UNIFORMLY DISCONTINUOUS GROUPS OF ISOMETRIES OF THE PLANE

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ABSTRACT. This paper discusses 2-dimensional locally Euclidean geometries and how these geometries can describe musical chords.

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1. LOCALLY EUCLIDEAN GEOMETRIES

Definition 1.1. A *metric space* is a set X and a function $d : X \times X \longrightarrow X$, called the *metric* of X such that for all elements $x, y, z \in X$, the following properties are satisfied:

a) d is positive definite: $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$
b) d is symmetric: $d(x, y) = d(y, x)$
c) d satisfies the triangle inequality: $d(x, y) \leq d(x, z) + d(z, y)$

If $X = \mathbb{R}^2$ and d is the Euclidean distance function, that is,

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

then the metric space we get is the 2-dimensional Euclidean plane. We may have metrics on $\mathbb{R}^2$ other than the Euclidean metric. Another example of a metric on $\mathbb{R}^2$ is the taxicab metric: $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$.

Definition 1.2. Let (X, d_X) and (Y, d_Y) be metric spaces. An *isometry* between X and Y is a function $f : X \longrightarrow Y$ such that $d_X(x, y) = d_Y(f(x), f(y))$ for all $x, y \in X$. When $Y = X$, we call f an isometry of X.

Example 1.3. Isometries of the Euclidean plane

a) Translation by a vector $v = (v_1, v_2)$: $f(x_1, x_2) = (x_1 + v_1, x_2 + v_2)$
b) Rotation about the origin by an angle θ counter clockwise is given by multiplying the coordinates by a matrix of the following form

$$f(x_1, x_2) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

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c) Reflection across the x_1-axis: $f(x_1, x_2) = (x_1, -x_2)$.
d) A glide reflection is the composition of a translation and a reflection across the translation axis. For example, fix $a \in \mathbb{R}$, then $f(x_1, x_2) = (x_1 + a, -x_2)$ translates points horizontally by a and then reflects the points about the x_1-axis.

Definition 1.4. A metric space (X, d) is a *2-dimensional locally Euclidean geometry* if there exists a number $r > 0$ such that for any $x \in X$, the disc centered at x with radius r, $D(x, r) := \{w \in X |\, d(w, x) \leq r\}$, can be mapped by an isometry to a disc of radius r on the Euclidean plane.

One can think that X is identical with the plane in sufficiently small regions.

Example 1.5. The Euclidean plane is of course a locally Euclidean geometry.

We will construct locally Euclidean geometries in the next section as quotients of the plane by a group of isometries.

2. Uniformly Discontinuous Groups of Isometries of the Plane

Definition 2.1. A *binary operation* $*$ on X is a function $f\colon X \times X \longrightarrow X$. We will denote $f(x, y)$ as $x * y$.

Definition 2.2. A *group* is a set Γ that is closed under a binary operation $*$ such that

a) For all elements $a, b, c \in \Gamma$, $*$ is associative: $(a * b) * c = a * (b * c)$.
b) There exists an element $e \in \Gamma$ such that for all $x \in \Gamma$, we have $e * x = x * e = x$. We say that e is the *identity element*.
c) For all elements $a \in \Gamma$, there exists an element $a' \in \Gamma$ such that $a * a' = a' * a = e$. We say that a' is the *inverse* of a.

Example 2.3. Some examples of groups are

a) The set of integers $\mathbb{Z}$ with addition as the binary operation.
b) The set of permutations of n objects S_n, with binary operation composition of functions.

Definition 2.4. Let X be a metric space. A non-empty set Γ is a *group of isometries* of X if Γ is a set of isometries that is also a group under the operation of composition. That is,

a) The composition of two isometries in Γ is still an isometry in Γ
b) For any isometry $f \in \Gamma$, there exists $f^{-1} \in \Gamma$ such that f^{-1} is the inverse of f.

Definition 2.5. A group of isometries of the plane is called *uniformly discontinuous* if there exists a positive k such that if f is an isometry in Γ and x is any point in the plane such that $x \neq f(x)$, then $d(x, f(x)) \geq k$.

Example 2.6. Rotations do not belong to uniformly discontinuous groups because a point that is closer to the fixed point of rotation will need a smaller k than a point farther from the point of rotation. In this case we cannot find a uniform k for all elements x in the plane.

Example 2.7. Uniformly discontinuous groups of isometries

(1) Let $T_{(v_1,v_2)}$ be the translation of the Euclidean plane by the vector (v_1, v_2). Then $\Gamma = \{T_{(n,0)} | n \in \mathbb{Z}\}$ is a uniformly discontinuous group of isometries.
(2) The group $\Gamma = \{T_{(a,b)} | a, b \in \mathbb{Z}\}$ containing the isometries that are obtained by horizontal translation by a multiple of a and vertical translation by a mutliple of b is a uniformly discontinuous group of isometries.

