LOCAL WELLPOSEDNESS FOR THE NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. The nonlinear Schrödinger equation is a nonlinear partial differential equation, solutions to which are complex-valued functions of d-dimensional space and of time. We examine the problem of, given initial data at time 0, extending it to a function satisfying the nonlinear Schrödinger equation for some time interval around 0. In particular, we are interested in whether this solution exists, how smooth it is or how quickly it decays, whether it is the unique solution for that initial data, and whether small changes in the data produce only small changes in the solution. We concentrate on cases where relatively little is assumed about the nature of the initial datum, introducing Duhamel's formula and an iteration principle to build up solutions.

CONTENTS

1. INTRODUCTION

Let u_0 be a complex-valued function of $\mathbb{R}^d$. Then our goal is to study functions $u(x,t) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$ satisfying

$$i\partial_t u + \frac{1}{2}\triangle u = \mu|u|^{p-1}u \qquad \text{and} \qquad u(x,0) = u_0(x), \tag{1.1}$$

where $\triangle$ is the Laplacian operator $\sum_{i=1}^d \partial_{x_i}^2$. This is the *nonlinear Schrödinger equation*, abbreviated NLS, with initial datum u_0. Here μ is a real constant; as we will see in section 4 the sign of μ has a large effect on global theory. We are most interested in the cases where p is an odd integer, and so the nonlinearity (right-hand side) is a smooth function of u and can be expressed as $\mu u^{(p+1)/2}\bar{u}^{(p-1)/2}$.

NLS comes from the theory of quantum mechanics, explaining the abundance of symmetries and conservation laws that we'll explore below. However, it is also interesting from a purely mathematical perspective. In the discussion that follows,

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we will try finding solutions to (1.1) given that u_0 lies in some Banach space. If the solutions found are guaranteed for only some time interval $[-T, T]$, they are called *local*, while if they exist for all times, the are called *global*. Other natural questions to ask are whether the solutions are unique and whether small changes in the initial datum produce small changes in the solution. Together, such results are known as *wellposedness theories* for an equation.

Section 2 discusses some symmetries and conservation laws of NLS. Section 3 presents the tools needed for the local existence results proved here. Section 4 deals with local wellposedness for fairly regular initial data. In Section 5 we obtain local wellposedness for much rougher data. Section 5 gives a brief summary of further results. The exposition here follows closely that in [4].

2. Symmetries and Conservation Laws

Before we proceed to a more technical discussion, it may be useful to look at some of the symmetries of the equation.

A symmetry is a transformation that, if applied to a solution of (1.1), produces another solution. The simplest examples are space and time translation:

$$u(x,t) \longmapsto u(x-y, t-s) \tag{2.1}$$

There is also a space rotation symmetry, a phase rotation symmetry, and a time reversal symmetry:

$$u(x,t) \longmapsto u(R_\theta x, t) \tag{2.2}$$

$$u(x,t) \longmapsto e^{i\theta} u(x,t) \tag{2.3}$$

$$u(x,t) \longmapsto \overline{u(x,-t)}. \tag{2.4}$$

The first of these follows since the Laplacian commutes with rotations, the second because the nonlinearity is radial, and the third by direct computation (with the i in the time derivative term playing a key role). Another vital symmetry is scaling; observe that for $\lambda > 0$

$$\begin{aligned} \triangle\left[u(\lambda x, \lambda^q t)\right] &= \lambda^2 (\triangle u)(\lambda x, \lambda^q t) \\ \partial_t\left[u(\lambda x, \lambda^q t)\right] &= \lambda^q (\partial_t u)(\lambda x, \lambda^q t). \end{aligned}$$

Therefore, setting $q = 2$ gives that

$$\begin{aligned} \lambda^2\left[i(\partial_t u)(\lambda x, \lambda^2 t) + 1/2(\triangle u)(\lambda x, \lambda^2 t)\right] &= (i\partial_t + 1/2\triangle)u(\lambda x, \lambda^2 t) \\ &= \mu |u(\lambda x, \lambda^2 t)|^{p-1} u(\lambda x, \lambda^2 t), \end{aligned}$$

so then $u(x,t) \longmapsto \lambda^s u(\lambda x, \lambda^2 t)$ is a symmetry if the power of λ on the left side of (3.1), $s + 2$, equals the power of λ on the right, sp. Solving, this gives a scaling symmetry for $s = 2/(p-1)$.

Symmetries have many uses in PDE theory. For example, there is a close connection between symmetries and conservation laws, which in turn can be used for uniqueness and regularity results as well as for proving global existence. One conserved quantity for NLS is mass (or total probability), which is given by

$$M(t) = \int_{\mathbb{R}^d} |u(x,t)|^2 dx.$$

The other important conservation law is for energy, given by

$$E(t) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla u(x,t)|^2 + \frac{2\mu}{p+1}|u(x,t)|^{p+1} dx.$$

The former is assotiated with phase rotation symmetry, while the latter is connected to time traslation symmetry. Other symmetries do produce additional conserved quantities, but these are harder to apply. See Section 6 for further discussion.

Also, the scaling symmetry gives a good indication of what kind of local existence and uniqueness result is available for a given exponent p and quality of initial datum. See Section 5 for a discussion of the heuristic.

3. Tools

A common technique to prove existence results in differential equations is iteratively applying some operator to a function and controlling how far it is from solving the equation at each step. Then if these functions converge to some other function in the same space, it will solve the equation, at least in some weakened sense. A classic example is Picard's theorem for ODE, which can be proved by the contraction mapping principle. In the setting of nonlinear PDE, it is harder to find the right space to perform the iteration in, the operator to use, and how to control the error at each step. The tools presented here will be assembled in the next section to perform such an iteration.

First, a contraction mapping principle tailored to nonlinear situations:

Proposition 3.1. **(Iteration Principle)** *Let $\mathcal{N}$ and $\mathcal{S}$ be Banach spaces, $D : \mathcal{N} \to \mathcal{S}$ a linear operator, and $N : \mathcal{S} \to \mathcal{N}$ a (not necessarily linear) operator. Furthermore, assume D is bounded with bound*

$$\|Df\|_{\mathcal{S}} \le C\|f\|_{\mathcal{N}} \tag{3.2}$$

for all f in $\mathcal{N}$, and that N has $N(0) = 0$ and satisfies

$$\|N(g) - N(g')\|_{\mathcal{N}} \le \frac{1}{2C}\|g - g'\|_{\mathcal{S}} \tag{3.3}$$

for all g, g' in $B_\epsilon = \{g \in \mathcal{S} : \|g\|_{\mathcal{S}} \le \epsilon\}$ for some $\epsilon > 0$. Then for all u^* in $B_{\epsilon/2}$, there is a unique solution u in B_ϵ to the equation

$$u = u^* + DN(u), \tag{3.4}$$

and if $v \in B_\epsilon$ solves $v = v^* + DN(v)$ for $v_* \in B_{\epsilon/2}$,

$$\|u - v\|_{\mathcal{S}} \le 2\|u^* - v^*\|_{\mathcal{S}}. \tag{3.5}$$

Proof. First, the uniqueness: assume $u' \in B_\epsilon$ is another solution of (3.4). Then

$$\|u - u'\|_{\mathcal{S}} = \|DN(u) - DN(u')\|_{\mathcal{S}} \le \frac{1}{2}\|u - u'\|_{\mathcal{S}},$$

which means we must have $u = u'$.

