1 = g_0$ by assumption. □

Example 2.5. In a convex set in $\mathbb{R}^n$, all loops are equivalent to the trivial loop. This is because given any two loops f_0 and f_1, we can always define the homotopy $f_t = (1-t)f_0 + tf_1$, which tells us in particular that any loop is homotopic to the constant loop.

Given the above proposition, we can now consider only different homotopies in the space rather than specific paths. From now on we shall refer to the homotopy class represented by a loop f by $[f]$. Of particular importance are paths whose endpoints coincide.

Definition 2.6. A path f is called a **loop** if $f(0) = f(1)$.

Definition 2.7. For a family of loops in space with common endpoint x_0, we refer to x_0 as the **basepoint**.

Inspired by the homotopy we've created in the previous proof, we shall now define an operation on paths. It is essentially defined so that paths are traversed sequentially, each twice as fast, in order for the path product to be traversed entirely in the same unit of time.

Definition 2.8. Given two paths $f, g : I \to X$ such that $f(1) = g(0)$, the **product path** $f \cdot g$ is a path in X defined by

$$f \cdot g = \begin{cases} f(2s), & 0 \leq s \leq \frac{1}{2} \\ g(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

By restricting the product above to loops having the same basepoint, we get a well defined operation on homotopy classes, since the homotopy class of the product path $f \cdot g$ is independent of the representative paths chosen from the homotopy classes of f and g. We can now state one of the most essential theorem in algebraic topology.

Theorem 2.9. *Given a space X, the set of homotopy classes [f] of loops based at $x_0 \in X$ is a group under the product path $[f] \cdot [g] = [f \cdot g]$*

We call this group the **fundamental group** of X based at x_0, written $\pi_1(X, x_0)$. We sometimes omit the basepoint and write simply $\pi_1(X)$ when X is path-connected, since in this case the fundamental groups of X based at each point will be isomorphic.

Proof. We wish to show that the above product on homotopy classes satisfy the group axioms. Given f, g, h loops based at x_0, we wish to prove first that $(f \cdot g) \cdot h = f \cdot (g \cdot h)$. To see this, we define a **reparametrization** of a path f to be the compostion $f\phi$ where $\phi : I \to I$ is a continuous map such that $\phi(0) = 0$ and $\phi(1) = 1$. We see that $f\phi$ is homotopic to f by the homotopy $f\phi_t$ where $\phi_t(s) = (1-t)\phi(s)+ts$. We observe that $f \cdot (g \cdot h)$ is a reparametrization of $(f \cdot g) \cdot h$ given by the function

$$\phi(s) = \begin{cases} \frac{1}{2}s, & 0 \leq s \leq \frac{1}{2} \\ s - \frac{1}{4}, & \frac{1}{2} \leq s \leq \frac{3}{4} \\ 2s-1, & \frac{3}{4} \leq s \leq 1 \end{cases}$$

The two paths are homotopic, thus their homotopy classes are equal and path product is associative.

The two sided identity is the constant path defined by $c_{x_0}(s) = x_0$ for all $s \in I$. Since f is a loop, $f(0) = f(1) = c_{x_0}(s)$ for all $s \in I$, so we can see that $f \cdot c_{x_0}$ is a reparametrization of f by the map

$$\phi(s) = \begin{cases} 2s, & 0 \leq s \leq \frac{1}{2} \\ 1, & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Similarly we see that $c_{x_0} \cdot f$ is a reparametrization of f thus the constant map satisfies the role of the identity in $\pi_1(X, x_0)$.

To verify the two sided inverse property, we define the **inverse path** of f to be $\bar{f}(s) = f(1-s)$. Now consider the identity path $i : I \to I$. Its inverse $\bar{i}$ is also a path on I, and $i \cdot \bar{i}$ is a loop based at 0. Since I is convex, there exists a homotopy H in I between $i \cdot \bar{i}$ and the constant path c_0 at 0. Then fH is a path homotopy between $fc_0 = c_{x_0}$ and $(fi) \cdot (f\bar{i}) = f \cdot \bar{f}$. We use a similar argument to prove that $\bar{f} \cdot f$ is homotopic to c_{x_0}. ☐

As its name suggests, the fundamental group is an essential algebraic invariant of topological spaces. We say a space is **simply connected** if it is path-connected and has trivial fundamental group. In our example above, we've shown that convex sets in $\mathbb{R}^n$ are simply connected.