Definition 2.8. An *equivalence relation* $\sim$ on a set X is a binary relation that satisfies the following properties:

a) Reflexive: $x \sim x$ for all $x \in X$,
b) Symmetric: if $x \sim y$ then $y \sim x$,
c) Transitive: if $x \sim y$ and $y \sim z$, then $x \sim z$

Definition 2.9. An equivalence class $[x]$ on $x \in X$ is the set of elements $y \in X$ such that $y \sim x$: $[x] = \{y \in X : y \sim x\}$

Using the notions of equivalence classes, we may begin to explore set theoretic quotient spaces of uniformly discontinuous groups of isometries of the plane.

Definition 2.10. Let X be a space with an equivalence relation $\sim$. The set theoretic quotient $X/\sim$ is the space of all equivalence classes of X.

We see that all points in the same equivalence class of X correspond to the same point in $X/\sim$. One can think of $X/\sim$ as the space we get by identifying all points in the same equivalence class.

For a uniformly discontinuous group of isometries Γ, we define a binary relation $\sim$ as $x \sim y$ if and only if $y = f(x)$, for some $f \in \Gamma$. We see that this is an equivalence relation since

a) $x \sim x$ since the identity map is in Γ
b) If $x \sim y$, then $y = f(x)$ for some $f \in \Gamma$. Since Γ is a group, $f^{-1} \in \Gamma$ and we have $f^{-1}(y) = x$, so $y \sim x$. That is, $\sim$ is symmetric.
c) If $x \sim y$ and $y \sim z$, that is, $f(x) = y$ and $g(y) = z$ for some $f, g \in \Gamma$, then $h := f \circ g \in \Gamma$ and $h(x) = z$ for, so $x \sim z$.

We denote $\mathbb{R}^2/\sim$ as $\mathbb{R}^2/\Gamma$. We are going to show that $\mathbb{R}^2/\Gamma$ is a locally Euclidean geometry.

Example 2.11. From Example 2.7,

(1) We identify points $x, y \in \mathbb{R}^2$ such that $f(x) = y$ for $f \in \Gamma = \{T_{(n,0)} | n \in \mathbb{Z}\}$. We obtain an infinite cylinder.
(2) By identifying equivalent points under $\Gamma = \{T_{(a,b)} | a, b \in \mathbb{Z}\}$, we obtain a torus, $\mathbb{R}^2/\Gamma$.

Now that we have defined set theoretic quotient spaces of the Euclidean plane by uniformly discontinuous groups of isometries, we are now going to define a way to measure distances between points in these spaces.

Let Γ be a uniformly discontinuous group of isometries and let $M = \mathbb{R}^2/\Gamma$. Let $\mathbf{A}$ and $\mathbf{B}$ be two points in M. For two points a and a' in $\mathbb{R}^2$ that belong to the same equivalance class $\mathbf{A}$, i.e. $[a] = [a'] = \mathbf{A}$ and a point $b \in \mathbb{R}^2$ such that $[b] = \mathbf{B}$, we may find that although a and a' become the same point, we have $d(a, b) \neq d(b, a')$. To rectify this problem, we will say that under the action of a uniformly discontinuous group, $d(\mathbf{A}, \mathbf{B})$ is the shortest distance between any of the points that are equivalent to $\mathbf{A}$ and $\mathbf{B}$. That is, for equivalent classes $\mathbf{A}$ and $\mathbf{B}$

$$d(\mathbf{A}, \mathbf{B}) = \inf_{a \in \mathbf{A}, b \in \mathbf{B}} d(a, b).$$

Fix a point $a_0 \in \mathbf{A}$. We claim that

$$d(\mathbf{A}, \mathbf{B}) = \inf_{b \in \mathbf{B}} d(a_0, b).$$

Clearly $\inf_{b\in \mathbf{B}} d(a_0, b) \geq \inf_{a\in\mathbf{A}, b\in\mathbf{B}} d(a, b)$. Now we need to show that the reverse inequality holds. To see this, pick sequence $a_n \in \mathbf{A}$ and $b_n \in \mathbf{B}$ such that $d(a_n, b_n) \to d(\mathbf{A}, \mathbf{B})$. For each n, there is an isometry $f_n \in \Gamma$ that takes a_n to a_0, that is, $f_n(a_n) = a_0$. Then $d(a_n, b_n) = d(f_n(a_n), f_n(b_n))$ since each f_n is an isometry and thus preserves distance. Let $b'_n = f_n(b_n) \in \mathbf{B}$. So $b'_n \in \mathbf{B}$ and $d(a_0, b'_n) = d(a_n, b_n)$. Since $d(a_n, b_n) \to d(\mathbf{A}, \mathbf{B})$, it follows that $\inf_{b\in\mathbf{B}} d(a_0, b) \leq d(\mathbf{A}, \mathbf{B})$. So $d(\mathbf{A}, \mathbf{B}) = \inf_{a\in\mathbf{A}, b\in\mathbf{B}} d(a, b)$.

