A FOUNDATIONAL CATEGORY

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Abstract. We present several axioms in an attempt to characterize $\mathcal{S}$et . A category which satisfies these axioms has many properties which can be seen to be analogous to properties of $\mathcal{S}$et , and, in fact, the category is equivalent to $\mathcal{S}$et .

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1. Introduction.

In his 1964 paper, "An Elementary Theory of the Category of Sets", William Lawvere presented a set theory with axioms given in purely categorical language. The irreducible term in the usual formalization of set theory is usually the inclusion relation. However, the irreducible term in Lawvere's formalization is the morphism.

We demonstrate that the natural numbers, familiar concepts of functions and subsets, and the axiom of choice can be understood in a category which satisfies Lawvere's axioms. Then we show that any such category which is additionally complete and locally small is equivalent to $\mathcal{S}$et.

2. Preliminaries and axioms.

In the following section, we give several of the axioms which will characterize categories equivalent to $\mathcal{S}$et and several definitions so that we can describe the properties of the categories in familiar set theoretic terms. We also prove several results which follow from the axioms.

We begin with a category $\mathscr{S}$ which has the following axioms. By a category we mean a collection of arrows and objects with the usual relations. However, in this paper we stress the fact that the objects of a category are only special instances of

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arrows: those which are identity arrows. For this reason, we denote both an object and its identity arrow by the same name: for any object X in $\mathscr{S}$

$$X : X \to X$$

is the identity arrow of X.

Axiom 2.1. All finite limits, co-limits, and exponentials exist.

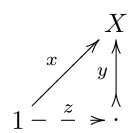
Remark 2.2. We only consider objects in $\mathscr{S}$ as up to isomorphism, and thus all limits, co-limits, and exponentials are considered unique.

In particular, we have a unique initial object 0 and a unique terminal object 1. By !, we will mean the canonical map from any object to 1. For any objects A and B, by π_A we mean the canonical projection $A \times B \to A$, and by ι_A we mean the canonical injection $A \to A + B$.

Definition 2.3. An object G is a *generator* in a category $\mathscr{C}$ if for any $f, g : A \to B$ in $\mathscr{C}$, $f = g$ if and only if for all $a : G \to A$ we have $fa = ga$. An object G' is a *co-generator* if for any $f, g : A \to B$ in $\mathscr{C}$, $f = g$ if and only if for all $b : B \to G'$ we have $bf = bg$.

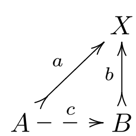
Axiom 2.4. 1 is a generator.

Definition 2.5. Given arrows x and y with y monic, there exists a unique $z : 1 \dashrightarrow \cdot$ such that the following commutes,



we say that x is an *member* of y, denoted $x \in y$ as usual. If y is an identity, it is clear that y is monic and that there exists such a z, i.e. x, for any $x : 1 \to X$. In this case, we say x is an *element* of X, denoted $x \in X$ as usual.

Definition 2.6. If given arrows $a : A \rightarrowtail X$ and $b : B \rightarrowtail X$ there exists a unique $c : A \dashrightarrow B$ such that the following commutes,



we write $a \subset b$. If b is an object, that is if $b = X : X \to X$, it is clear that b is monic and that there exists such a c, i.e. a, for any $a : A \to X$. In this case, we say x is a *subset* of X, denoted $x \subset X$ as usual. We sometimes refer to the domains of the arrows and write $A \subset X$ if $a \subset X$ so that our notation resembles familiar set notation.

Axiom 2.7. For any $f : A \to B$ if A has elements, then there exists $g : B \to A$ such that $f = fgf$.

Remark 2.8. If f is monic, this axiom says there exists g such that $gf = A$, and if f is epic, $fg = B$.

Axiom 2.9. Every object but 0 has elements.

Remark 2.10. Otherwise, $0 \cong 1$. In fact, if $!$ is the canonical arrow $0 \to 1$ and there exists $x : 1 \to 0$, we have $x! : 0 \to 0$ and $!x : 1 \to 1$. However by the universal properties of 1 and 0 there exist only one endomorphism, the identity, on either of these objects. Thus $x! = 0$ and $!x = 1$.

Axiom 2.11. Every element of a co-product is an member of one of the injections.

Axiom 2.12. There exists an object with more than one element.

Remark 2.13. If for an arrow m with domain A we have $ma = ma'$ implies $a = a'$ for all $a, a' \in A$, m is monic. In fact, let $x, y : B \rightrightarrows A$. Then we have $mx = my$ implies $mxb = myb$ for all $b \in B$. By hypothesis, we have $xb = yb$, and by axiom 2.7, $x = y$.

