# CATEGORIES OF PROBABILITY SPACES

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Abstract. In this paper, we will show how several important ideas of probability theory can be fit into the framework of category theory by considering two different categories of probability spaces. Once we have defined these categories, we can see several important structures of probability spaces arise in their structure; specifically, we can identify conditional probabilities, subprobability spaces, independent probabily spaces, and independent and disjoint probability spaces. We define an equivalence of categories between a subcategory and a category of very specific monoids.

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## 1. The First Category

We begin by recalling the definition of a probability space:

**Definition 1.1.** A *probability space* is a triple $(\Omega, \mathcal{F}, P)$ consisting of:
(i) A nonempty set $\Omega$, which we call the sample space,
(ii) A $\sigma$-algebra $\mathcal{F}$ of subsets of $\Omega$,
(iii) A function $P : \mathcal{F} \to [0, 1]$, called the probability measure, such that:
(a) $P$ is countably additive (for $F_1, F_2, ... \in \mathcal{F}$ disjoint, we have $P(\bigcup F_i) = \Sigma P(F_i))$,
(b) $P(\Omega) = 1$.

We now give a few examples of probability spaces which will be useful throughout:

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**Example 1.2.** We can define a probability space corresponding to a coin flip by taking $\Omega = \{H, T\}$, $\mathcal{F} = 2^{\Omega}$, the power set of $\Omega$, and $P(H) = P(T) = \frac{1}{2}$. Let $C$ denote this probability space.

**Example 1.3.** We can also define a probability space corresponding to a $q$-biased coin flip by taking $\Omega = \{H, T\}$, $\mathcal{F} = 2^{\Omega}$, and $P(H) = q$, $P(T) = 1 - q$. Let $C_q$ denote this probability space.

**Example 1.4.** We can define a probability space corresponding to the roll of a die by taking $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F} = 2^{\Omega}$, $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$. Let $D_6$ denote this probability space.

**Example 1.5.** We can similarly define a probability space corresponding to an $n$-sided die roll (for $n \in \mathbb{N}$), by taking $\Omega = \{1, ..., n\}$, $\mathcal{F} = 2^{\Omega}$, $P(1) = \cdots = P(n) = \frac{1}{n}$. Let $D_n$ denote this probability space.

The above are all very simple examples of probability spaces; in fact, they are discrete probability spaces:

**Definition 1.6.** A *discrete probability space* is a probability space $(\Omega, \mathcal{F}, P)$ such that $\Omega$ is finite or countably infinite, and $\mathcal{F}$ is the power set of $\Omega$.

We often write discrete probability spaces as $(\Omega, P)$, since $\mathcal{F}$ is always $2^{\Omega}$.

Although most of this paper will apply to general probability spaces, we will consider primarily finite, discrete probability spaces in order to gain an intuition for the structures. Later on, restricting to finite discrete probability spaces will allow us to give an equivalence with a certain category of monoids.

We also give a definition of what we call the empty space:

**Definition 1.7.** The empty space is the triple $(\emptyset, \{\emptyset\}, P_{\emptyset})$, where $P_{\emptyset}(\emptyset) = 0$.

Note that this is not an actual probability space, since that would require $P_{\emptyset}(\emptyset) = 1$, and by additivity we must have $P_{\emptyset}(\emptyset) = 0$. However, we will be able to define morphisms between the empty space and probability spaces very easily and still satisfy the definition of a category. The importance of this object will become clear in the next section; it will be the zero object in our category.

We now give a possible definition of a morphism of probability spaces:

**Definition 1.8.** A *morphism* between probability spaces $(\Omega, \mathcal{F}, P)$ and $(\Omega', \mathcal{F}', P')$ is a function $\phi : \mathcal{F} \to \mathcal{F}'$ which preserves unions and intersections, and satisfies $\phi(\emptyset) = \emptyset$.

We can now define the first category of probability spaces which we will consider in this paper.

**Definition 1.9.** *Prob* is the category with objects probability spaces and the empty space, and morphisms as defined above.

It may seem strange at first that the definition of a morphism doesn't include any specific conditions to preserve the probability measure on the first space. However, since probability measures are countably additive and the morphism must preserve unions, it does necessarily preserve some of the structure of the probability measure.

It also may seem strange that we require $\emptyset \mapsto \emptyset$, while not requiring $\Omega \mapsto \Omega'$. The reason for this will be apparent when we consider subprobability spaces and quotients in the next section.

The rest of this section will be our justification for taking this as the definition of a morphism, since other ways to define the morphisms do not allow for the structures that we will consider. However, other definitions do result in other interesting properties worth considering, one of which we will discuss in the second part of this paper.

We now give some examples of maps of probability spaces that are morphisms in $Prob$:

**Example 1.10.** We can define a morphism $\phi : C \to C_q$ for any $q$ by simply taking $\{H\} \mapsto \{H\}$ and $\{T\} \mapsto \{T\}$. It is simple to check that this is a morphism, and is, in fact, an isomorphism.