Although the distance $d(a, b)$ is positive if $a \in \mathbf{A}$, $b \in \mathbf{B}$ and $\mathbf{A} \neq \mathbf{B}$, it is still possible that the infimum is never realized. For example, the sequence $\frac{1}{n}$ for $n \in \mathbb{N}$ is positive as n approaches infinity, but the infimum is not realized. We must thus determine if the infimum is realized in our set theoretic quotient space defined by uniformly discontinuous groups of isometries of the plane.

We fix a', a point in $\mathbf{A}$. We want to show that there exists a point b' in $\mathbf{B}$ such that $d(a', b') \leq d(a', b)$ for every b in $\mathbf{B}$. Then $d(a', b')$ is the minimum of $d(\mathbf{A}, \mathbf{B})$.

Lemma 2.12. *Let Γ be a uniformly discontinuous group of isometries of the plane, which defines a equivalence relation between points in the plane as above. Let $\boldsymbol{A}$ be a set of equivalent points on the plane and let D be a disc in the plane. There are finitely many points in $\boldsymbol{A}$ contained in D. Thus, the infimum of $d(\boldsymbol{A}, \boldsymbol{B})$ is always attained.*

Proof. Let D be a disc of radius R. If D contains points in $\mathbf{A}$, let M be the points of $\mathbf{A}$ in D. Let D_i be discs of radius $k/2$ centered at A_i in M, for k the uniform distance between equivalent points under Γ. Thus, D_i will not overlap. Let D' be the disc with radius $R + k/2$. D' contains all D_i.

The ratio of the surface area of D' to D_i is $\dfrac{\pi(R + k/2)^2}{\pi(k/2)^2} = \dfrac{4(R + k/2)^2}{k^2}$

This ratio is a finite number and represents an upper bound of the number of A_i in the disc D. Thus any disc in the plane contains at most a finite number of points in $\mathbf{A}$. A disc centered on a' will thus contain finitely many points of B. If we take the radius of the disc large enough to contain some point in $\mathbf{B}$, we may choose b' in the disc such that $d(a', b') \leq d(a', b)$ for every b in $\mathbf{B}$. The infimum of $d(\mathbf{A}, \mathbf{B})$ is realized by $d(a', b')$. □

Theorem 2.13. *The space $\mathbb{R}^2/\Gamma$ with the function d defined above is a metric space.*

Proof. We now check that the 3 axioms of a metric space apply to our geometry.

a) $d(\mathbf{A}, \mathbf{B})$ is positive definite. Since $d(a, b) \geq 0$ for all a, b, it follows that $d(\mathbf{A}, \mathbf{B}) = \inf_{a\in\mathbf{A}, b\in\mathbf{B}} d(a, b) \geq 0$. We need to show $d(\mathbf{A}, \mathbf{B}) \neq 0$ for $\mathbf{A} \neq \mathbf{B}$. For $\mathbf{A} \neq \mathbf{B}$, the sets $\mathbf{A}$ and $\mathbf{B}$ do not have points in common. For otherwise, suppose there exists a point c in both $\mathbf{A}$ and $\mathbf{B}$. Then every point in both $\mathbf{A}$ and $\mathbf{B}$ is equivalent to c and all points in $\mathbf{A}$ are equivalent to all points in $\mathbf{B}$. Then, $\mathbf{A} = \mathbf{B}$, which is a contradiction.

   Let a' be any point of $\mathbf{A}$. A disc D with radius $k/2$ centered on a' will contain at most one point b in $\mathbf{B}$. If there were two points b and b' in $\mathbf{B}$

in D, we would have $d(b, b') < k$, which is a contradiction, since $b \neq b'$ and $d(b, b') \geq k$.