Example 2.10. A very important computation is that the fundamental group of S^1 is isomorphic to the free group on one generator. An intuitive way to see this fact is to imagine the real line as a helix wound up above S^1. Now consider a path on the helix starting at 0 going up n times around the helix if $n > 0$ or down n times if $n < 0$, ($n \in \mathbb{N}$). Projecting these paths onto the circle below gives us all loops in S^1. When we apply the product path to these loops, we get that the fundamental group of S^1 is isomorphic to the group of integers under addition, which is itself isomorphic to the free group generated by one element. We will return to this example in the section about covering spaces.

A critical property of the fundamental group is its relation to maps between spaces. Specifically, basepoint-preserving maps between topological spaces induce homomorphisms on their fundamental groups.

Definition 2.11. Let X, Y be topological spaces, $x_0 \in X, y_0 \in Y$ be basepoints in their respective spaces, and let $\psi : X \to Y$ be a continuous map such that $\psi(x_0) = y_0$. The map ψ induces a map $\psi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ called **induced**

homomorphism and defined as composing loops in X with ψ. The resulting path in Y is a loop since $\psi(f(0)) = \psi(f(1))$.

Proposition 2.12. *ψ_* is a group homomorphism.*

Proof. Given two loops f, g in X based at x_0, direct computation shows that

$$\begin{aligned}\psi(f \cdot g) =& \begin{cases} \psi(f(2s)), & 0 \leq s \leq \frac{1}{2} \\ \psi(g(2s-1)), & \frac{1}{2} \leq s \leq 1 \end{cases} \\ =& \psi(f) \cdot \psi(g)\end{aligned}$$

□

3. Van Kampen's Theorem

The Van Kampen theorem provides a means of computing fundamental group by decomposing a space into a union of simpler spaces, whose fundamental groups are already understood. Repeated use of this theorem will allow us to compute the fundamental group of a very large number of spaces.

Definition 3.1. Let A, B be topological spaces such that $A \subset B$. An **inclusion map** is a map $i : A \to B$ that sends an element from A to itself, but considered as an element in B. We sometimes use the notation $A \hookrightarrow B$ for the inclusion map of A in B.

Definition 3.2. Given two groups G and H, the free product $G * H$ is the set of elements $s_1 s_2 \ldots s_n$, where s_i is an element of either G or H. Such a word can be reduced only by removing an instance of the identity element (in either G or H), or by replacing a consecutive pair of elements in the same group by their product in that group.

Theorem 3.3. *Let U_1, U_2 be open subsets of X such that $U_1 \cup U_2 = X$ and $U_1 \cap U_2$ is path connected. Let $x_0 \in U_1 \cup U_2$ be a basepoint, and let $i_1 : \pi_1(U_1 \cup U_2) \to \pi_1(U_1)$, $i_2 : \pi_1(U_1 \cup U_2) \to \pi_1(U_2)$ be the homomorphisms respectively induced by the inclusion maps $(U_1 \cup U_2) \hookrightarrow U_1$ and $(U_1 \cup U_2) \hookrightarrow U_2$. Then the homomorphism $\Phi : \pi_1(U_1) * \pi_1(U_2) \to \pi_1(X)$ is surjective and has a normal subgroup N generated by elements of the form $i_1(\omega_\alpha) i_2^{-1}(\omega_\alpha)$ where ω_α are loops in $U_1 \cup U_2$. Thus Φ induces an isomorphism $\pi_1(X) \approx (\pi_1(U_1) * \pi_1(U_2))/N$.*

The proof of this theorem is somewhat long and technical, so we shall instead give a brief outline of it. Surjectivity of Φ is given by considering a loop f in the space based at some point x_0. We then divide I into subintervals so that when we restrict f to each subinterval, it lies in only one U_i. The product of these loop sections with appropriate paths in $U_1 \cap U_2$ gives us a product of loops each lying in a single U_i that is homotopic to f. Thus Φ is surjective.