**Example 1.11.** We can also define a morphism $\phi : C \to D_6$ by taking $\{T\} \mapsto \{1, 2, 3\}$ and $\{H\} \mapsto \{4, 5, 6\}$. This is a monomorphism.

**Example 1.12.** We can define another morphism $\phi : C \to D_6$ by taking $\{T\} \mapsto \{1\}$ and $\{H\} \mapsto \{2\}$. In this way $C$ is a subprobability space of $D_6$. We can also define a morphism $\psi : D_6 \to C$ by taking $\{1\} \mapsto \{T\}$, $\{2\} \mapsto \{H\}$, $\{3\}, \{4\}, \{5\}, \{6\} \mapsto \emptyset$. These kinds of maps will be the subject of the next section.

It is simple to check that taking objects to be probability spaces (and the empty space) and morphisms to be as above, we do, in fact, get a category.

1.1. **Subprobability Spaces and Conditional Probability Spaces.** Recall that given a probability space $(\Omega, \mathcal{F}, P)$ and a subset $B \subset \Omega$ such that $P(B) > 0$, we can define another probability space by considering the conditional probabilities of events given $B$. The resulting probability space, denoted $(B, \mathcal{F}_B, P_B)$, is given by setting $\mathcal{F}_B = \{A \cap B | A \in \mathcal{F}\}$ and $P_B(A \cap B) = P(A|B)$. As an example, consider the following:

**Example 1.13.** Consider the subset $\{1, 2\} \subset \{1, 2, 3, 4, 5, 6\}$. Then the corresponding conditional probability space is $(\{1, 2\}, \mathcal{F}_{\{1,2\}}, P_{\{1,2\}})$, where $\mathcal{F}_{\{1,2\}} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ and $P(\{1\}) = P(\{2\}) = \frac{1}{2}$.

It is easy to check that the inclusion $\phi_B : (B, \mathcal{F}_B, P_B) \to (\Omega, \mathcal{F}, P)$ given by $\phi_B : A \cap B \mapsto A \cap B$ is a morphism in $Prob$. In addition, this provides an important example of a map which wouldn't be a morphism in $Prob$ had we required $\phi(\Omega) = \Omega'$. While the image of the morphism isn't itself a probability space, since $P(B) \neq 1$, it is kind of a scaled version of the probability space, so we are not losing anything by taking this definition. It is also simple to check that $\phi_B$ is a monomorphism, so that $(B, \mathcal{F}_B, P_B)$ is a subobject of $(\Omega, \mathcal{F}, P)$ as we would expect. We can see this in the following example:

**Example 1.14.** We define a morphism $\phi_{\{1,2\}} : (\{1, 2\}, \mathcal{F}_{\{1,2\}}, P_{\{1,2\}}) \to D_6$ simply by taking $\{1\} \mapsto \{1\}$ and $\{2\} \mapsto \{2\}$. This is a monomorphism.

We can also define a morphism $\psi_B : (\Omega, \mathcal{F}, P) \to (B, \mathcal{F}_B, P_B)$ by taking $\psi_B : A \mapsto A \cap B$. This map is an epimorphism, and so it is a categorical quotient of $(\Omega, \mathcal{F}, P)$. We can once again see this in the example:

**Example 1.15.** We define a morphism $\psi_{\{1,2\}} : D_6 \to (\{1, 2\}, \mathcal{F}_{\{1,2\}}, P_{\{1,2\}})$ by taking $\{1\} \mapsto \{1\}$, $\{2\} \mapsto \{2\}$, and $\{3\}, \{4\}, \{5\}, \{6\} \mapsto \emptyset$. This is an epimorphism.

It is now our goal to show that the subobject $(B^c, \mathcal{F}_{B^c}, P_{B^c})$ is the kernel of the quotient $(B, \mathcal{F}_B, P_B)$. The intuition of this statement should be clear from the above definitions and examples, however it takes more work to define this rigorously, and it is here that we make use of the empty space.

**Proposition 1.16.** *The empty space is a zero object in Prob.*

*Proof.* Given any probability space $(\Omega, \mathcal{F}, P)$, we can define a map $\phi : (\emptyset, \{\emptyset\}, P_\emptyset) \to (\Omega, \mathcal{F}, P)$ by taking $\emptyset \mapsto \emptyset$. It is easy to see that this preserves unions and intersections, and so is a morphism in *Prob*. Since $\emptyset \mapsto \emptyset$ is forced by the definition of a morphism, $\phi$ is the only morphism $(\emptyset, \{\emptyset\}, P_\emptyset) \to (\Omega, \mathcal{F}, P)$. Thus $(\emptyset, \{\emptyset\}, P_\emptyset)$ is initial in *Prob*.