For the case of no points of $\mathbf{B}$ in D, then we take $c = k/2$. If there is one point of $\mathbf{B}$ in D, then $c = d(a', b)$ for b in $\mathbf{B}$. Thus, there is always a positive number c such that $d(a', b) \geq c$ for all points b in $\mathbf{B}$. Since $d(a, b)$ is the minimum of $d(a', b)$ where a' is a fixed point of $\mathbf{A}$ and b any point of $\mathbf{B}$, we have $d(a, b) \geq c > 0$.

b) $d(\mathbf{A}, \mathbf{B})$ is symmetric because

$$d(\mathbf{A}, \mathbf{B}) = \inf_{a \in \mathbf{A}, b \in \mathbf{B}} d(a, b) = \inf_{a \in \mathbf{A}, b \in \mathbf{B}} d(b, a) = d(\mathbf{B}, \mathbf{A}).$$

c) $d(\mathbf{A}, \mathbf{B})$ satisfies the triangle inequality. We must show that

$$d(\mathbf{A},\mathbf{C}) \leq d(\mathbf{A},\mathbf{B}) + d(\mathbf{B},\mathbf{C}).$$

Let $a \in \mathbf{A}$ and let $b \in \mathbf{B}$ such that $d(a, b) = d(\mathbf{A}, \mathbf{B})$. Such a point b exists since the distance $d(\mathbf{A}, \mathbf{B})$ can be realized by Lemma 2.12. Similarly, let $c \in \mathbf{C}$ such that $d(b, c) = d(\mathbf{B}, \mathbf{C})$ and $c' \in \mathbf{C}$ such that $d(a, c') = d(\mathbf{A}, \mathbf{C})$. Then

$$d(\mathbf{A}, \mathbf{C}) \leq d(a, c) \leq d(a, b) + d(b, c) = d(\mathbf{A}, \mathbf{B}) + d(\mathbf{B}, \mathbf{C}).$$

The first inequality is because $d(\mathbf{A}, \mathbf{C})$ is the minimum distance between a and points in $\mathbf{C}$. Therefore, the triangle equality holds.

□

We have determined that the set theoretic quotient space of a uniformly discontinuous group of isometries on the plane is a metric space. In other words, this set of (equivalent) points and the shortest distance between equivalent points is a geometry.

We finally check that the geometry is locally Euclidean. We will show that a disc centered on any point p in our geometry is identical to a disc in the plane.

Theorem 2.14. *If Γ is a uniformlly discontinuous group of isometries of the plane, then $\mathbb{R}^2/\Gamma$ is a locally Euclidean geometry. Hence, to every uniformly discontinuous group of isometries of the plane there corresponds a 2-dimensional locally Euclidean geometry.*

Proof. Let P be a point in $M = \mathbb{R}^2/\Gamma$, and p a point on the plane such that $p \in P$. Let D be a disc of radius $k/4$ centered at p in the plane. Let a be in the disc D.

By definition, $d(A, P) \leq d(a, p) \leq k/4$. Thus, A belongs to a disc centered or P with radius $k/4$. We map D to this disc on our geometry by taking a to A.

Now we show that this map is an isometry. Let a and b be points on D, with $\mathbf{A} = [a]$, the set of equivalent points to a, and $\mathbf{B} = [b]$. We must show that $d(A, B) = d(a, b)$ for any a and b in D.

By definition and Lemma 2.12, we have $d(\mathbf{A},\mathbf{B}) = \min_{b' \in \mathbf{B}} d(a, b')$. Suppose $d(a, b)$ is not the minimum. Then $d(a, b') < d(a, b)$ for some $b' \in \mathbf{B}$. Since a and b are in D, $d(a, b) \leq k/2$. Then $d(a, b') < k/2$. From the triangle inequality, $d(b, b') \leq d(a, b) + d(a, b') < k/2 + k/2 = k$. This is a contradiction because Γ is a uniformly discontinuous group. So, $d(a, b)$ is the least of $d(a, b')$. It follows that $d(\mathbf{A}, \mathbf{B}) = d(a, b)$ and our map preserves distance.

Finally, we must show that any point of our geometry in the disc D' centered at P with radius $k/4$ corresponds to a point of disc D in the plane. Let $\mathbf{A}$ be in D'.

Let a be the closest element of $\mathbf{A}$ to p on the plane. Since $d(a,p) \leq d(a',p)$ for any a' in $\mathbf{A}$, $d(a,p) = d(A,P)$. Since $d(\mathbf{A},\mathbf{P}) \leq k/4$, we have $d(a,p) \leq k/4$ and a is in D. Thus, any point $\mathbf{A}$ in D' on our geometry corresponds to a in D on the plane.