Now begin by iterating:

$$\begin{aligned} u_1 &= u^* \\ u_2 &= u^* + DN(u^*) \\ u_3 &= u^* + DN(u^* + DN(u^*)) \\ &\vdots \\ u_n &= u^* + DN(u_{n-1}). \end{aligned}$$

Note that, using assumptions (3.2) and (3.3),

$$\|u_2 - u_1\|_{\mathcal{S}} = \|DN(u^*)\|_{\mathcal{S}} \le \frac{1}{2}\|u^*\|_{\mathcal{S}} \le \frac{\epsilon}{4}$$

and

$$\|u_n - u_{n-1}\|_{\mathcal{S}} = \|u^* + DN(u_{n-1}) - u^* - DN(u_{n-2})\|_{\mathcal{S}} \le \frac{1}{2}\|u_{n-1} - u_{n-2}\|_{\mathcal{S}},$$

provided u_{n-1} and u_{n-2} are in B_ϵ. Induction on n gives

$$\|u_n - u_{n-1}\|_{\mathcal{S}} \le \frac{\|u^*\|_{\mathcal{S}}}{2^{n-1}} \le \frac{\epsilon}{2^n}$$

as well as the necessary fact that u_n is in B_ϵ. Then, using the triangle inequality, if $m < n$

$$\|u_n - u_m\|_{\mathcal{S}} \le \sum_{i=m}^{n-1} \|u_{i+1} - u_i\|_{\mathcal{S}} \le \sum_{i=m}^{n-1} \frac{\epsilon}{2^i} \le \frac{\epsilon}{2^{m-1}}.$$

This shows the sequence u_n is Cauchy, and since $\mathcal{S}$ is complete it has some limit u. Writing u as a telescoping sum gives the following bound (which then shows that it belongs to B_ϵ):

$$\|u\|_{\mathcal{S}} = \|u_1 + \sum_{n=1}^{\infty} u_{n+1} - u_n\|_{\mathcal{S}} \le \|u_1\|_{\mathcal{S}} + \sum_{n=1}^{\infty} \|u_{n+1} - u_n\|_{\mathcal{S}} \le \sum_{n=0}^{\infty} \frac{\|u^*\|_{\mathcal{S}}}{2^n} = 2\|u^*\|_{\mathcal{S}}.$$

To see that u solves (3.4), observe that DN is continuous on B_ϵ, so

$$u - DN(u) = \lim_{n\to\infty} u_n - DN(\lim_{n\to\infty} u_n) = \lim_{n\to\infty} [u_n - DN(u_n)] = \lim_{n\to\infty} u^* = u^*$$

where the limits are in the $\mathcal{S}$ topology.

Finally, let v solve (3.4) starting with $v^* \in B_{\epsilon/2}$. We know, using uniqueness, that $v \in B_\epsilon$ Then

$$\|u - v\|_{\mathcal{S}} \le \|u^* - v^*\|_{\mathcal{S}} + \|DN(u) - DN(v)\|_{\mathcal{S}} \le \|u^* - v^*\|_{\mathcal{S}} + \frac{1}{2}\|u - v\|_{\mathcal{S}}.$$

Doing this repeatedly gives

$$\|u - v\|_{\mathcal{S}} \le \sum_{n=0}^{\infty} \frac{1}{2^n}\|u^* - v^*\|_{\mathcal{S}} = 2\|u^* - v^*\|_{\mathcal{S}},$$

giving the estimate (3.5). □

Note that this isn't an 'unconditional' uniqueness: we had to assume that the second solution u' is small to make the bound (3.3) apply. When we apply the iteration principle to get local existence, we will want to upgrade the uniqueness. For this, a helpful tool is the continuity argument below. The idea is that if one statement is known to be true at one time, and if it provides information about nearby times, it can be extended to all times.

Proposition 3.6. (Continuity Principle) *Assume that for each time t in some interval $I \subset \mathbb{R}$, we have statements $H(t)$ and $C(t)$ that satisfy the following:*

(1) There is a time $t \in I$ such that $H(t)$ is true.
(2) If for some $t \in I$, $H(t)$ is true, then $C(t)$ is also true.
(3) If for some $t \in I$, $C(t)$ is true, then there is a neighborhood of t where H is true.
(4) The set of times for which C is true is closed, i.e. it contains all of its limit points.

Then $C(t)$ holds for all $t \in I$.

Proof. Let Ω be the set on which C holds. By (4), Ω is closed. If $t \in \Omega$, then by (3) there is a neighborhood V of t on which H holds. By (2), C also holds on V, so $V \subset \Omega$. Thus Ω is open. By (1) Ω is nonempty, so we must have $\Omega = I$. □

It remains to discuss what choices should be made for the spaces $\mathcal{S}$ and $\mathcal{N}$ and the operators between them. The spaces need to capture both decay and regularity properties of solutions, as well as interact well with both the linear and nonlinear parts of the equation. It is often necessary to have spaces that treat space and time differently; we will use the notation $X_t Y_x$ to refer to the normed space with the norm $\|(\|f\|_{Y(\mathbb{R}^d)})\|_{X(\mathbb{R})}$, where the inner norm is a function of time.

We will mostly use L^p spaces and *fractional Sobolev spaces* . The latter, denoted by $H^s(\mathbb{R}^d)$, are roughly speaking spaces of L^2 functions whose s^{th} derivatives are in L^2. Roughly, because L^2 functions are defined almost everywhere, making it harder to discuss differentiation, and also because s need not be an integer. A better definition is functions f in L^2 for which

$$\|f\|_{H^s} = \|(1 + |\xi|^2)^{s/2}\hat{f}(\xi)\|_{L^2}$$

is finite, where $\hat{f}(\xi)$ is the Fourier transform of f. The motivation is that this assures, by Plancharel's theorem, that $f \in L^2$ and also that $|\xi|^s\hat{f} \in L^2$. But using the principle that $\widehat{\partial_{x_i} f} = i\xi_i\hat{f}$, this can be intepreted as saying the "s^{th} derivative" of f is in L^2. It is easy to see from this definition that H^s is a Banach, and in fact a Hilbert, space under this norm. Less obvious is the Banach algebra property:

Proposition 3.7. (H^s is a Banach Algebra)

(1) If $s > 0$, the following Leibniz rule holds:

$$\|fg\|_{H^s} \le C\left(\|f\|_{H^s}\|g\|_{L^\infty} + \|g\|_{H^s}\|f\|_{L^\infty}\right)$$

for all $f, g \in H^s$ and some $C > 0$ that depends only on s and the dimension.