To show that Φ has a normal subgroup N, we choose to factorize an element $[f]$ into individual loops in each of the U_i such that the product of all the loops is homotopic to $[f]$. A factorization of $[f]$ is thus a word in $(\pi_1(U_1) * \pi_1(U_2))$ mapped to $[f]$ by Φ. We now show the uniqueness of such a factorization by simultaneously combining adjacent loops in the factorization that lie in the same space U_i and by systematically considering loops in the intersection to be in one space or the other (say, consider all loops in $U_1 \cap U_2$ to be only in U_1). By applying this method, we can show that any two factorizations of a loop $[f]$ are in fact equivalent. By definition of

N, equivalent factorizations of $[f]$ give us the same element in $(\pi_1(U_1) * \pi_1(U_2))/N$. Since the factorization is unique, the induced map $(\pi_1(U_1) * \pi_1(U_2))/N \to \pi_1(X)$ is injective, thus the kernel of Φ is exactly N.

Example 3.4. The fundamental group of the wedge sum $\bigvee_{i=1}^{n} S_i^1$ of n copies of S^1 is isomorphic to the free group on n generators. If x_j is a basepoint in S_j^1, the wedge sum $\bigvee_{i=1}^{n} S_i^1$ is the union of all the S_i^1 with their respective basepoints identified to a single point x_0. For each x_j there exists an open neighbourhood U_j such that U_j deformation retracts onto x_j. We've seen previously that $\pi_1(S^1)$ is the free group on one element. We now inductively apply the Van Kampen theorem on $A = (\bigvee_{i=1}^{k} S_i^1) \cup U_{k+1}$ and $B = S_{k+1}^1 \cup (\bigvee_{i=1}^{k} U_k)$. The intersection of these two spaces is $\bigvee_{i=1}^{k+1} U_{k+1}$, which is simply connected since it deformation retracts to the basepoint x_0. A similar argument also shows that $\pi_1(B) = \pi_1(S^1)$, thus we have an isomorphism $\Phi : \pi_1(A) * \pi_1(S^1) \to \pi_1(\bigvee_{i=1}^{k+1} S_i^1)$. So $\pi_1(\bigvee_{i=1}^{n} S_i^1)$ is the free group on n generators, one for each copy of S^1.

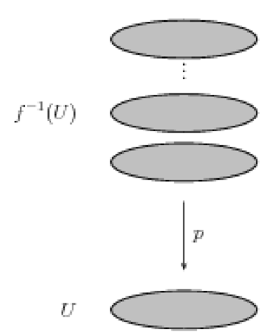
4. Covering Spaces

While the Van Kampen theorem is useful for computing the fundamental group of unions of spaces whose fundamental group we already know, it is not very helpful if we do not actually know the fundamental group of any spaces. In an earlier section we asserted without proof that $\pi_1(S^1)$ was isomorphic to the free group with one generator. The notion of covering space will finally give us the tools needed to directly compute non-trivial fundamental groups.

Definition 4.1. Let $p : \tilde{X} \to X$ be a surjective map between two topological spaces. We say an open set $U \subset X$ is **evenly covered** by p if $p^{-1}(U)$ can be written as the union of disjoint open sets $V_\alpha \subset \tilde{X}$ such that $p|V_\alpha$ is a homoemorphism onto U.

Definition 4.2. Let $p : \tilde{X} \to X$ be surjective map. If every $x \in X$ has a neighborhood U such that U is evenly covered by p, then we call p a **covering map** and refer to $\tilde{X}$ as the **covering space**. We sometimes refer to the topological space as a covering space of some space X; the existence of a covering map is implicit.

It is often useful to visualize the set $p^{-1}(U)$ as a 'stack of pancakes' of identically shaped and sized copies of U floating above it.



As we hinted in our intuitive explanation about the fundamental group of the circle, a space X and its covering space are related by what are called **lifts**. Lifts give us the correspondence between paths in X and paths in $\tilde{X}$.