We have shown that any 2-dimensional locally Euclidean geometry corresponds to a uniformly discontinuous group of isometries of the plane. □

Example 2.15. There are only five 2-dimensional locally Euclidean geometries up to "topological" equivalence:

(1) The plane corresponds to $\Gamma = 1$, the trivial group.
(2) The cylinder corresponds to $\Gamma = \{T_{(n,0)} |\ n \in \mathbb{Z}\}$,
(3) The torus corresponds to $\Gamma = \{T_{(a,b)} |\ a, b \in \mathbb{Z}\}$
(4) The Mobius band corresponds to $\Gamma = \{f |\ f(x, y) = (x+n, (-1)^n y), n \in \mathbb{Z}\}$
(5) Klein bottle $= \mathbb{R}^2/\Gamma = f(x, y) = (x + n, (-1)^n(y + m)$ for $n, m \in \mathbb{Z}$.

This claim will not be proved here but is a natural direction for further study.

3. Examples in Music: The Geometry of Musical Chords

In Western music, composers use *harmony* and *counterpoint* to create aesthetic works that can be feasibly performed.

Chords are sets of notes played simultaneously, and tradition permits that only certain combinations of notes and sequences may be used. When choosing chords and chord sequences, composers are guided by concerns of harmony and acoustic consonance. Consonant chords are stable and sound pleasant. Counterpoint (or *voice-leading*) connects consonant chords to form simultaneously occurring melodies. These simultaneous voices must move independently and efficiently: not parallel to each other, and by short distances.

The reason why harmony and counterpoint can be used together, however, is not clear. Theorists have not found a way to explain how counterpoint connects consonant chords. The recent work of a geometer named Dmitri Tymozcko has uncovered an explanation for a long tradition of musical decision making. Tymoczko argues that harmony and counterpoint are indeed related; he has constructed n-dimensional orbifolds in which points are n-note chords. The structure of the orbifold illuminates a broad range of musical styles. Chords that can be connected efficiently are close to each other in this space. This shows that composers have uniformly and unknowingly used nearly-symmetrical chords to connect harmony with counterpoint.

We consider the space of *pitch classes.* Two pitches are in the same pitch class if they are one ore more octaves apart. For f the fundamental frequency of a pitch, the equation

$$p = 69 + 12 \log_2(f/440)$$

gives us a linear pitch space where middle C has $p = 60$, an octave has size 12, and a semitone (distance between adjacent notes) has size 1. There are 12 semitones in Western music. We assign C = 0, C♯ = 1, D = 2, ... B = 11 and have C = 12 = 0 be an equivalent point.

We thus have a set theoretic quotient space in $\mathbb{R}$ that is constructed by a uniformly discontinuous group with $k = 12$. $\mathbb{R}/12\mathbb{Z}$ is a circle (Figure 1).

A *chord* is an unordered multiset of pitch classes. We denote the C major chord by {0, 4, 7} and the F-major chord by {0, 5, 9}. The musical term *transposition*

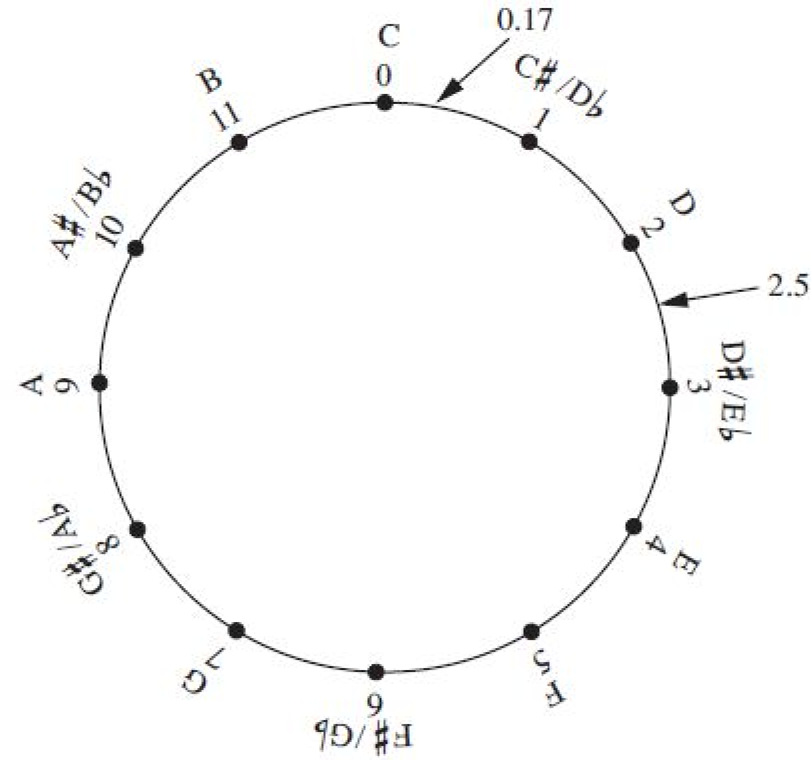

FIGURE 1. $\mathbb{R}/12\mathbb{Z}$

is related to the mathematical term *translation*. It is addition in $\mathbb{R}/12\mathbb{Z}$. The C-major and F-major chords are *transpositionally equivalent (T-equivalent)* because $\{0, 5, 9\} = \{7 + 5, 0 + 5, 4 + 5\}$, i.e. $\mathbf{T}_5(\{0, 4, 7\}) = \{5, 9, 0\}$.