Observe that we can also define a map $\psi : (\Omega, \mathcal{F}, P) \to (\emptyset, \{\emptyset\}, P_\emptyset)$ by taking $F \mapsto \emptyset$ for all $F \in \mathcal{F}$. This also trivially preserves unions and intersections, and so is a morphism in *Prob*. Also, for all $F \in \mathcal{F}$, we must have $F \mapsto \emptyset$ because there is no other possible set to send it to, so $\psi$ is the only morphism $(\Omega, \mathcal{F}, P) \to (\emptyset, \{\emptyset\}, P_\emptyset)$. This shows that $(\emptyset, \{\emptyset\}, P_\emptyset)$ is terminal in *Prob*, and thus that $(\emptyset, \{\emptyset\}, P_\emptyset)$ is a zero object in *Prob*.

□

By definition, the zero morphism from $(\Omega, \mathcal{F}, P)$ to $\Omega', \mathcal{F}', P')$ is given by $0(F) = \emptyset$ for all $F \in \mathcal{F}$. Recall that the kernel of a morphism $f : X \to Y$ is the equalizer of $0_{XY}$ and $f$. We can now prove what we wanted; we can prove that the subobject $(B^c, \mathcal{F}_{B^c}, P_{B^c})$ is the kernel of the map $\psi_B : (\Omega, \mathcal{F}, P) \to (B, \mathcal{F}_B, P_B)$ as defined above.

**Proposition 1.17.** *The map $\phi_{B^c} : (B^c, \mathcal{F}_{B^c}, P_{B^c}) \to (\Omega, \mathcal{F}, P)$ is the kernel of the map $\psi_B : (\Omega, \mathcal{F}, P) \to (B, \mathcal{F}_B, P_B)$.*

*Proof.* Write $0_{(B^c, \mathcal{F}_{B^c}, P_{B^c})(B, \mathcal{F}_B, P_B)}$ as $0_{B^cB}$ for simplicity. It's clear that $0_{B^cB} = \psi_B \phi_{B^c}$.

Now suppose we have a map $h : (\Omega', \mathcal{F}', P') \to (\Omega, \mathcal{F}, P)$ such that $\psi_B h = 0_{B^cB}$. We must show that there is unique $h' : (\Omega', \mathcal{F}', P') \to (B^c, \mathcal{F}_{B^c}, P_{B^c})$ such that $h = \phi_{B^c} h'$. It is enough to show that such a morphism exists, since it must be unique because $\phi_{B^c}$ is a monomorphism.

As $\psi_B h = o_{B^cB}$ implies that $h(\Omega') \subset B^c$, we can define $h'$ by taking $h'(F) = h(F) \subset B^c$ for all $F \in \mathcal{F}'$. It is simple to check that this is a morphism, and we clearly have $h = \phi_{B^c} h'$, completing the proof.

□

This makes sense intuitively: because considering the conditional probability space given by $B$ is the same as assuming that $B^c$ does not happen, it should be the quotient by the subobject corresponding to $B^c$.

1.2. **Coproducts.** In this section, we show that *Prob* has countable coproducts.

**Definition 1.18.** Suppose we have two probability spaces, $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$. We define $(\Omega_1, \mathcal{F}_1, P_1) \amalg (\Omega_2, \mathcal{F}_2, P_2)$ to be $(\Omega_1 \amalg \Omega_2, \mathcal{F}_1 \amalg \mathcal{F}_2, P_1 \amalg P_2)$, where $\Omega_1 \amalg \Omega_2$ is the disjoint union of sets $\Omega_1$ and $\Omega_2$, $\mathcal{F}_1 \amalg \mathcal{F}_2 = \{F_1 \amalg F_2 | F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$, and $(P_1 \amalg P_2)(F_1 \amalg F_2) = \frac{1}{2}(P_1(F_1) + P_2(F_2))$.

We define morphisms $i_1 : (\Omega_1, \mathcal{F}_1, P_1) \to (\Omega_1, \mathcal{F}_1, P_1) \amalg (\Omega_2, \mathcal{F}_2, P_2)$ and $i_2 : (\Omega_2, \mathcal{F}_2, P_2) \to (\Omega_1, \mathcal{F}_1, P_1) \amalg (\Omega_2, \mathcal{F}_2, P_2)$ by $i_1 : F_1 \mapsto F_1$ and $i_2 : F_2 \mapsto F_2$. It is very simple to show that these are, in fact, morphisms.

We now show that this gives us a coproduct.

**Proposition 1.19.** *The probability space* $(\Omega_1, \mathcal{F}_1, P_1) \amalg (\Omega_2, \mathcal{F}_2, P_2)$ *with the maps* $i_1$ *and* $i_2$*, as defined above, is the coproduct of the probability spaces* $(\Omega_1, \mathcal{F}_1, P_1)$ *and* $(\Omega_2, \mathcal{F}_2, P_2)$*.*

*Proof.* Suppose we have morphisms $f : (\Omega_1, \mathcal{F}_1, P_1) \to (\Omega, \mathcal{F}, P)$ and $g : (\Omega_2, \mathcal{F}_2, P_2) \to (\Omega, \mathcal{F}, P)$, for some probability space $(\Omega, \mathcal{F}, P)$. Then we can define a map $h : (\Omega_1, \mathcal{F}_1, P_1) \amalg (\Omega_2, \mathcal{F}_2, P_2) \to (\Omega, \mathcal{F}, P)$ by taking $h : F_1 \amalg F_2 \mapsto f(F_1) \cup g(F_2)$.