(2) If $s > d/2$, we also have

$$\|fg\|_{H^s} \le C\|f\|_{H^s}\|g\|_{H^s}$$

for some (different) $C = C(s, d)$.

Proof. We do not prove (1). For integer s, this follows from the weak derivative formulation of H^s. A more common approach is through Littlewood-Paley theory; see [4] for details.

For (2), it suffices to prove the Sobolev embedding $\|f\|_{L^\infty} \leq C\|f\|_{H^s}$, which is of independent interest. Perform the following estimate (for f, say, a Schwartz function):

$$\begin{aligned}
|f(x)| &= |\int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi i x\cdot\xi} d\xi| && \text{(Fourier inversion)}\\
&\leq \int_{\mathbb{R}^n} |\hat{f}(\xi)|(1+|\xi|^2)^{s/2}(1+|\xi|^2)^{-s/2} d\xi\\
&\leq \left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2(1+|\xi|^2)^{s} d\xi\right)^{1/2}\left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{-s} d\xi\right)^{1/2} && \text{(Cauchy-Schwarz)}\\
&= \|f\|_{H^s}\left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{-s} d\xi\right)^{1/2}.
\end{aligned}$$

Since $s > d/2$, the integral converges, so taking the supremum gives the embedding inequality for f Schwartz. But these are dense in H^s (see [1]), so the embedding extends to the entire space with the same bound. □

The operator N will be the nonlinearity operator $u \mapsto \mu|u|^{p-1}u$. To motivate the operator D, it helps to write the equation (1.1) in integral form.

Proposition 3.8. (Duhamel's Formula) *Let $u(x,t) : \mathbb{R}^d \times I \to \mathbb{C}$ be a solution of (1.1) that is Schwartz in space and C^1 in time. Then it satisfies the integral equation*

$$u(x,t) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi - 2\pi^2 i|\xi|^2 t}\widehat{u_0} d\xi - i\int_0^t \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi - 2\pi^2 i|\xi|^2(t-s)}\widehat{F(u)} d\xi ds \tag{3.9}$$

where $F(u) = \mu|u|^{p-1}u$ is the nonlinearity and the hats are space Fourier transforms.

Proof. Start with (1.1) and take the space Fourier transform, rearranging terms and pulling the time derivative out of the integral sign:

$$-i(\widehat{F(u)} + 2\pi^2|\xi|^2\hat{u}) = \widehat{\partial_t u} = \partial_t \hat{u}$$

This is an ordinary differential equation in time, and it's easy to check that, given initial data $\hat{u}(\xi, 0) = \widehat{u_0}(\xi)$, it has the solution

$$\hat{u}(\xi,t) = e^{-2\pi^2 i|\xi|^2 t}\widehat{u_0}(\xi) - ie^{-2\pi^2 i|\xi|^2 t}\int_0^t e^{2\pi^2 i|\xi|^2 s}\widehat{F(u)}(\xi,s) ds.$$

Now take the inverse space Fourier transform of both sides, using Fubini's theorem to rearrange the integrals. □

Note that we only needed u to have $C_t^1 C_x^2$ regularity and enough decay to have Fourier inversion and the differentiation rule ($u, F(u), \partial_t u, \partial_{x_i} u, \triangle u \in L_x^1$ and $\hat{u}, \widehat{F(u)} \in L_\xi^1$). The converse is true, provided u has that much regularity. Now if u^* is the first term in (3.9), $N(u) = F(u)$, and $D(f)$ is the second term, (3.9) is a form of equation (3.4).

There is an estimate on the nonlinearity that will prove useful in the following sections.

Lemma 3.10. *(1) The following is true for any $a, b \in \mathbb{C}$ and $p \geq 1$:*

$$|a|a|^{p-1} - b|b|^{p-1}| \leq C(p)|a-b|(|a|^{p-1} + |b|^{p-1}). \tag{3.11}$$

(2) If $p > 1$ is an odd integer and $u, v \in H^s(\mathbb{R}^d)$ for $s > d/2$, we have:

$$\|u|u|^{p-1} - v|v|^{p-1}\|_{H^s} \leq C(p)\|u - v\|_{H^s}(\|u\|_{H^s}^{p-1} + \|v\|_{H^s}^{p-1}). \tag{3.12}$$

Proof. We first prove (1). Note that if $|a| = |b|$, $|a| = 0$, or $|b| = 0$, this is obvious. Then assume without loss of generality that $|a| < |b|$ and note that dividing both sides of (3.11) by $b|b|^{p-1}$ makes it equivalent to

$$|1 - x|x|^{p-1}| \leq C(p)|1 - x|(1 + |x|^{p-1}), \tag{3.13}$$

where $x = a/b$ and so $|x| < 1$.

Next, assume p is an integer. Then write as a telescoping sum and apply the triangle inequality:

$$|1-x|x|^{p-1}| \leq |1-x|+|x-x|x||+|x|x|-x|x|^2|+\ldots+|x|x|^{p-2}-x|x|^{p-1}| \leq \sum_{n=0}^{p-1} |x|^n|1-x|,$$

using the reverse triangle inequality at the end. Since $|x|^n < 1$, this establishes (3.13) with $C(p) = p$.

If p is not an integer, we interpolate. Let $\lfloor p \rfloor < p < \lceil p \rceil$ be the the two integers neighboring p. Then observe that either $|1 - x|x|^{p-1}| \leq |1 - x|x|^{\lfloor p \rfloor - 1}|$ or $|1 - x|x|^{p-1}| \leq |1 - x|x|^{\lceil p \rceil - 1}|$ (or both): think about these quantities as distances between the point 1 and points on the line segment $\{tx : t \in [0, 1]\}$ in the complex plane. Then $|1 - x|x|^{p-1}|$ lies in between the other two points on the line, and so can't be further away than both of them from 1 (the distance function $t \mapsto |1 - tx|$ has no local maxima). But then applying the integer case proved above,

$$\begin{aligned}
|1 - x|x|^{p-1}| &\leq |1 - x|x|^{\lceil p \rceil - 1}| + |1 - x|x|^{\lfloor p \rfloor - 1}| \\
&\leq C(p)|1 - x|(1 + |x|^{\lceil p \rceil - 1} + |x|^{\lfloor p \rfloor - 1}) \\
&\leq C(p)|1 - x|(1 + |x|^{p-1} + 1) \\
&\leq 2C(p)|1 - x|(1 + |x|^{p-1}),
\end{aligned}$$

as desired, using the fact that $|x|^{\lfloor p \rfloor - 1} < 1$ (since we assumed $p > 1$) in the penultimate step. This completes the proof of (3.11).