Definition 4.3. Let X, Y be spaces and $p : \tilde{X} \to X$ be a covering space of X. A **lift** of a map $f : Y \to X$ is a map $\tilde{f} : Y \to \tilde{X}$ such that $p\tilde{f} = f$.

Proposition 4.4. *Given a covering space $p : \tilde{X} \to X$, a homotopy $f_t : Y \to X$ and a lift $\tilde{f}_0 : Y \to \tilde{X}$ lifting of f_0, there exists a unique homotopy $\tilde{f}_t : Y \to \tilde{X}$ starting at $\tilde{f}_0$ lifting f_t.*

Proof. Let $F : Y \times I \to X$ be our homotopy map. We first construct a lift $\tilde{F} : N \times I \to \tilde{X}$ for N a neighborhood of a fixed point y_0 in Y. F is continuous, thus for every $(y, t) \in Y \times I$, there exists a neighborhood $N_t \times (a_t, b_t)$ such that $F(N_t \times (a_t, b_t)) \subset U_\alpha$, where U_α is an open set in X such that U_α is openly covered by p. Since $\{y_0\} \times I$ is compact, we have a finite number of such neighborhoods covering it. Hence we can choose a single neighborhood N of y_0 and a partition $0 = t_0 < t_1 < \ldots < t_m = 1$ of I so that for each i, $F(N \times [t_i, t_{i+1}])$ is contained in some U_α, which we will now refer to as U_i. We will now construct $\tilde{F}$ inductively, assuming it is constructed on $N \times [0, t_i]$. On $N \times [t_i, t_{i+1}]$, by definition there exists an open set $\tilde{U}_i \in \tilde{X}$ containing the point $\tilde{F}_0(y_0, t_i)$ that projects homeomorphically onto U_i by p. Up to replacing N with a smaller neighborhood, we can assume that $\tilde{F}(N \times \{t_i\})$ is contained in $\tilde{U}_i$. We can now define $\tilde{F}$ on $N \times [t_i, t_{i+1}]$ to be the composition of F with $p^{-1} : U_i \to \tilde{U}_i$. This process terminates after finitely many iterations, thus giving us a completely defined lift $\tilde{F} : N \times I \to X$ for some neighborhood N of y_0.

We prove uniqueness in two parts. We begin with the special case where Y is a single point. For ease of notation, we can simply write I for $\{y\} \times I$. Let us suppose $\tilde{F}$ and $\tilde{F}'$ are two lifts of $F : I \to X$ such that $\tilde{F}(0) = \tilde{F}'(0)$. Once again we choose a partition of I such that for each i, $F([t_i, t_{i+1}]) \subset U_i$. Assume by induction that on $[0, t_i]$ $\tilde{F} = \tilde{F}'$. $\tilde{F}([t_i, t_{i+1}])$ is connected, so it must lie in a single $\tilde{U}_i$. Similarly $\tilde{F}'([t_i, t_{i+1}])$ lies in a single $\tilde{U}_j$. However $\tilde{F}(t_i) = \tilde{F}'(t_i)$ so in fact $U_j = U_i$. Since p is injective on $\tilde{U}_i$ and $p\tilde{F} = p\tilde{F}'$, we conclude that $\tilde{F} = \tilde{F}'$ on the whole interval $[t_i, t_{i+1}]$, thus completing the inductive step.

We now finalize by remarking that since our $\tilde{F}$ defined as above on $N \times I$ is unique when restricted to $\{y\} \times I$, it must agree whenever two such sets $N \times I$ overlap. So $\tilde{F}$ is well-defined and unique on all of $Y \times I$. □

Corollary 4.5. *(Path lifting property) Given a covering space $p : \tilde{X} \to X$, for each path $f : I \to X$ and each pre-image $\tilde{x}_0$ of $f(0) = x_0$, there is a unique path $\tilde{f} : I \to \tilde{X}$ lifting f starting at $\tilde{x}_0$.*

To prove this corollary, we simply consider a path to be homotopic to a point and apply the proposition above.

Before we go on and compute the fundamental group of the circle, we shall first prove a proposition which we shall need later on.