To see that $h$ is a morphism, observe that:

$$\begin{aligned}
h(\emptyset) &= \emptyset \cup \emptyset \\
&= \emptyset, \\
h(F_1 \cap F_1') &= f(F_1 \cap F_1') \\
&= f(F_1) \cap f(F_1') \\
&= h(F_1) \cap h(F_2), \\
h(F_1 \cup F_1') &= f(F_1 \cup F_1') \\
&= f(F_1) \cup f(F_1') \\
&= h(F_1) \cup h(F_1'), \\
&\text{etc.}
\end{aligned}$$

$hi_1(F_1) = h(F_1) = f(F_1)$ and $hi_2(F_2) = h(F_2) = g(F_2)$.

To show uniqueness, suppose that there is another such morphism $h'$ satisfying $h'i_1 = f$ and $h'i_2 = g$. Then by definition:

$$\begin{aligned}
h'(F_1 \amalg F_2) &= h'(F_1) \cup h'(F_2) \\
&= h'i_1(F_1) \cup h'i_2(F_2) \\
&= f(F_1) \cup g(F_2) \\
&= h(F_1 \amalg F_2)
\end{aligned}$$

Therefore this is the coproduct.

□

We consider an example to see what this probability space is:

**Example 1.20.** Let $(\Omega_1, \mathcal{F}_1, P_1)$ be the probability space corresponding to a three sided die roll (or what it would be, if it were possible to make one!), so $\Omega_1 = \{1, 2, 3\}$, $\mathcal{F}_1$ is the power set of $\Omega_1$, and $P_1(\{1\}) = P_1(\{2\}) = P_1(\{3\})$. Let $(\Omega_2, \mathcal{F}_2, P_2)$ also be the probability corresponding to a three sided die roll. To avoid confusion, we write $\Omega_2 = \{4, 5, 6\}$. Then $(\Omega_1, \mathcal{F}_1, P_1) \amalg (\Omega_2, \mathcal{F}_2, P_2) = (\Omega_1 \amalg \Omega_2, \mathcal{F}_1 \amalg \mathcal{F}_2, P_1 \amalg P_2)$, where $\Omega_1 \amalg \Omega_2 = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F}_1 \amalg \mathcal{F}_2$ is the power set of $\Omega_1 \amalg \Omega_2$, and $(P_1 \amalg P_2)(\{1\}) = (P_1 \amalg P_2)(\{2\}) = (P_1 \amalg P_2)(\{3\}) = (P_1 \amalg P_2)(\{4\}) = (P_1 \amalg P_2)(\{5\}) = (P_1 \amalg P_2)(\{6\})$. This is just a six sided die roll!

This example clearly shows that taking the coproduct of two probability spaces is the same as considering $\Omega_1$ and $\Omega_2$ as mutually exclusive events in the sample space $\Omega_1 \amalg \Omega_2$, each weighted with probability $\frac{1}{2}$, and taking the resulting probability space.

Arbitrary finite coproducts follow inductively.

We can also form a coproduct of countably many probability spaces $\{(\Omega_i, \mathcal{F}_i, P_i)\}_{i=1}^{\infty}$ by taking $\coprod(\Omega_i, \mathcal{F}_i, P_i) = (\coprod \Omega_i, \coprod \mathcal{F}_i, \coprod P_i)$, where $\coprod \Omega_i$ is the disjoint union of sets $\Omega_i$, $\coprod \mathcal{F}_i = \{\coprod F_i | F_i \in \mathcal{F}_i\}$, and $(\coprod P_i)(\coprod F_i) = \sum_{i=1}^{\infty} \frac{1}{2^i} P_i(F_i)$. Note that permutations of the probability spaces give the same coproduct up to isomorphism; since we can define isomorphisms that change the weight of the $\Omega_i$'s.

1.3. **Equivalence with a Category of Monoids.** For this section, we restrict to the full subcategory of *Prob* on the probability spaces with finite $\sigma$-algebras. Denote this category $Prob^{<\infty}$. We will show that $Prob^{<\infty}$ is equivalent to a very specific category of abelian monoids.

**Definition 1.21.** Let $Mon_{Prob}$ be the category with objects all finite idempotent partially ordered abelian monoids with unique factorization into minimal elements, and morphisms all monoid homomorphisms between them which preserve greatest common divisors.

We now construct a functor $F : Prob^{<\infty} \to Mon_{Prob}$ which, intuitively, translates unions in the $\sigma$-algebra into multiplication in a monoid.

Let $(\Omega, \mathcal{F}, P)$ be an object of $Prob^{<\infty}$. Define $F_a = \bigcap_{F\in\mathcal{F}, a\in F} F$ for $a \in \Omega$. Then $F(\Omega, \mathcal{F}, P)$ is defined to be the free abelian monoid generated by the distinct $F_a$ modulo relations $F_a^2 = F_a$ for all $F_a$. It is clear that this is a finite idempotent partially ordered abelian monoid with unique factorization into minimal elements. How $F$ should act on morphisms is then clear.