In the geometry of musical chords, the elements of an ordered n-note chord may be represented as the coordinates of a point on the n-torus $(\mathbb{R}/12\mathbb{Z})^n$, or $\mathbf{T}^n$.

There is more than one way to measure distance between points in this space. We may use the taxicab or Euclidean metric, or simply compare distances without quantifying them as real numbers. Let a, b be elements of $\mathbb{R}/12\mathbb{Z}$. The *norm* of a, $|a|_{12\mathbb{Z}}$, is the smallest real number $|x|$ such that x and a are congruent modulo $12\mathbb{Z}$. The *distance* between a and b is $|b - a|_{12\mathbb{Z}}$. The *displacement multiset* is $|b_j - a_i|$ for all (a_i, b_j) in the voice leading A $\longrightarrow$ B. In two dimensions, the space of ordered pairs of notes is a torus (Figure 2).

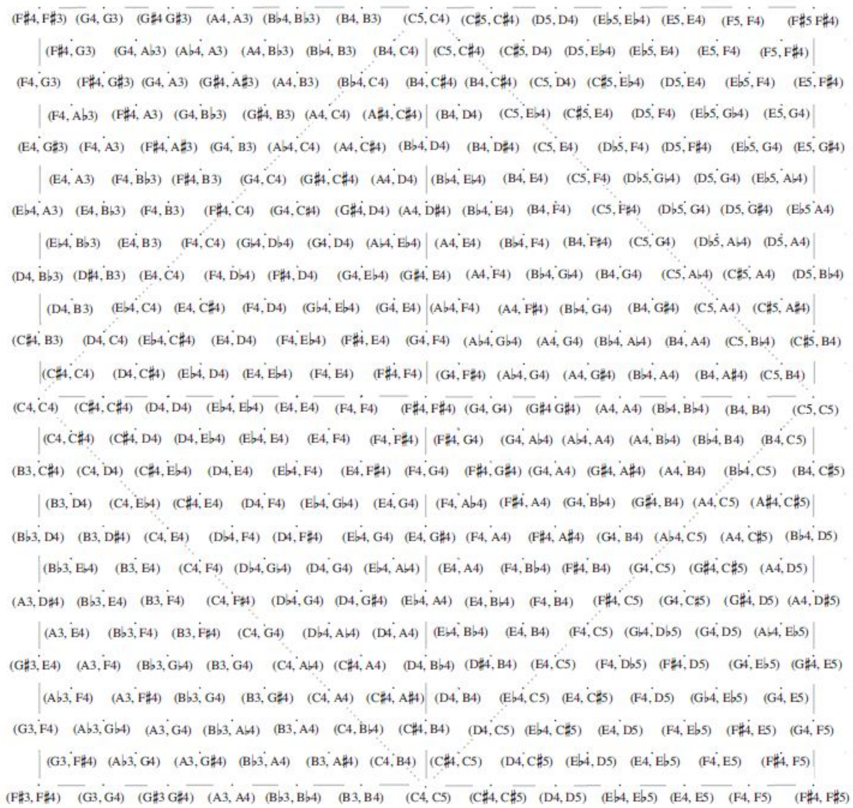

FIGURE 2. $\mathbb{R}/\Gamma = \{T_{(12,12)}\}$

These points represent two note chords (*dyads*), or all the combinations of any two notes. Since this space is constructed by a uniformly discontinuous group of isometries, we have a locally Euclidean geometry.

To represent voice-leadings between unordered chords, we will identify all points (chords) with the same notes, regardless of order. We thus form the set theoretic quotient space $\mathbf{T}^n/\mathbf{S}_n$, by identifying all points $(x_1, x_2, ..., x_n)$ and $(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)})$, where σ is some permutation of integers from 1 to n. This space is called a *global-quotient orbifold*, which results from identifying all the points that lie in the orbits of a uniformly discontinuous group acting on a locally Euclidean space. Orbifolds have *singularities* where the space is not locally Euclidean.

In the case of dyads ($n = 2$), we fold the torus along the AB diagonal and we have a triangle. If we cut along line CD and glue AC to CB, we finish with a Möbius band (Figure 3 and Figure 4).