In (3.12), there is a technical issue that the H^s norm of $|f|$ and f may be quite different; this prevents us from simply applying (3.11). The same argument can be modified slightly, however, to avoid dealing with absolute values. Note that, from the basic fact about the Fourier transform, $\|f\|_{H^s} = \|\bar{f}\|_{H^s}$. Using this, together with the expansion of $|u|^2 = u\bar{u}$ and the Banach algebra property 2.7(2), we get

$$\begin{aligned}\|u|u|^{p-1} - v|v|^{p-1}\|_{H^s} &= \|u^{(p+1)/2}\bar{u}^{(p-1)/2} - v^{(p+1)/2}\bar{v}^{(p-1)/2}\|_{H^s} \\ &\le \|u^{(p+1)/2}\bar{u}^{(p-1)/2} - u^{(p+1)/2}\bar{u}^{(p-3)/2}\bar{v}\|_{H^s} \\ &\quad + \|u^{(p+1)/2}\bar{u}^{(p-3)/2}\bar{v} - u^{(p+1)/2}\bar{u}^{(p-5)/2}\bar{v}^2\|_{H^s} \\ &\quad + \ldots + \|uv^{(p-1)/2}\bar{v}^{(p-1)/2} - v^{(p+1)/2}\bar{v}^{(p-1)/2}\|_{H^s} \\ &\le \|u-v\|_{H^s} \sum_{n=0}^{p-1} \|u\|_{H^s}^{p-1-n}\|v\|_{H^s}^{n} \\ &\le \|u-v\|_{H^s}\left[\|u\|_{H^s}^{p-1} + \sum_{n=1}^{p-2}\left(\frac{p-1}{p-1-n}\|u\|_{H^s}^{p-1} + \frac{p-1}{n}\|v\|_{H^s}^{p-1}\right) + \|v\|_{H^s}^{p-1}\right] \\ &\le C(p)\|u-v\|_{H^s}\left(\|u\|_{H^s}^{p-1} + \|v\|_{H^s}^{p-1}\right),\end{aligned}$$

where the penultimate step uses Young's inequality on all but the first and last terms of the sum. □

4. Local Wellposedness for Regular Data

Now we have the machinery to prove some wellposedness results. Rather than look for solutions to (1.1) directly, we instead look for solutions to the integral equation (3.9). There are several reasons for doing this: first, if a function solves (3.9) and lies in some space that guarantees sufficient regularity, it will automatically solve (1.1). Existence then becomes a two-part question, asking first whether some weakened solution exists and then, if it does, whether it has some additional regularity. Other times, there may be no pointwise solution to (1.1), but there is a solution to (3.9) that captures interesting behavior of a physical system, such as discontinuities or singularities. We are also interested in what can be said about u if very little regularity is assumed for u_0. In this case, solutions of (3.9) generally won't solve (1.1) in any useful way because they won't have enough derivatives for (1.1) to even make sense.

We begin with solutions for fairly regular data:

Theorem 4.1. *Let $s > d/2$, p be an odd positive integer, $\mu = \pm 1$, and assume u_0 is in H^s_x. Then we have:*

- *There exists a $u \in C^0_t H^s_x$ that solves (3.9) on some interval I centered around* 0
- *u is the unique function in $C^0_t H^s_x$ that solves (3.9)*
- *The size of the interval I depends only on the norm of u_0 (and d, p, and s), so for each R there is a $T = T(R, d, s, p)$ such that if $\|u_0\|_{H^s_x} \le R$, there is a unique $u \in C^0_t H^s_x(\mathbb{R}^d \times [-T, T])$ that solves (3.9)*
- *The solution u depends continuously on the data; in fact, the solution map $Q : u_0 \mapsto u$ satisfies*

$$\|Q(u_0) - Q(v_0)\|_{C^0_t H^s_x(\mathbb{R}^d\times[-T,T])} \le C\|u_0 - v_0\|_{\{u_0\in H^s_x : \|u_0\|_{H^s_x}\le R\}} \tag{4.2}$$

for some $C > 0$.

Proof. Fix $R > 0$. Let $\mathcal{S} = \mathcal{N} = C_t^0 H_x^s(\mathbb{R}^d \times [-T, T])$ for T which will be chosen shortly. Let $D : \mathcal{N} \to \mathcal{S}$ be

$$Df = -i \int_0^t \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi - 2\pi^2 i |\xi|^2 (t-s)} \hat{f}(\xi, s) d\xi ds.$$

Clearly D is linear. To check that it satisfies (3.2), compute the H_x^s norm:

$$\begin{aligned}
\|Df\|_{H_x^s}^2(t) &= \|(1+|\xi|^2)^{s/2}\widehat{Df}\|_{L_\xi^2}^2 \\
&= \int_{\mathbb{R}^d} \left(\int_0^t (1+|\xi|)^{s/2} e^{-2\pi^2 i |\xi|^2 (t-s)} \hat{f}(\xi, s) ds \right)^2 d\xi \\
&\leq t \int_{\mathbb{R}^d} \int_0^t (1+|\xi|)^s |e^{-2\pi^2 i |\xi|^2 (t-s)} \hat{f}(\xi, s)|^2 ds d\xi && \text{(Jensen)} \\
&= t \int_{\mathbb{R}^d} \int_0^t (1+|\xi|)^s |\hat{f}(\xi, s)|^2 ds d\xi && \text{(exponent is imaginary)} \\
&= t \int_0^t \|f\|_{H_x^s}^2(t) ds && \text{(Fubini)} \\
&\leq t^2 \|f\|_{C_t^0 H_x^s}^2. && \text{(sup-norm)}
\end{aligned}$$

Combining this with the fact that Df is automatically continuous in t (being an integral), we get

$$\|Df\|_{\mathcal{S}} \leq T \|f\|_{\mathcal{N}}.$$

Let u^* be given by the first term of Duhamel's formula:

$$u^*(x,t) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi - 2\pi^2 i |\xi|^2 t} \widehat{u_0} d\xi.$$

If we have $\|u_0\|_{H_x^s} \leq R$, then

$$\|u^*\|_{H_x^s}(t) = \left(\int_{\mathbb{R}^d} (1+|\xi|^2)^s |e^{-2\pi^2 i |\xi|^2 t} \widehat{u_0}(\xi)| d\xi^2 \right)^{1/2} = \|u_0\|_{H_x^s} \leq R, \tag{4.3}$$

since the exponential term has absolute value 1. Then $\|u^*\|_{\mathcal{S}} \leq R$ regardless of the choice of T.

Let $N : \mathcal{S} \to \mathcal{N}$ be given by $N(f) = \mu |f|^{p-1} v$. Clearly $N(0) = 0$. To check (3.3) with $\epsilon = 2R$, apply (3.12) (here using the assumptions $s > d/2$ and p odd):

$$\begin{aligned}
\|N(f) - N(g)\|_{\mathcal{N}} &= \sup_{t \in [-T,T]} \||f|^{p-1} f - |g|^{p-1} g\|_{H_x^s} \\
&\leq C(p) \sup_{t \in [-T,T]} \|f - g\|_{H^s} \left(\|f\|_{H^s}^{p-1} + \|g\|_{H_x^s}^{p-1} \right) \\
&\leq C(p) \|f - g\|_{\mathcal{S}} \left(\|f\|_{\mathcal{S}}^{p-1} + \|g\|_{\mathcal{S}}^{p-1} \right) \\
&\leq 2C(p)(2R)^{p-1} \|f - g\|_{\mathcal{S}}.
\end{aligned}$$

Let $T(R) = 1/(4C(p)(2R)^{p-1})$. Now apply the iteration principle (3.1) with $C = 2C(p)(2R)^{p-1}$, observing that the fixed-point formula (3.4) becomes (3.9), while combining the bound (3.5) and the equality (4.3) gives the continuity result (4.2).