We now define the functor $G : Mon_{Prob} \to Prob^{<\infty}$. Let $M$ be an object of $Mon_{Prob}$. Taking $\Omega$ to be the set of minimal elements in $M$, $\mathcal{F}$ to be the power set of $\Omega$, $P(a) = \frac{1}{|\Omega|}$ for any minimal element $a \in M$ (extended to be countably additive), gives us a probability space. How $G$ should act on morphisms is obvious, since preserving greatest common divisors is equivalent to preserving intersections.

**Proposition 1.22.** *$G : Mon_{Prob} \to Prob^{<\infty}$ is an equivalence of categories.*

*Proof.* It is clear from the definition of morphism in each category that $G$ is full and faithful; we show that $G$ is essentially surjective. Suppose we are given an object $(\Omega, \mathcal{F}, P)$ of *Prob*. Consider $GF(\Omega, \mathcal{F}, P) = (\Omega', \mathcal{F}', P')$, where $\Omega' = \{F_a | a \in \Omega\}$, $\mathcal{F}'$ is the power set of $\Omega'$, and $P(F_a) = \frac{1}{|\Omega'|}$. We obtain an isomorphism $\phi : (\Omega, \mathcal{F}, P) \to (\Omega', \mathcal{F}', P')$ simply by taking the morphism generated by the mapping $F_a \mapsto \{F_a\}$. $\square$

It is interesting to observe that this equivalence of categories factors as the equivalence of categories with the underlying $\sigma$-algebra of the probability space, followed by the isomorphism of categories between the finite sigma algebras and the category of monoids defined above.

We end this section by giving a few examples to illustrate this equivalence.

**Example 1.23.** The monoid corresponding to a $q$-biased coin flip (for any $q$) has the multiplication table (we drop set brackets for simplicity):

| | $\emptyset$ | $H$ | $T$ | $HT$ |
|---|---|---|---|---|
| $\emptyset$ | $\emptyset$ | $H$ | $T$ | $HT$ |
| $H$ | $H$ | $H$ | $HT$ | $HT$ |
| $T$ | $T$ | $HT$ | $T$ | $HT$ |
| $HT$ | $HT$ | $HT$ | $HT$ | $HT$ |

**Example 1.24.** The monoid corresponding to a 3 sided die roll has the multiplication table:

| | $\emptyset$ | 1 | 2 | 3 | $1\cdot 2$ | $1\cdot 3$ | $2\cdot 3$ | $1\cdot 2\cdot 3$ |
|---|---|---|---|---|---|---|---|---|
| $\emptyset$ | $\emptyset$ | 1 | 2 | 3 | $1\cdot 2$ | $1\cdot 3$ | $2\cdot 3$ | $1\cdot 2\cdot 3$ |
| 1 | 1 | 1 | $1\cdot 2$ | $1\cdot 3$ | $1\cdot 2$ | $1\cdot 3$ | $1\cdot 2\cdot 3$ | $1\cdot 2\cdot 3$ |
| 2 | 2 | $1\cdot 2$ | 2 | $2\cdot 3$ | $1\cdot 2$ | $1\cdot 2\cdot 3$ | $2\cdot 3$ | $1\cdot 2\cdot 3$ |
| 3 | 3 | $1\cdot 3$ | $2\cdot 3$ | 3 | $1\cdot 2\cdot 3$ | $1\cdot 3$ | $2\cdot 3$ | $1\cdot 2\cdot 3$ |
| $1\cdot 2$ | $1\cdot 2$ | $1\cdot 2$ | $1\cdot 2$ | $1\cdot 2\cdot 3$ | $1\cdot 2$ | $1\cdot 2\cdot 3$ | $1\cdot 2\cdot 3$ | $1\cdot 2\cdot 3$ |
| $1\cdot 3$ | $1\cdot 3$ | $1\cdot 3$ | $1\cdot 2\cdot 3$ | $1\cdot 3$ | $1\cdot 2\cdot 3$ | $1\cdot 3$ | $1\cdot 2\cdot 3$ | $1\cdot 2\cdot 3$ |
| $2\cdot 3$ | $2\cdot 3$ | $1\cdot 2\cdot 3$ | $2\cdot 3$ | $2\cdot 3$ | $1\cdot 2\cdot 3$ | $1\cdot 2\cdot 3$ | $2\cdot 3$ | $1\cdot 2\cdot 3$ |
| $1\cdot 2\cdot 3$ | $1\cdot 2\cdot 3$ | $1\cdot 2\cdot 3$ | $1\cdot 2\cdot 3$ | $1\cdot 2\cdot 3$ | $1\cdot 2\cdot 3$ | $1\cdot 2\cdot 3$ | $1\cdot 2\cdot 3$ | $1\cdot 2\cdot 3$ |

## 2. The Second Category

We now will consider another possible definition for a category of probability spaces, which will give rise to categorical translations of different structures from probability theory than we saw in *Prob*. For this category, we take the same objects (probability spaces and the empty space), but we take a different definition of morphism, and so of our category:

**Definition 2.1.** A *morphism* $\phi : (\Omega, \mathcal{F}, P) \to (\Omega', \mathcal{F}', P')$ of probability spaces in $MeasProb$ is a measurable function, i.e. a function $\phi : \Omega \to \Omega'$ such that $\phi^{-1}(F') \in \mathcal{F}$ for all $F' \in \mathcal{F}'$. $MeasProb$ is the category with objects probability spaces and the empty space, and morphisms measurable functions.