Proposition 4.6. *Given a space X and a covering space $p : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$, the induced map $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective.*

Proof. The proof is a simple application of our previous proposition. Elements in the kernel of p_* are represented by loops $\tilde{f}_0 : I \to \tilde{X}$ such that there exists a homotopy $f_t : I \to X$ of $f_0 = p\tilde{f_0}$ to the trivial loop f_1. By the above proposition, there exists a unique homotopy $\tilde{f_t}$ starting at $\tilde{f_0}$ lifting f_t. Thus $[\tilde{f_0}] = 0$ and p_* is injective. □

Definition 4.7. Let $p : \tilde{X} \to X$ be a covering space, $x_0 \in X$ be a basepoint. Choose a point $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = x_0$. Given an element $[f]$ of $\pi_1(X, x_0)$, let $\tilde{f}$ be the corresponding lift of f starting at $\tilde{x}_0$.

If we denote the endpoint $\tilde{f}(1)$ by $\Phi([f])$, then Φ is a well defined map $\Phi : \pi_1(X, x_0) \to p^{-1}(x_0)$ called the **lifting correspondence** derived from the covering map p.

Proposition 4.8. *Let $p : \tilde{X} \to X$ be a covering map and $\Phi : \pi_1(X, x_0) \to p^{-1}(x_0)$ be a lifting correspondence based at $\tilde{x}_0$. If $\tilde{X}$ is simply connected, Φ is bijective.*

Proof. Let $[f], [g]$ be elements in $\pi_1(X, x_0)$ such that $\Phi([f]) = \Phi([g])$, and let $\tilde{f}, \tilde{g}$ be lifts of f and g, respectively, beginning at $\tilde{x}_0$. Then $\tilde{f}(1) = \tilde{g}(1)$. Since $\tilde{X}$ is simply connected, there exists a path homotopy between $\tilde{F}$ between $\tilde{f}$ and $\tilde{g}$. Then $p\tilde{F}$ is a path homotopy in X between f and g; thus Φ is injective.

Φ is surjective since given $\tilde{x}_1 \in p^{-1}(x_0)$, there is a path $\tilde{f}$ in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_1$. Then $f = p\tilde{f}$ is a loop in X based at x_0. Thus Φ is surjective. □

We are now finally ready to prove that the fundamental group of the circle is isomorphic to the additive group of integers.

Example 4.9. We shall use without proof the fact that $p : \mathbb{R} \to S^1$ given by the equation $p(x) = (\cos(2\pi x), \sin(2\pi x))$ is a covering map. We refer to our intuitive notion explained earlier of 'wrapping' $\mathbb{R}$ around S^1 as a helix. Choose our $\tilde{x}_0$ to be $0 \in \mathbb{R}$, and let $x_0 = (1, 0)$. We thus have that $p^{-1}(1, 0) = \mathbb{Z}$. Since $\mathbb{R}$ is simply connected, the lifting correspondence $\Phi : \pi_1(S^1, x_0) \to \mathbb{Z}$ is bijective according to our above proposition.

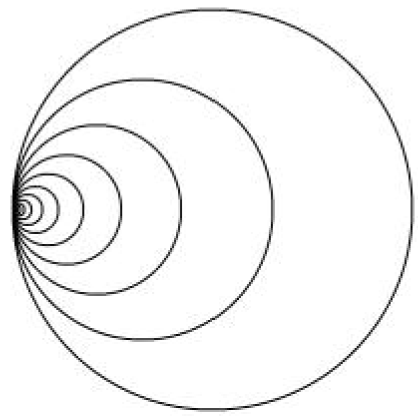
We now need to prove that Φ is a homomorphism to complete the proof. Given $[f], [g]$ in $\pi_1(X, x_0)$, let $\tilde{f}$ and $\tilde{g}$ be their respective lifts beginning at 0. Let $n = \tilde{f}(1)$ and $m = \tilde{g}(1)$. Then by definition $\Phi([f]) = n$ and $\Phi([g]) = m$.