The iteration principle automatically gives uniqueness provided both solutions are in $B_{2R} \subset S$, but we can improve that. Let u be the solution we constructed (so we have $\|u\|_s \leq 2R$) and u' be some other solution. Let $H(t)$ be the statement

$\|u'\|_{C_t^0 H_x^s(\mathbb{R}^d\times[-t,t])} \leq 2(R+\delta)$ and $C(t)$ the statement that $\|u'\|_{C_t^0 H_x^s(\mathbb{R}^d\times[-t,t])} \leq 2R$.

3.6(1) holds: at $t=0$ we have $u'=u_0$, which has H_x^s norm at most $R/2$. 3.6(2) follows for all times less than $T(R+\delta)$, since for those times

$$\|u'\|_{C_t^0 H_x^s(\mathbb{R}^d\times[-T(R+\delta),T(R+\delta)])} \leq 2(R+\delta),$$

so the automatic uniqueness result gives that $u'=u$ on $[-T(R+\delta),T(R+\delta)]$, and so in fact

$$\|u'\|_{C_t^0 H_x^s(\mathbb{R}^d\times[-T(R+\delta),T(R+\delta)])} = \|u\|_{C_t^0 H_x^s(\mathbb{R}^d\times[-T(R+\delta),T(R+\delta)])} \leq 2R.$$

3.6(3) and (4) are obvious from the fact u' is continuous in time.

Thus we have that $\|u'\|_{C_t^0 H_x^s(\mathbb{R}^d\times[-T(R+\delta),T(R+\delta)])} \leq 2R$. By making δ go to 0, the same holds for $T(R)$ instead of $T(R+\delta)$, and so the automatic uniqueness applies to give $u=u'$ on the entire interval for any u' in $\mathcal{S}$. □

Now consider all pairs (u,I) of intervals I containing 0 and solutions $u \in C_t^0 H_x^s(\mathbb{R}^d\times I)$ for a given initial datum u_0. Two such pairs (u_1,I_1) and (u_2,I_2) with $I_1 \subset I_2$ will have $u_1=u_2$ on I_1 by the uniqueness result above. Therefore, there is a unique maximal interval and unique solution on it. Now assume this interval contains an endpoint T. Then $\|u\|_{H_x^s}(T)$ is finite, so we can use Theorem 4.1 to find an interval around T with an $C_t^0 H_x^s$ solution. But this lets us extend the solution interval I, contradicting maximality. Thus the maximal interval is open, and if it's finite the solution's H_x^s norm must blow up at the endpoint. If the norm stays bounded, the solution can be uniquely and continuously extended.

The following result will help characterize what forms of blowup are allowed.

Theorem 4.4. (Persistence of Regularity) *Let u be a solution to (2.9) in $C_t^0 H_x^s(\mathbb{R}^d\times I)$ for a time interval I containing* 0*, $p>1$ an odd integer, and $s>0$. Then we have the estimate*

$$\|u\|_{C_t^0 H_x^s(\mathbb{R}^d\times I)} \leq \|u_0\|_{H_x^s} \exp(C\|u\|^{p-1}_{L_t^{p-1}L_x^\infty(\mathbb{R}^d\times I)}) \tag{4.5}$$

for some C that depends on d, s, and p. Conversely, if $s>d/2$ and $|I|<\infty$, u is in $L_t^{p-1}L_x^\infty(\mathbb{R}^d\times I)$.

Proof. Write $u = u^* + DN(u)$ using Duhamel's formula. From the computation (4.3), $\|u^*\|_{H_x^s}(t) = \|u_0\|_{H_x^s}$. Using Minkowski's integral inequality (see [2]),

$$\begin{aligned}\|DN(u)\|_{H_x^s} &= \| \int_0^t (1+|\xi|)^{s/2} e^{-2\pi^2 i|\xi|^2(t-s)} \widehat{F(u)} ds\|_{L_\xi^2} \\ &\leq \int_0^t \|(1+|\xi|)^{s/2}\widehat{F(u)}\|_{L_\xi^2} ds \\ &= \int_0^t \||u|^{p-1}u\|_{H_x^s} ds.\end{aligned}$$

Then using 3.7(1), $\||u|^{p-1}u\|_{H_x^s} \leq C\|u\|_{L_x^\infty}^{p-1}\|u\|_{H_x^s}$, giving the inequality

$$\|u\|_{H_x^s}(t) \leq \|u_0\|_{H_x^s} + C\int_0^t \|u\|_{L_x^\infty}^{p-1}\|u\|_{H_x^s} ds$$

Applying Gronwall's inequality (See [1] or [4]) then gives the conclusion.

For the converse, note that the Sobolev embedding (see proof of 3.7(2)) gives that $\|u\|_{L_x^\infty} \leq C\|u\|_{H_x^s}$, and then $\|u\|_{L_t^{p-1}L_x^\infty(\mathbb{R}^d\times I)} \leq |I|^{\frac{1}{p-1}}\|u\|_{C_t^0 L_x^\infty(\mathbb{R}^d\times I)} < \infty$. □

Now take two regularities $s > s' > d/2$ and assume $u_0 \in H^s_x$. Then the solution can be extended in H^s_x and $H^{s'}_x$ for exactly the same distance. For assume the H^s_x norm blows up at some point T. Then by (4.5), the $L^{p-1}_t L^\infty_x(\mathbb{R}^d \times [0,T])$ norm of u is infinite. But if $\|u\|_{C^0_t H^{s'}_x(\mathbb{R}^n \times [0,T])}$ had been finite, by the second part of Theorem 4.4 the $L^{p-1}_t L^\infty_x(\mathbb{R}^d \times [0,T])$ norm of u would also be finite; this is a contradiction. The idea behind persistence of regularity is that, provided u is an H^s solution to the integral equation, it is controlled by the $L^{p-1}_t L^\infty_x(\mathbb{R}^d \times [0,T])$ norm, which doesn't depend on s, and can be extended for exactly as long as that norm stays finite.

5. Local Wellposedness for Rough Data

The theory above is very effective when the initial data is regular. However, when the regularity of u_0 is $s \leq d/2$, this argument fails. These spaces are not Banach algebras, and dealing with the nonlinearity in the H^s norm becomes impossible. The H^s norms, however, don't capture the more subtle effects in the Duhamel formula. It turns out that using spaces that involve integration in time gives additional control over the solution. The tool needed comes from harmonic analysis.