We call the category $MeasProb$ to indicate that this is the full subcategory of the category of measure spaces and measurable functions obtained by taking only measure spaces which are, in fact, probability spaces or the empty space.

We now give some examples of morphisms in this new category.

**Example 2.2.** The map $\phi : C \to D_6$ given by taking $H \mapsto 1$ and $T \mapsto 2$ is a morphism in $MeasProb$. This is very similar to a morphism in $Prob$.

**Example 2.3.** Likewise, the map $\phi : D_6 \to C$ by taking $1, 2, 3 \mapsto H$ and $4, 5, 6 \mapsto T$ is a morphism in $MeasProb$. Once again, this is similar to a morphism in $Prob$.

**Example 2.4.** We can define a morphism $\phi : C \to D_6$ in $MeasProb$ by taking $H \mapsto 1$ and $T \mapsto 1$. Note that this is not a morphism in $Prob$ because it fails to preserve intersections. This morphism doesn't make much sense from the perspective of probability theory, since it suggests that two disjoint events can become a single event.

**Example 2.5.** In $Prob$, we had a morphism $\phi : C \to D_6$ given by taking $\{H\} \mapsto \{1, 2, 3\}$ and $\{T\} \mapsto \{4, 5, 6\}$. This does not correspond to a morphism in $MeasProb$, since it doesn't arise as a function on the sample spaces.

As before, we now consider monomorphisms and epimorphisms, which express the ideas of subprobability spaces and conditional probability spaces.

2.1. **Subprobability Spaces and Conditional Probability Spaces.** Given a probability space $(\Omega, \mathcal{F}, P)$ and a subset $B \subset \Omega$ with $P(B) > 0$, we once again consider the subprobability space $(B, \mathcal{F}_B, P_B)$, where $\mathcal{F}_B = \{A \cap B | A \in \mathcal{F}\}$ and $P_B(A \cap B) = P(A|B)$. Then we can once again define a morphism $\phi_B : (B, \mathcal{F}_B, P_B) \to (\Omega, \mathcal{F}, P)$ by taking $b \mapsto b$ for all $b \in B$. This is clearly a monomorphism, so $(B, \mathcal{F}_B, P_B)$ is, once again, a subobject of $(\Omega, \mathcal{F}, P)$.

While we just saw that subprobability spaces work out mostly the same way they did in the previous category, conditional probability spaces do not work out as neatly. When we defined a morphism $(\Omega, \mathcal{F}, P) \to (B, \mathcal{F}_B, P_B)$ in the previous category, we had $B^c \mapsto \emptyset$, so there is no obvious choice of where to map the elements in $B^c$. We can resolve this problem somewhat clumsily by replacing $B$ by $B \amalg \{0\}$, and letting $\{0\} \in \mathcal{F}_B$ and $P(\{0\}) = 0$. Then we can define $\psi : \Omega \to B \amalg \{0\}$ by taking $b \mapsto b$ for all $b \in B$ and $\bar{b} \mapsto \{0\}$ for all $\bar{b} \notin B$. This gives a morphism of probability spaces in $MeasProb$.

Now, we would like to be able to say that the subprobability space corresponding to $B^c$ is the kernel of the map to the conditional probability space corresponding to $B$. However, the empty space is no longer a zero object. It is still clearly an initial object, since the empty set is initial in $Set$. However, since the empty set is a strict initial object in set, there are no functions $X \to \emptyset$ unless $X \simeq \emptyset$. Thus by the definition of a morphism in $MeasProb$, the empty space cannot be terminal. But $MeasProb$ does have a new terminal object given by $(\{1\}, \{\emptyset, \{1\}\}, P)$, where $P(\{1\}) = 1$. Thus, we do not have a notion of kernel in $MeasProb$, so there is no hope of repeating the results we obtained in $Prob$.

2.2. **Coproducts.** We can define a coproduct in the category $MeasProb$ just as we did in the category $Prob$.

**Definition 2.6.** Given probability spaces $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$, we once again define a probability space $(\Omega_1, \mathcal{F}_1, P_1) \amalg (\Omega_2, \mathcal{F}_2, P_2) = (\Omega_1 \amalg \Omega_2, \mathcal{F}_1 \amalg \mathcal{F}_2, P_1 \amalg P_2)$, where $\Omega_1 \amalg \Omega_2$ is the disjoint union of sets $\Omega_1$ and $\Omega_2$, $\mathcal{F}_1 \amalg \mathcal{F}_2 = \{F_1 \amalg F_2 | F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$, and $(P_1 \amalg P_2)(F_1 \amalg F_2) = \frac{1}{2}(P_1(F_1) + P_2(F_2))$.