Lemma 2.14. *An object S is the sum of objects A and B if and only if there are arrows $a : A \to S$ and $b : B \to S$ such that any element of S is an member of either a or b but not both.*

Proof. Let $a : A \to S$ and $b : B \to S$ be arrows such that any element of S is an member of either a or b but not both. Let $f, g : Z \rightrightarrows A$ be arrows such that $af = ag$. Then we see that for any $z \in Z$, $afz = agz : 1 \to S$. By hypothesis, there is a unique $\bar{z}$ such that $a\bar{z} = afz = agz$, and thus $fz = gz$. By axiom 2.4, we see that $f = g$, and thus a, and similarly b, is monic.

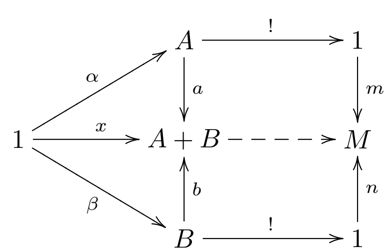
By the universal property of $A + B$ there exists $f : A + B \to S$ such that $f\iota_A = a$ and $f\iota_B = b$. Suppose $fx = fy$ for $x, y \in A + B$ and that $x, y \in A$ without loss of generality. Then there exist $\bar{x}, \bar{y}$ such that $x = \iota_A\bar{x}$, $y = \iota_A\bar{y}$. Then $a\bar{x} = f\iota_A\bar{x} = fx = fy = f\iota_A\bar{y} = a\bar{y}$ which implies $\bar{x} = \bar{y}$ since a is monic. Therefore $x = \iota_A\bar{x} = \iota_A\bar{y} = y$, and we see that f is a monomorphism.

Now let $g : S \to A + B$ such that $gf = A$ as guaranteed by axiom 2.7. Let $x \in S$. Then, by hypothesis and without loss of generality, there exists $\bar{x} : 1 \to A$ such that $a\bar{x} = x$. We have $fgx = fga\bar{x} = fgf\iota_A\bar{x} = f\iota_A\bar{x} = a\bar{x} = x$. Thus, by axiom 2.4, $fg = S$ so S is isomorphic to the sum of A and B.

Therefore, S is the sum of A and B if there are arrows $a : A \to S$ and $b : B \to S$ such that any element of S is an member of either a or b but not both.

Let S be the sum of A and B. By axiom 2.11, any element of S is an element of A or B.

Let $x \in S$ and suppose $x \in a$ and $x \in b$. Let M be an object with two distinct elements, m and n, as guaranteed by axiom 2.12. Then there exist α and β such that the following diagram commutes.



We see that $m!\alpha = n!\beta$, and thus $m = n$. By contradiction, we cannot have both $x \in a$ and $x \in b$.

Therefore, S is the sum of A and B only if any element of S is an member of either a or b but not both.

$\square$

Remark 2.15. We see that the sum of two objects is analogous to the disjoint union of two sets.

We denote the co-product of 1 and 1 as 2:

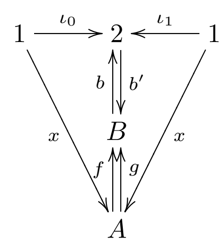
$$1 \xrightarrow{\iota_0} 2 \xleftarrow{\iota_1} 1$$

Lemma 2.16. *2 is a co-generator in* $\mathscr{S}$.

Proof. Let $f, g : A \to B$.

If $f = g$ then clearly $bf = bg$ for all $b : B \to 2$.

Conversely, suppose $f \neq g$. By axiom 2.4, there exists a $x : 1 \to A$ such that $fx \neq gx$. Let $b' = \langle fx, gx \rangle : 2 \to B$ so that $fx = b'\iota_0$ and $gx = b'\iota_1$. Since 2 has elements, by the axiom 2.7 there exists $b : B \to 2$ such that $b' = bb'b$. Thus we have $fx = b'\iota_0 = b'bb'\iota_0 = b'bfx$ and similarly $gx = b'bgx$. Thus we see that $b'bf \neq b'bg$ and so $bf \neq bg$.



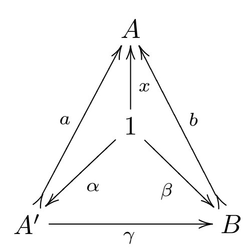
Thus, the contrapositive says that $f = g$ only if $bf = bg$ for all $b : B \to b$. $\square$

Lemma 2.17. *Let* $a, b \subset A$. *Then* $a \subset b$ *if and only if for all* $x \in A$ *we have* $x \in a$ *implies that* $x \in b$.

Proof. If $a \subset b$ and $x \in a$ there exist unique α, γ by definition making the outer and left triangles commute below. Define

$$\beta \equiv \gamma\alpha$$

We have $\beta b = a\alpha = x$. Thus β makes the right triangle commute below, and since b is monic, β is unique in this respect. Therefore $x \in b$.