Proposition 5.1. (Strichartz Estimates) *A pair of exponents (q,r) is called* admissible *if $2 \leq q, r \leq \infty$, $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, and q is not 2 when $d = 2$. Then, for any admissible (q,r) and $(\tilde{q},\tilde{r})$ we have*

$$\| \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi - 2\pi^2 i|\xi|^2 t} \widehat{u_0} d\xi \|_{L^q_t L^r_x} \leq C(d,q,r)\|u_0\|_{L^2_x} \tag{5.2}$$

and

$$\| \int_0^t \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi - 2\pi^2 i|\xi|^2 (t-s)} \widehat{F} d\xi ds \|_{L^q_t L^r_x} \leq C(d,q,r,\tilde{q},\tilde{r})\|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}, \tag{5.3}$$

where $\tilde{q}'$ and $\tilde{r}'$ are the dual exponents for $\tilde{q}$ and $\tilde{r}$.

The proof can be approached either using varients of the Fourier restriction problem or a "'localization"' lemma for certain integral operators. For the former, see [3]; for the latter, a sketch and further references can be found in [4].

These powerful estimates allow us to give wellposedness results even if the initial datum is assumed to be only in L^2_x.

Theorem 5.4. *Let $1 < p < 1 + \frac{4}{d}$, $\mu = \pm 1$, and assume u_0 is in L^2_x. Then we have:*

- *There exists a $u \in C^0_t L^2_x$ that solves (2.9) on some interval I centered around 0*
- *u is the unique function in some space $\mathcal{S} \subset C^0_t L^2_x$, to be determined below, that solves (3.9)*
- *The size of the interval I depends only on the norm of u_0 (and d, p, and s), so for each R there is a $T = T(R,d,s,p)$ such that if $\|u_0\|_{L^2_x} \leq R$, there is a unique $u \in \mathcal{S} \subset C^0_t L^2_x(\mathbb{R}^d \times [-T,T])$ that solves (2.9)*
- *The solution u depends continuously on the data; the solution map Q : $u_0 \mapsto u$ satisfies*

$$\|Q(u_0) - Q(v_0)\|_{\mathcal{S}} \leq C\|u_0 - v_0\|_{\{u_0 \in L^2_x : \|u_0\|_{L^2_x} \leq R\}} \tag{5.5}$$

for some $C > 0$.

Note that this only holds for $1 < p < 1 + \frac{4}{d}$. A different kind of wellposedness is possible for $p = 1 + \frac{4}{d}$ and is given in Theorem 5.8. We leave the significance of the value $1 + \frac{4}{d}$ until after the proofs.

Proof. The idea is to let $\mathcal{S}$ be the space governed by all the Strichartz norms:

$$\|u\|_{\mathcal{S}(\mathbb{R}^d \times I)} = \sup_{(q,r) \text{ admissible}} \|u\|_{L^q_t L^r_x(\mathbb{R}^d \times I)}$$

Then the dual space of $\mathcal{S}$, which will be called $\mathcal{N}$, defined by

$$\|f\|_{\mathcal{N}(\mathbb{R}^d \times I)} = \sup_{g \in \mathcal{S}, \|g\|_s < 1} \int_{\mathbb{R}^d \times I} fg dx dt \le \|f\|_{L^{q'}_t L^{r'}_x(\mathbb{R}^d \times I)}$$

for any (q,r) admissible, using Hölder's inequality. Then writing the solution in the usual Duhamel form $u = u^* + DN(u)$, we want to apply (5.2) and (5.3) to bound the $\mathcal{S}$ norm of u^* and $ND(u)$, respectively. However, that requires the constants in those inequalities to depend only on dimension, not on exponent pair. When $d \neq 2$, the interval over which q ranges is compact (in the extended reals), and taking the supremum over q generates such a constant. If $d = 2$, unfortunately the endpoint $(2, \infty)$ is not admissible, so we truncate the definition of $\mathcal{S}$:

$$\|u\|_{\mathcal{S}(\mathbb{R}^2 \times I)} = \sup_{(q,r) \text{ admissible, } q \ge 2+\eta} \|u\|_{L^q_t L^r_x(\mathbb{R}^2 \times I)},$$

with $\eta > 0$ to be chosen below.

Now for the iteration argument. Assume $\|u_0\|_{L^2} \le R$. Then

$$\|u^*\|_{\mathcal{S}(\mathbb{R}^d \times I)} \le C(d)\|u_0\|_{L^2_x} \le 2C(d)R$$

using (5.2) with the unified constant. Note this doesn't depend on the interval $[-T, T]$, which we will choose later. Thus in the iteration fix $\epsilon \ge 2C(d)R$.

Next,

$$\|Df\|_{\mathcal{S}(\mathbb{R}^d \times I)} \le C(d)\|f\|_{\mathcal{N}(\mathbb{R}^d \times I)} \tag{5.6}$$

this time using (5.3) with unified constant and the definition of the $\mathcal{N}$ norm.

For the N operator, take $u, v \in B_\epsilon \subset \mathcal{S}(\mathbb{R}^d \times I)$. Then take the exponent pair $(q,r) = (\frac{4(p+1)}{d(p-1)}, p+1)$. We have, for $1 < d < 1 + \frac{4}{d}$, that $2 < r = p+1$ and $p + 1 < q < \infty$. Also,

$$\frac{2}{q} + \frac{d}{r} = \frac{d(p-1)}{2(p+1)} + \frac{d}{p+1} = \frac{d(p+1)}{2(p+1)} = \frac{d}{2},$$

meaning (q,r) is admissible (if $d = 2$, choose η at this point to guarantee $q > 2+\eta$).

Note that $r' = r/(r-1) = r/p$. Now estimate the $L^{r'}_x$ norm of $|N(u) - N(v)|$:

$$\begin{aligned}
\|N(u) - N(v)\|_{L^{r'}_x(\mathbb{R}^d)} &\le C\||u-v|(|u|^{p-1} + |v|^{p-1})\|_{L^{r'}_x(\mathbb{R}^d)} && \text{(using (3.11))} \\
&\le C\|u-v\|_{L^{pr'}_x(\mathbb{R}^d)} \||u|^{p-1} + |v|^{p-1}\|_{L^{pr'/(p-1)}_x(\mathbb{R}^d)} && \text{(Hölder)} \\
&\le C\|u-v\|_{L^r_x(\mathbb{R}^d)} \left(\||u|^{p-1}\|_{L^{r/(p-1)}_x(\mathbb{R}^d)} + \||v|^{p-1}\|_{L^{r/(p-1)}_x(\mathbb{R}^d)}\right) && \text{(Minkowski)} \\
&= C\|u-v\|_{L^r_x(\mathbb{R}^d)} \left(\|u\|^{p-1}_{L^r_x(\mathbb{R}^d)} + \|v\|^{p-1}_{L^r_x(\mathbb{R}^d)}\right)
\end{aligned}$$