As before, we can define maps $i_1 : (\Omega_1, \mathcal{F}_1, P_1) \to (\Omega_1, \mathcal{F}_1, P_1) \amalg (\Omega_2, \mathcal{F}_2, P_2)$ and $i_2 : (\Omega_2, \mathcal{F}_2, P_2) \to (\Omega_1, \mathcal{F}_1, P_1) \amalg (\Omega_2, \mathcal{F}_2, P_2)$ by taking $i_1 : a \mapsto a$ for all $a \in \Omega_1$ and $i_2 : b \mapsto b$ for all $b \in \Omega_2$. It is simple to show that these are morphisms in $MeasProb$.

We now show that this is the coproduct.

**Proposition 2.7.** *$(\Omega_1, \mathcal{F}_1, P_1) \amalg (\Omega_2, \mathcal{F}_2, P_2)$ with the morphisms $i_1$ and $i_2$ is the coproduct of the probability spaces $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$.*

*Proof.* Suppose we have another probability space $(\Omega, \mathcal{F}, P)$ and morphisms $f : (\Omega_1, \mathcal{F}_1, P_1) \to (\Omega, \mathcal{F}, P)$ and $g : (\Omega_2, \mathcal{F}_2, P_2) \to (\Omega, \mathcal{F}, P)$. We can define a map $h : (\Omega_1, \mathcal{F}_1, P_1) \amalg (\Omega_2, \mathcal{F}_2, P_2) \to (\Omega, \mathcal{F}, P)$ by taking $h(a) = a$ for all $a \in \Omega_1$ and $h(b) = b$ for all $b \in \Omega_2$.

To see that $h$ is a morphism, observe that:

$$\begin{aligned} h^{-1}(F) &= \{a \in F_1 | f(a) \in F\} \cup \{b \in F_2 | g(b) \in F\} \\ &= f^{-1}(F) \cup g^{-1}(F) \\ &\in \mathcal{F}_1 \amalg \mathcal{F}_2 \end{aligned}$$

Also, $hi_1(a) = h(a) = f(a)$ for all $a \in \Omega_1$, and $hi_2(b) = h(b) = g(b)$ for all $b \in \Omega_2$.

To show uniqueness, suppose there is another morphism $h' : (\Omega_1, \mathcal{F}_1, P_1) \amalg (\Omega_2, \mathcal{F}_2, P_2) \to (\Omega, \mathcal{F}, P)$ such that $h'i_1 = f$ and $h'i_2 = g$. Then we must have $h'(a) = h'i_1(a) = f(a)$ for all $a \in F_1$ and $h'(b) = h'i_2(b) = g(b)$ for all $b \in F_2$. Thus $h' = h$, and we do have the coproduct in the category $MeasProb$.

$\square$

The coproduct has the same interpretation as it did in the previous category; the coproduct of two probability spaces is the probability space obtained by considering their sample spaces as disjoint events in a larger probability space.

We form countable coproducts the same way as for the previous category.

2.3. **Product.** Considering products in the category $MeasProb$ gives us another way to translate a relationship in probability theory into the language of category theory.

**Definition 2.8.** Suppose we are given two probability spaces, $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$. We define $(\Omega_1, \mathcal{F}_1, P_1) \times (\Omega_2, \mathcal{F}_2, P_2) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, P_1 \times P_2)$ by taking $\Omega_1 \times \Omega_2$ to be the cartesian product of sets $\Omega_1$ and $\Omega_2$, $\mathcal{F}_1 \times \mathcal{F}_2$ the $\sigma$-algebra generated by sets $F_1 \times F_2$ where $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$, and $(P_1 \times P_2)(F_1 \times F_2) = P_1(F_1)P_2(F_2)$ (extended to be countably additive).

We can define morphisms $p_1 : (\Omega_1, \mathcal{F}_1, P_1) \times (\Omega_2, \mathcal{F}_2, P_2) \to (\Omega_1, \mathcal{F}_1, P_1)$ and $p_2 : (\Omega_1, \mathcal{F}_1, P_1) \times (\Omega_2, \mathcal{F}_2, P_2) \to (\Omega_2, \mathcal{F}_2, P_2)$ by taking $p_1 : (a_1, a_2) \mapsto a_1$ and $p_2 : (a_1, a_2) \mapsto a_2$. These are morphisms, since $p_1^{-1}(F) = \{(a_1, a_2) | a_1 \in F\} = F \times \Omega_2$ and $p_2^{-1}(F) = \{(a_1, a_2) | a_2 \in F\} = \Omega_1 \times F$.

We now show that we do have a product:

**Proposition 2.9.** $(\Omega_1, \mathcal{F}_1, P_1) \times (\Omega_2, \mathcal{F}_2, P_2)$ *with the morphisms* $p_1$ *and* $p_2$ *is the product of the probability spaces* $(\Omega_1, \mathcal{F}_1, P_1)$ *and* $(\Omega_2, \mathcal{F}_2, P_2)$.