Tymoczko found that the paths on Figure 4 uniquely represent voice-leading between dyads. The paths are either line segments inside the orbifold or line-segments that are reflected off the orbifold's singular boundary. Paths that are parallel to the boundary are not independent. Also, the pitches on the edge of the orbifold contain the chords with duplicate pitch classes. For example, $(0, 1) \longrightarrow (1, 0)$ corresponds to the path that starts and ends at the same point after being

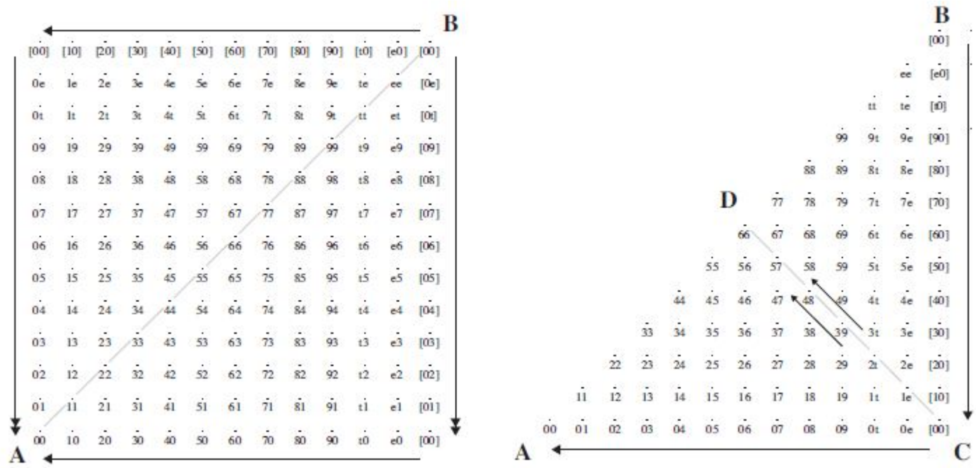

FIGURE 3. Identifying

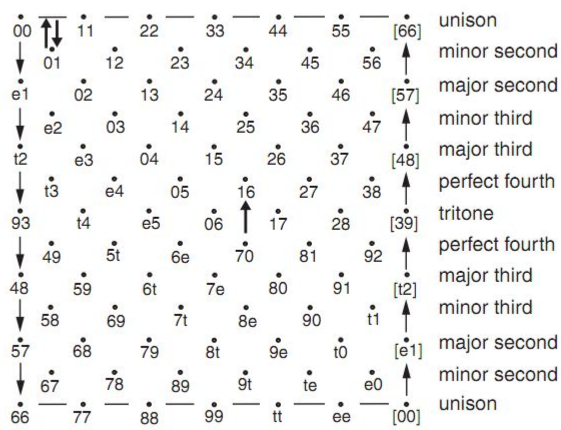

FIGURE 4. The orbifold $\mathbf{T}^2/\mathbf{S}_2$, with Euclidean distance

reflected off of the boundary. These reflected paths represent voice leadings that contain voice crossings, which composers avoid. Furthermore, they are inefficient because the identity transformation is more minimal than the reflected path.

We may generalize to higher dimensions, or n-note chords. Tymozcko worked up to 3-note (Figure 5) and 4-note (Figure 6) chords. These are prisms that are constructed by identifying the faces with a 360°/n twist. As in 2 dimensions, the singular boundary acts as a mirror for crossing voice leading paths. Chords containing only one pitch still lie along the boundary and chords that divide the octave into n equal parts lie in the center of the orbifold. Line segments that are parallel to the boundary represent voice leadings that are not independent, and line segments that are perpendicular to the boundary represent independent voice leadings.