Now note that the $\mathcal{N}$ norm is controlled by the $L_t^{q'}L_x^{r'}$ norm, allowing:

$$\begin{aligned}
\|N(u)-N(v)\|_{\mathcal{N}(\mathbb{R}^d\times I)} &\le \|N(u)-N(v)\|_{L_t^{q'}L_x^{r'}(\mathbb{R}^d\times I)} \\
&\le C\|\|u-v\|_{L_x^r(\mathbb{R}^d)}\left(\|u\|^{p-1}_{L_x^r(\mathbb{R}^d)}+\|v\|^{p-1}_{L_x^r(\mathbb{R}^d)}\right)\|_{L_t^{q'}(I)} \\
&\le C\left(\|\|u-v\|_{L_x^r(\mathbb{R}^d)}\|u\|^{p-1}_{L_x^r(\mathbb{R}^d)}\|_{L_t^{q'}(I)}+\|\|u-v\|_{L_x^r(\mathbb{R}^d)}\|v\|^{p-1}_{L_x^r(\mathbb{R}^d)}\|_{L_t^{q'}(I)}\right) \\
&\le C|I|^{1-pq'/q}\left(\|\|u-v\|_{L_x^r(\mathbb{R}^d)}\|u\|^{p-1}_{L_x^r(\mathbb{R}^d)}\|_{L_t^{q/p}(I)}+\|\|u-v\|_{L_x^r(\mathbb{R}^d)}\|v\|^{p-1}_{L_x^r(\mathbb{R}^d)}\|_{L_t^{q/p}(I)}\right) \\
&\le C|I|^{1-pq'/q}\left(\|u-v\|_{L_t^qL_x^r(\mathbb{R}^d\times I)}\|u\|^{p-1}_{L_t^qL_x^r(\mathbb{R}^d\times I)}+\|u-v\|_{L_t^qL_x^r(\mathbb{R}^d\times I)}\|v\|^{p-1}_{L_t^qL_x^r(\mathbb{R}^d\times I)}\right) \\
&\le 2C|I|^{1-pq'/q}\epsilon^{p-1}\cdot\|u-v\|_{L_t^qL_x^r(\mathbb{R}^d\times I)} \\
&\le 2C|I|^{1-pq'/q}\epsilon^{p-1}\cdot\|u-v\|_{\mathcal{S}(\mathbb{R}^d\times I)}.
\end{aligned}$$

The second line uses the above estimate on the $L_x^{r'}$ norm, the third Minkowski's inequality. The fourth line is a Hölder inequality in time, writing each function as 1 times itself and giving the function the exponent $q/q'p$ and the 1 an exponent of $1/(1-pq'/q)$. The fifth line is another Hölder inequality in time, this time with the two functions being the two norms and the exponents being p and $p/(p-1)$. The sixth line uses that $u, v \in B_\epsilon$, while the last the definition of $\mathcal{S}$.

Now note that $pq'/q = p/(q-1) < p/(r-1) = 1$, so the exponent on the factor of $|I|$ is positive. Then by choosing a sufficiently small time T, we get

$$\|N(u)-N(v)\|_{\mathcal{N}(\mathbb{R}^d\times[-T,T])} < \frac{1}{2C(d)}\|u-v\|_{\mathcal{S}(\mathbb{R}^d\times[-T,T])}. \tag{5.7}$$

Thus conditions (3.2) and (3.3) are satisfied, and the iteration principle applies to give a unique solution in $\mathcal{S}$ to (3.9), with the conclusion (3.5) giving the desired continuity (5.5). The uniqueness can be upgraded to the entire space $\mathcal{S}$ using a continuity argument identical to the one in the proof of Theorem 4.1. □

The key point in the proof of Theorem 5.4 that in the range of $1 < p < 1 + 4/d$, known as the subcritical region, the Strichartz estimates give improved control over the nonlinearity. In the supercritical range of $p > 1 + 4/d$, local existence is more complicated and may fail. In the critical case, it is possible to have existence, but the control over duration comes from the shape as well as size of the initial datum.

Theorem 5.8. *Let $p = 1 + \frac{4}{d}$, $\mu = \pm 1$. Then for ever $R > 0$, there exists a $\gamma > 0$ such that if u_1 is in L_x^2 with $\|u_1\|_{L^2} \le R$, and I is chosen so that*

$$\|\int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi - 2\pi^2 i|\xi|^2 t}\widehat{u_1}d\xi\|_{L_t^{1+p}L_x^{1+p}(\mathbb{R}^d\times I)} \le \gamma, \tag{5.9}$$

then for all u_0 with $\|u_1 - u_0\|_{L_x^2(\mathbb{R}^d)} \le \gamma$ there is a solution to (2.9), u, which is unique in the Strichartz space $\mathcal{S}$ as defined in the proof of (3.9). The solution map Q satisfies

$$\|Q(u_0)-Q(v_0)\|_{\mathcal{S}} \le C\|u_0-v_0\|_{\{u_0\in L_x^2:\|u_0-u_1\|_{L_x^2}\le\gamma\}} \tag{5.10}$$

for some $C > 0$.

First, we check to make sure the statement makes sense.

Lemma 5.11. *There exists an interval I such that (5.9) holds.*

Proof. Since $\|u_1\|_{L^2}$ is finite and the the pair $(p+1, p+1)$ is admissible, (5.2) guarantees that

$$\| \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi - 2\pi^2 i|\xi|^2 t}\widehat{u_1} d\xi\|_{L_t^{1+p} L_x^{1+p}(\mathbb{R}^d\times\mathbb{R})} \leq C\|u_1\|_{L_x^2}$$

is also finite. Let $I_n = [-\frac{1}{n}, \frac{1}{n}]$. Then

$$g_n(x,t) \equiv \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi - 2\pi^2 i|\xi|^2 t}\widehat{u_1} d\xi 1_{I_n}(t)$$

converge to 0 almost everywhere in $\mathbb{R}^d\times\mathbb{R}$. They are dominated in $L_t^{p+1}L_x^{p+1}$ by

$$\left|\int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi - 2\pi^2 i|\xi|^2 t}\widehat{u_1} d\xi\right|,$$

so by the dominated convergence theorem they converge to 0 in $L_t^{p+1}L_x^{p+1}$ and we can find g_n with $\|g_n\|_{L_t^{p+1}L_x^{p+1}} < \gamma$, as desired. □

Proof of Theorem 5.8. We recycle elements of the proof of (5.4). The new iteration will use the same space $\mathcal{N}$ and operators D and N. Leave $\mathcal{S}$ the Strichartz space; the iteration, however, will use the space $\mathcal{S}'$, which is equipped with the norm

$$\delta\|u\|_{\mathcal{S}(\mathbb{R}^d\times I)} + \|u\|_{L_t^{p+1}L_x^{p+1}(\mathbb{R}^d\times I)}, \tag{5.12}$$

with γ and δ to be picked later.