Consider $\tilde{\tilde{g}}(x) = n + \tilde{g}(x)$ a path in $\mathbb{R}$. This path is a lifting of $\tilde{g}$ beginning at n since $p(n + x) = p(x)$ for all $x \in \mathbb{R}$. Then the product $\tilde{f} \cdot \tilde{\tilde{g}}$ is well defined, and is a lift of $f \cdot g$. The endpoint of this path is $\tilde{\tilde{g}}(1) = n + m$. Then by definition $\Phi([f] \cdot [g]) = n + m = \Phi([f]) + \Phi([g])$.

As we have seen in our previous calculation, it is fairly advantageous to have a simply connected covering space. A necessary condition on the original space is that of **semi-locally simple connectedness**. Roughly speaking, this condition imposes a lower bound on the size of 'holes' in X. It is a fairly general condition, and spaces that do not satisfy it are usually considered pathological.

Definition 4.10. Let X be a space, and $p : \tilde{X} \to X$ be a covering space. We say X is **semi-locally simply connected** if for each point $x \in X$, there exists a neighborhood U each loop in U is nullhomotopic (i.e, the inclusion induced map $\pi_1(U, x) \to \pi_1(X, x)$ is trivial).

Example 4.11. A standard example of a non semi-locally simply connected space is the so called Hawaiian Earrings, the union of circles with center $(\frac{1}{n}, 0)$ and radius $\frac{1}{n}$ for $n \in \mathbb{N}$.



This next proposition is important in classifying the various covering spaces of a space. We shall however not supply its proof since it involves notions about the construction of simply-connected covering spaces. It is sufficient to say that such spaces exist as quotient spaces of simply-connected covering spaces.

Proposition 4.12. *Let X be a path connected and semi-locally simply connected space. Then for any subgroup $G \leq \pi_1(X)$ there exists a covering space $p : \tilde{X}_G \to X$ such that $p_*(\pi_1(X_G, \tilde{x}_0)) = G$ for an appropriately chosen basepoint $\tilde{x}_0 \in X_G$*

5. Graphs

The usual objective of homotopy theory is to reduce problems of topology to those of algebra. In our final section, we will be doing the reverse by using topological properties of linear graphs to prove the Nielsen-Schreier Theorem, stating that every subgroup of a free group is free.

Definition 5.1. A **linear graph** is a space X obtaining by attaching to a discrete set X^0 of points a collection of 1-cells e_α (spaces homeomorphic to the open interval $(0, 1)$). We create X by identifying the endpoints of the closed intervals I_α with points in X^0. We call points in X^0 **vertices** and the e_α **edges**.

Under this definition edges do not include their endpoints. They are open subsets of X. Their closure $\bar{e}_\alpha$ is homeomorphic to I or S^1, depending on whether the two endpoints of the edge are distinct or not. Since X is defined as a quotient space of the disjoint union $X^0 \bigcup_\alpha I_\alpha$, X has the **weak topology**, meaning a subset of X is closed (resp. open) if and only if it intersects the closure $\bar{e}_\alpha$ of each edge E_α in a closed (resp. open) set in $\bar{e}_\alpha$.

A **subgraph** of X is a closed subset $Y \subset X$ that is a union of vertices and edges such that if $e_\alpha \in Y$, then $\bar{e}_\alpha \in Y$.

Definition 5.2. A **tree** is a subgraph of a graph X such that any two vertices are connected by exactly one path. Equivalently, a tree is a subgraph that is contractible to a single point.

We say a tree is **maximal** if it contains all the vertices of X.

Proposition 5.3. *Every connected graph contains a maximal tree, and every tree is contained in a maximal tree.*

We shall omit the proof of the above proposition, but it is a standard exercise in graph theory.

Definition 5.4. Let X, Y be topological spaces. We say X is **homotopy equivalent** to Y if there exists maps $f : X \to Y$ and $g : Y \to X$ such that fg is homotopic to id_Y and gf is homotopic to id_X.

An important property of homotopy equivalent spaces is that their fundamental group are isomorphic. To prove this we shall use a fact about homotopies that do not fix the basepoint.