*Proof.* Suppose we have another probability space $(\Omega, \mathcal{F}, P)$ and morphisms $f : (\Omega, \mathcal{F}, P) \to (\Omega_1, \mathcal{F}_1, P_1)$ and $g : (\Omega, \mathcal{F}, P) \to (\Omega_2, \mathcal{F}_2, P_2)$. Then we can define a map $h : (\Omega, \mathcal{F}, P) \to (\Omega_1, \mathcal{F}_1, P_1) \times (\Omega_2, \mathcal{F}_2, P_2)$ by taking $h(a) = (f(a), g(a))$.

To see that $h$ is a morphism, observe that:

$$
\begin{aligned}
h^{-1}(\bigcup(F_1 \times F_2)) &= \bigcup h^{-1}(F_1 \times F_2) \\
&= \bigcup\{a \in \Omega | f(a) \in F_1, g(a) \in F_2\} \\
&= \bigcup(f^{-1}(F_1) \cap g^{-1}(F_2)) \\
&\in \mathcal{F}
\end{aligned}
$$

Also, $p_1 h(a) = p_1(f(a), g(a)) = f(a)$ and $p_2 h(a) = p_2(f(a), g(a)) = g(a)$, so $p_1 h = f$ and $p_2 h = g$.

To show uniqueness, suppose there is another morphism $h'$ such that $p_1 h' = f$ and $p_2 h' = g$. Denote $h'(a) = (x, y)$. Then $p_1 h'(a) = x$, so $x = f(a)$, and $p_2 h'(a) = y$, so $y = g(a)$. Therefore $h = h'$, and this is the product.

□

We consider an example of this probability space:

**Example 2.10.** Let $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$ both be the probability space corresponding to a fair coin flip, $C$. Then
$(\Omega_1, \mathcal{F}_1, P_1) \times (\Omega_2, \mathcal{F}_2, P_2) = (\{(H,T), (T,H), (H,H), (T,T)\}, \mathcal{F}, P)$, where $\mathcal{F}$ is the power set of $\{(H,T), (T,H), (H,H), (T,T)\}$, and
$P(\{(H,T)\}) = P(\{(T,H)\}) = P(\{(H,H)\}) = P(\{(T,T)\}) = \frac{1}{4}$. This is the probability space obtained by considering two independent coin flips.

This is true for any probability spaces $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$: the product $(\Omega_1, \mathcal{F}_1, P_1) \times (\Omega_2, \mathcal{F}_2, P_2)$ is the probability space obtained by considering the two probability spaces as representing independent random variables, and taking the resulting probability space.

We can similarly define a product of countably many probability spaces, $\{(\Omega_i, \mathcal{F}_i, P_i)\}_{i=1}^{\infty}$. We do this by taking $\prod_{i=1}^{\infty}(\Omega_i, \mathcal{F}_i, P_i) = (\prod \Omega_i, \prod \mathcal{F}_i, \prod P_i)$, where $\prod \Omega_i$ is the cartesian product of the sets $\Omega_i$, $\prod \mathcal{F}_i$ is the $\sigma$-algebra generated by the sets $\prod F_i$ where $F_i \in \mathcal{F}_i$, and $(\prod P_i)(\prod F_i) = \prod(P_i(F_i))$ (extended to be countably additive). The proof that this is a product follows similarly to the above.

## 3. Conclusion

The first category we considered, *Prob*, exhibited some structure of morphisms reminiscent of that of groups, since the kernel of the quotient map taking a probability space to a conditional probability space was a subprobability space in a very predictable way. I believe that there are more interesting observations to be made about morphisms in this category. Not surprisingly, we were able to define an equivalence of a restriction the category with a very specific category of monoids. In *Prob*, we were also able to construct coproducts, which exhibited a certain relationship between two probability spaces. By taking pushouts in this category, we should be able to exhibit different relationships between two probability spaces, but I have not been able to work out the details of this. The main issue with the first category is that (up to isomorphism) all it cares about is the $\sigma$-algebra of the probability space, which is obviously not ideal for working with probability spaces.

In the second category, we were much more limited when considering subprobability spaces and conditional probability spaces. However, we were able to define products as well as coproducts, both exhibiting different possible relationships between probability spaces. Considering pushouts and pullbacks in this category may give us a way to express every possible relationship between probability spaces, though I have not worked out the details of this.

In either case, the probability measure is neglected. In every category that I have considered that would preserved the probability measure in some stronger sense has proved to be painfully restrictive. Although ideally we would have a category that would express all of the above ideas and still be true to probability measure, I doubt that this is possible.

Finally, at no point in this paper was the requirement that the sample space have measure 1 essential to a result. Because of this, all of the paper can be applied to a (similar) category of general measure spaces, and it may be most worthwhile to consider only categories of measure spaces, with an appropriately chosen definition of morphism.

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## References

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