Write $u = u_0^* + DN(u)$ for the integral equation (2.9) with initial data u_0 with $\|u_0 - u_1\|_{L_x^2} \leq \gamma$ (Note: we changed notation slightly here; $*$ should now be thought of as an operator acting on initial data). Then

$$\begin{aligned}
\|u_0^*\|_{\mathcal{S}'(\mathbb{R}^d\times I)} &\leq \delta\|u_0^*\|_{\mathcal{S}(\mathbb{R}^d\times I)} + \|u_0^*\|_{L_t^{p+1}L_x^{p+1}(\mathbb{R}^d\times I)} \\
&\leq \delta C(d)\|u_0\|_{L_x^2} + \|(u_0-u_1)^*\|_{L_t^{p+1}L_x^{p+1}} + \|u_1^*\|_{L_t^{p+1}L_x^{p+1}} \\
&\leq \delta C(d)(R+\gamma) + \|u_0-u_1\|_{L_x^2} + \gamma \\
&\leq \delta C(d)(R+\gamma) + 2\gamma \equiv \epsilon/2,
\end{aligned}$$

where the first line follows form the definition of $\mathcal{S}'$; the second uses Strichartz estimate (5.2) and the triangle inequality; and the third uses (5.2) again, the bound (5.9), and the estimates on the L^2 norms of u_1 and $u_0 - u_1$. Then ϵ will be the radius of the ball over which we iterate.

To get (3.2), we need to do two separate estimates: one on the Strichartz norm in $\mathcal{S}$ and another on the norm in $L_x^{p+1}L_t^{p+1}$. For the first, copy the proof of (5.6) to get

$$\|Df\|_{\mathcal{S}(\mathbb{R}^d\times I)} \leq C(d)\|f\|_{\mathcal{N}(\mathbb{R}^d\times I)}.$$

For the second, use the same thing, observing that $L_x^{p+1}L_t^{p+1}\subset\mathcal{S}$. Then

$$\|Df\|_{\mathcal{S}'(\mathbb{R}^d\times I)} \leq C(d)(1+\delta)\|f\|_{\mathcal{N}(\mathbb{R}^d\times I)}.$$

To get (3.3), we first observe that the computation from the proof of (5.4) carries over to give

$$\|N(u)-N(v)\|_{\mathcal{N}(\mathbb{R}^d\times I)} \leq C|I|^{1-pq'/q}\cdot\|u-v\|_{L_x^{p+1}L_t^{p+1}(\mathbb{R}^d\times I)}(\|u\|^{p-1}_{L_t^{p+1}L_x^{p+1}}+\|v\|^{p-1}_{L_t^{p+1}L_x^{p+1}})$$

where u, v are in B_ϵ. Note that unfortunately $pq'/q = 1$ in the critical case, which makes it impossible to make the $\mathcal{N}$ norm small this way. Noting the L^{p+1} norms on the right are controlled by the $\mathcal{S}'$ norm, we have:

$$\|N(u) - N(v)\|_{\mathcal{N}} \leq C\|u - v\|_{L_x^{p+1}L_t^{p+1}}(2\epsilon^{p-1}) \leq 2C\epsilon^{p-1}\|u - v\|_{\mathcal{S}'}$$

Now observe that ϵ shrinks as γ and δ are made small, so by fixing them sufficiently small we get

$$\|N(u) - N(v)\|_{\mathcal{N}} \leq \frac{1}{4C(d)}\|u - v\|_{\mathcal{S}'},$$

enough to make the iteration work. Then, as usual, (3.4) gives a unique solution in S', and so in S, as desired and (3.5) becomes the bound (5.10). The continuity argument once again upgrades uniqueness to all of S', and so S (S being contained in S', albeit with a different norm). □

This argument can be streamlined if the solutions are only required to be in $L_t^{p+1}L_x^{p+1}$, since the Strichartz component of the $\mathcal{S}'$ wasn't needed to make the iteration work, but the above gives additional information: namely, the solution belongs to S, is unique there, and the bound (5.10) holds.

Unlike in the subcritical regime, where scaling the time interval shrank the $\mathcal{N}$ norm of the nonlinearity, in the critical case all the estimates are scale-invariant. Indeed, looking at the scaling symmetry from Section 2 gives a heuristic for behavior of a solution. Let u be a solution and $u_\lambda(x, t) = \lambda^{2/(p-1)}u(\lambda x, \lambda^2 t)$, which is also a solution (from Section 2). Then

$$\begin{aligned}\|u_\lambda\|_{L_x^2}(t) &= \lambda^{2/(p-1)}\left(\int_{\mathbb{R}^d}|u(\lambda x, \lambda^2 t)|^2 dx\right)^{1/2}\\ &= \lambda^{2/(p-1)-d/2}\left(\int_{\mathbb{R}^d}|u(y, \lambda^2 t)|^2 dy\right)^{1/2}\\ &= \lambda^{2/(p-1)-d/2}\|u\|_{L_x^2}(\lambda^2 t).\end{aligned}$$

The L_x^2 norm, then, is unaffected by scaling when $p = 1 + \frac{4}{d}$. For smaller p, contracting time reduces the size of the norm; this is the effect that was exploited in the proof of Theorem 5.4. For larger p, contracting time increases solution norm. This makes that region very difficult to work with.

More generally, this can be applied to data in other spaces (for example, H^1 or other Sobolev spaces). For a fixed regularity, if the norm of the solution is scale-invariant, i.e., it doesn't change under the scaling symmetry, we are in the critical case and can expect the kind of result obtained in Theorem 5.8. If contracting time makes the norm smaller, we are in the subcritical case and can expect a result along the lines of Theorem 5.4. If contracting time makes the norm larger, we are in the supercritical case and shouldn't expect this kind of wellposedness.

6. Further Results

Local wellposedness results along the lines of Theorems 5.4 and 5.8 can be reproduced for regularities H^s with s between the classical range $s > d/2$ and the $s = 0$ case dealt with above, provided p is subcritical or critical with respect to that regularity. It is not known whether the uniqueness result can be extended to $C_t^0L_x^2$ (we only proved uniqueness in the Strichartz space $\mathcal{S}$). For p larger than the critical (scale-invariant) value, local wellposedness can be shown to fail.

Global wellposedness results usually require coupling a conservation law with a local result. For example, to arrive at global wellposedness for classical solutions, one approach is to use a conservation law to bound the $L_t^{p-1}L_x^\infty$ and then use the discussion after the persistence of regularity result to extend the solution. NLS, at least in the (1.1) form with u smooth, enjoys the conservation laws stated in Section 2. The problem is that they aren't immediately obvious from the Duhamel formula (3.9), and it takes some additional regularity to make them work.

The L^2 solutions have enough regularity to justify the mass conservation law, which can then be combined with Theorem 5.4 to produce global wellposedness for subcritical exponents. It fails to work with Theorem 5.8, though, and so, for example, two-dimensional L^2 critical (so $p = 3$) global wellposedness is an open problem.

At the $H^{1/2}$ regularity, momentum conservation is justified, but is actually difficult to apply. At the H^1 regularity, energy conservation allows a much more complete critical wellposedness theory when the constant $\mu = +1$. This is known as the defocusing case, and has the advantage that the energy is always positive. The focusing case, where $\mu = -1$, can exhibit more complicated behavior for critical exponents and is being intensively studied by experts in the field.

For further reference see [4].

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