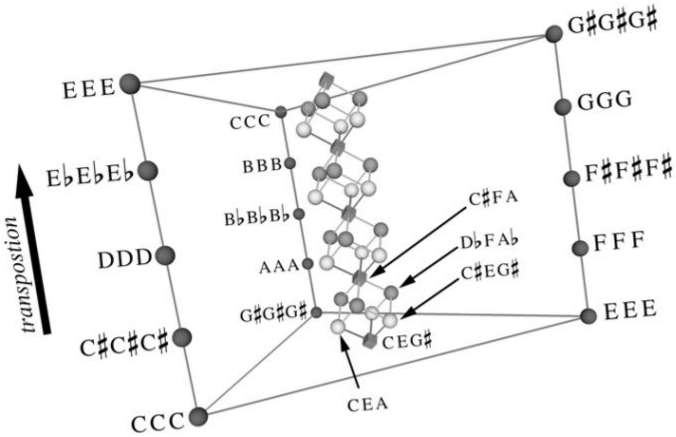

FIGURE 5. $\mathbf{T}^3/\mathbf{S}_3$

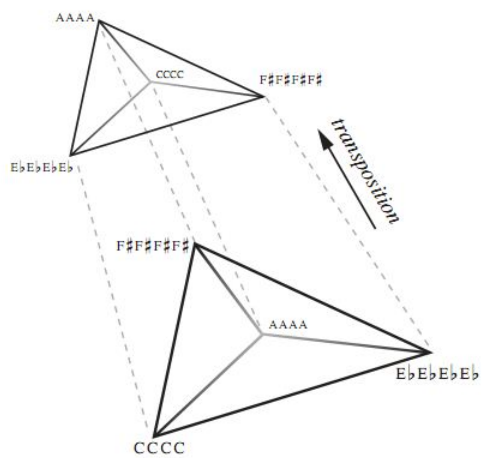

FIGURE 6. $\mathbf{T}^4/\mathbf{S}_4$

Tymozcko found that there always exists a minimal voice-leading between two chords that do not reflect off the boundary. It is an interesting fact that if two chords are close to each other on the orbifold, they can be connected by an efficient voice leading; the chords can be used in a harmonious but independent and efficient chord progression.

The size of an independent and efficient voice leading is determined by the chord's invariance, or symmetry under transformations. Transposition by x semitones preserves the distance of voice leadings, according to any normlike strict weak ordering. For example, $(a_1, a_2, ..., a_n) \longrightarrow (b_1, b_2, ..., b_n)$ has the same distance path as $(a_1 + x, a_2 + x, ..., a_n + x) \longrightarrow (b_1 + x, b_2 + x, ..., b_n + x)$. A *transpositionally invariant (T-invariant)* chord is a fixed point of such a transposition. For n-note chords, T-invariant chords exist when nx is congruent to 0, mod $12\mathbb{Z}$. Chords that lie close to T-invariant chords are called *nearly T-invariant.* Nearly T-invariant chords can be connected by efficient voice leadings because there are multiple transpositions located near fixed T-invariant chords.

A chord is T-invariant when its elements evenly divide the octave. Similarly, nearly T-invariant chords divide the octave into nearly equal parts. More evenly spaced chords are clustered in the center of the orbifold, and so, voice leading paths between T-equivalent forms are small. We now show that the chord that evenly divides an octave into n equal parts has the smallest possible voice leading to all of its transpositions.

Theorem 3.1. *Let A be any multiset of cardinality n. Let E be a multiset of cardinality n that divides pitch-class space into n equal parts. For all x, the minimal voice leading between A and* $\boldsymbol{T}_x(A)$ *is no smaller than the minimal voice-leading between E and* $\boldsymbol{T}_x(E)$.

Proof. Since E divides pitch space into n equal parts, E is invariant under transposition by $12/n$ semitones. Thus, there will be a voice leading between E and $\mathbf{T}_x(E)$ of the form $(e_1, e_2, ..., e_n) \longrightarrow (e_1 + c, e_2 + c, ..., e_n + c)$, where $c \equiv_{12\mathbb{Z}/n} x$.

If we choose c so that $|c|$ is as small as possible, the displacement multiset of $E \longrightarrow \mathbf{T}_x(E)$ is $\{|c|, |c|, ..., |c|\}$. The sum of the elements of this multiset is $n|c|$, where $n|c|$ is the smallest positive real number such that $nc \equiv_{12\mathbb{Z}} nx$.

Let $\sum A$ represent the sum of the components of A. Since A $\mathbf{T}_x(A)$ are transpositionally related by x, we have $\sum(\mathbf{T}_x(A) - A) \equiv_{12\mathbb{Z}} nx$. $\sum(\mathbf{T}_x(A) - A)$ represents nx. Since $n|c|$ is the smallest positive number such that $nc \equiv_{12\mathbb{Z}} nx$, $n|x|$ is therefore less than or equal to $n|c|$. Thus the elements of the displacement multiset of $A \longrightarrow \mathbf{T}_x(A)$ sum to no less than $n|c|$. Therefore, the size of $A \longrightarrow \mathbf{T}_x(A)$ can be no smaller than the minimal voice leading of $E \longrightarrow \mathbf{T}_x(E)$. □

This fact is important because acoustically consonant (harmonious) chords tend to be nearly T-invariant. Western composers have thus favoured the chords that lie in the center of the orbifold – chords that are linked by independent and efficient voice leadings. Traditional Western counterpoint has therefore depended on this relationship between harmonious, nearly-even chords and the efficient voice leadings that connect them. Hitherto, music theorists have had an incomplete understanding of acoustic consonance. Happily, Tymozcko's voice-leading orbifold explains the relationship that is responsible for the intuitive decisions and preferences that have guided composers of many different styles and periods.

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