Conversely, suppose that for all $x \in A$, $x \in a$ implies that $x \in b$. Let c be an arrow such that $bcb = b$ by axiom 2.7. Then define

$$\gamma \equiv ca$$

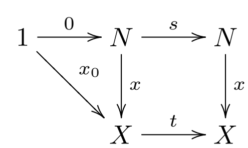
Since for every $\alpha \in A'$ there is a $\beta \in B$ such that $b\beta = a\alpha$ by hypothesis, we have $a\alpha = b\beta = bcb\beta = bca\alpha = b\gamma\alpha$ for all $\alpha \in A'$. By axiom 2.4, $a = b\gamma$, and γ is the unique such arrow since b is monic. Therefore, $a \subset b$. □

The next section will explore the final axiom.

3. The natural numbers.

By the following axiom, there is an object in $\mathscr{S}$ which respresents $\mathbb{N}$ with an initial element and successor morphism. We show that Peano's postulates hold for $\mathbb{N}$ in $\mathscr{S}$.

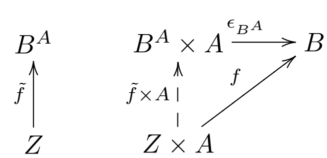
Axiom 3.1. (Natural numbers object.) There exists a triple $(N, 0, s)$ such that for all other triples (X, x_0, t), there exists a unique x such that the following diagram commutes:



N is universal among objects with a distinguished element and endomorphism.

The object N corresponds to the usual notion of the natural numbers with 0 as its first element and s as the successor function. The unique arrow x corresponds to the usual notion of a sequence in X which would be given in usual set notation as $(x(n))_{n\in\mathbb{N}}$ where $x(n) = (t^n \circ x_0)(1)$ for all $n \in \mathbb{N}$.

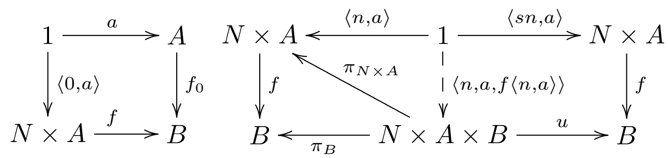
Recall that an exponential of two objects A and B is an object B^A and arrow $\epsilon_{B^A} \colon B^A \times A \to B$, called the evaluation, such that for any $f \colon Z \times A \to B$ there exists a unique $\tilde{f} \colon Z \to B^A$ such that the triangle below commutes:



An arrow denoted ϵ_{B^A} will always refer to the evaluation of B^A for any A, B.

When we have arrows $x : C \to A$ and $y : C \to B$, the arrow $C \to A \times B$ induced by the universal property of $A \times B$ is denoted as $\langle x, y \rangle$. When we have arrows $x \colon A \to C$ and $y \colon B \to D$, the arrow $A \times B \to C \times D$ induced by the arrows $x\pi_A \colon A \times B \to C$ and $y\pi_B \colon A \times B \to D$ is denoted just $\langle x, y \rangle$ for simplicity. It will be clear in context which is meant.

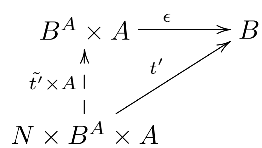
Theorem 3.2. *(Primitive recursion.) Given arrows $f_0 : A \to B$ and $u : N \times A \times B \to B$, there exists $f : N \times A \to B$ such that for any $n \in N$, $a \in A$ we have $f\langle 0, a \rangle = f_0 a$ and $f\langle sn, a \rangle = u\langle n, a, f\langle n, a \rangle \rangle$ as in the following diagrams:*



Proof. We construct f as follows. Let $\Delta = \langle D, D\rangle : D \to D \times D$ be the diagonal arrow for any object D, $\epsilon : B^A \times A \to B$ the evaluation arrow for the exponential B^A. Define $t' \colon N \times B^A \times A \to B$ to be the composite

$$N \times B^A \times A \xrightarrow{N\times B^A \times \Delta} N \times B^A \times A \times A \cong N \times A \times B^A \times A \xrightarrow{N \times A \times \epsilon} N \times A \times B \xrightarrow{u} B$$

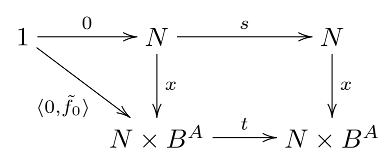
Let $\tilde{t'} : N \times B^A \to B^A$ be the exponential adjoint to t' so that the following diagram commutes:



Define

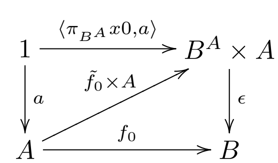
$$t \equiv \langle s\pi_N, \tilde{t'}\rangle : N \times B^A \to N \times B^A$$