Lemma 5.5. *If $f_t : X \to Y$ is a homotopy and h is the path $f_t(x_0)$ formed by the images of a basepoint $x_0 \in X$, then the three maps from the following diagram satisfy $f_{0*} = \beta_h f_{1*}$, where β_h is an isomorphism.*

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f_{1*}} & \pi_1(Y, f_1(x_0)) \\ & \searrow^{f_{0*}} & \downarrow \beta_h \\ & & \pi_1(Y, f_0(x_0)) \end{array}$$

Proof. Let $h_t(s) = h(ts)$. The map h_t is the restriction of h to $[0, t]$, reparametrized such that the domain of h_t is still $[0, 1]$. If ω is a loop in X based at x_0, then the product $h_t \cdot (f_t\omega) \cdot \bar{h}_t$ gives a homotopy of loops at $f_0(x_0)$. Looking at $t = 0$ and $t = 1$, we see that $f_{0*}([\omega]) = \beta_h(f_{1*}([\omega]))$. □

Using this lemma, we can now prove the following proposition.

Proposition 5.6. *Let $f : X \to Y$ be a homotopy equivalence between two spaces X and Y. Then the induced homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism for all x_0.*

Proof. Let $g : Y \to X$ be a homotopy inverse of f, in other words a map such that gf is homotopic to id_Y and fg is homotopic to id_X. Now consider the following maps.

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, gf(x_0)) \xrightarrow{f_*} \pi_1(Y, fgf(x_0))$$

Since $gf = id_X$, by the lemma $g_* f_* = \beta_h$ for some h, then the composition of the first two maps is an isomorphism. This implies in particular that f_* is injective. Applying the same reasoning to the second and third map tells us that g_* is injective. So since both maps are injective and their composition $f_* g_*$ is an isomorphism, then the first map f_* must be surjective as well. □

In our next proposition we shall use without proof the fact that the quotient space of a space X by a contractible subspace T is a homotopy equivalence. This is made intuitively obvious since the space we collapse already has the homotopy type of a point.

Proposition 5.7. *Given a connected graph X with maximal tree T, $\pi_1(X)$ is a free group on n generators, where n is the number of edges in $X - T$.*

Proof. Since the map $X \to X/T$ is a homotopy equivalence, the fundamental group of X is the same as that of X/T. Since our tree T contains all the vertices of X, when we collapse it we are left with a graph consisting of a single vertex and as many arcs as were in $X - T$. This space is homeomorphic to the wedge sum of circles, whose fundamental group we have calculated previously to be the free group on n elements, where n is the number of circles in the wedge, which in our case is the number of edges in $X - T$. □

We are almost ready to prove our final theorem. We just need a lemma about the covering spaces of graphs.

Lemma 5.8. *Every covering space of a graph is also a graph.*

Proof. Let $p : \tilde{X} \to X$ be the covering space. We shall see that we can construct $\tilde{X}$ according to our definition of a linear graph. First, we use the set $\tilde{X}^0 = p^{-1}(X^0)$ as our set of vertices. If we write X as a quotient space of the disjoint union $X^0 \bigcup_\alpha I_\alpha$, we can then apply the path lifting property to the resulting maps $I_\alpha \to X$ used to attach our 1-cells to X^0. We then obtain a unique lift $I_\alpha \to \tilde{X}$ passing through each point of $p^{-1}(x)$, for $x \in e_\alpha$ which define edges in $\tilde{X}$. Thus our covering space satisfies our definition of a graph. The topology between X and $\tilde{X}$ is the same by the property of p being a local homeomorphism. □

We now have all the tools we need to smoothly attain our goal of proving the Nielsen-Schreier Theorem.

Theorem 5.9. *Every subgroup of a free group is free.*

The proof is now merely a technicality.

Proof. Given a free group F, choose a graph X such that $\pi_1(X) \approx F$. We may take X to be the wedge sum of one circle for every generator of F. By proposition 4.12, for each subgroup $G \leq F$ there exists a covering space $p : \tilde{X} \to X$ such that $p_*(\pi_1(\tilde{X})) = G$. By proposition 4.6, p_* is injective, thus $\pi_1(\tilde{X}) \approx G$. By the preceding lemma, $\tilde{X}$ is a graph, thus its fundamental group is free by proposition 5.7. □

Acknowledgements. I would like to thank my mentors for the REU program, Jennifer Wilson and Jared Bass, for their outstanding support during the writing of this paper.

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