Then by axiom 3.1 there exists $x : N \to N \times B^A$ such that the following diagram commutes:

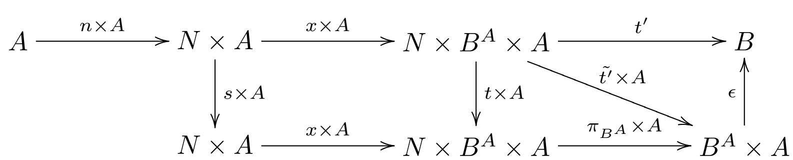


where $\tilde{f}_0 : 1 \to B^A$ is the exponential adjoint of f_0. Then define

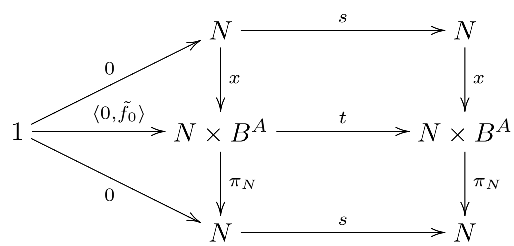
$$f \equiv \epsilon\langle \pi_{B^A} x, A\rangle : N \times A \to B.$$



In consideration of the above diagram, in which the right triangle commutes, we see have that $f\langle 0, a\rangle = \epsilon\langle \pi_{B^A} x, A\rangle\langle 0, a\rangle = \epsilon\langle \pi_{B^A} x0, a\rangle = \epsilon\langle \pi_{B^A}\langle 0, \tilde{f_0}\rangle, a\rangle = \epsilon\langle \tilde{f_0}, a\rangle = f_0 a$.



In consideration of the above commutative diagram, we have that $f\langle sn, a\rangle = \epsilon\langle \pi_{B^A} x, A\rangle\langle sn, a\rangle = \epsilon\langle \pi_{B^A} xsn, a\rangle = \epsilon\langle \pi_{B^A} txn, a\rangle = \epsilon\langle \pi_{B^A}\langle s\pi_N, \tilde{t'}\rangle xn, a\rangle = \epsilon\langle \tilde{t'} xn, a\rangle = t'\langle xn, a\rangle$. By the definitions of t', u, and f that $t'\langle xn, a\rangle = u(N \times A \times \epsilon)(N \times B^A \times \Delta)\langle xn, a\rangle = u(N\times A\times\epsilon)\langle xn, a, a\rangle = u\langle \pi_N xn, a, \epsilon\langle \pi_{B^A} xn, a\rangle\rangle = u\langle \pi_N xn, a, f\langle n, a\rangle\rangle$. Since $\pi_N\langle 0, \tilde{f_0}\rangle = 0$ and $\pi_N t = \pi_N\langle s\pi_N, \tilde{t'}\rangle = s\pi_N$ the following diagram commutes:



In particular, the outer pentagon commutes. Since it is clear that by replacing both instances of $\pi_N x$ with N, the outer pentagon still commutes, and by axiom 3.1 this arrow is unique, we see $\pi_N x = N$ and in particular $\pi_N xn = n$. Thus, we have that $f\langle sn, a\rangle = t'\langle xn, a\rangle = u\langle \pi_N xn, a, f\langle n, a\rangle\rangle = u\langle n, a, f\langle n, a\rangle\rangle$. □

Lemma 3.3. *There exists a* predecessor *function $p : N \to N$ such that it is left inverse to the successor function, and the predecessor of* 0 *is* 0.

Proof. By theorem 3.2, from the arrows $0\colon 1 \to N$ and $\pi_{N_1}\colon N \times 1 \times N \to N$ we can define $p\colon N \times 1 \to N$ such that for all $n \in N$ $p\langle 0, 1\rangle = 0 \circ 1$ and $p\langle sn, 1\rangle = \pi_{N_1}\langle n, s, f\langle n, 1\rangle\rangle$. In essence, from the arrows $0\colon 1 \to N$ and $\pi_{N_1}\colon N \times N \to N$ we can define $p\colon N \to N$ such that for all $n \in N$ $p0 = 0$ and $psn = n$. By axiom 2.4, we have $ps = N$. □

Theorem 3.4. *(Peano's postulates.)*

(i) $0 \in N$ and for any $n \in N, sn \in N$.

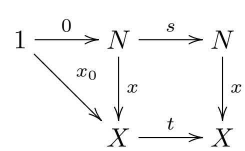
(ii) 0 is not the successor of any $n \in N$.

(iii) The successor function is injective.

(iv) Let $A \subset N$ such that $0 \in A$ and if $n \in A$ then $sn \in A$ for all $n \in N$. Then $A \cong N$.

Proof. (i) By the definition of $0, 0 \in N$ and for any $n \in N$, n is an arrow $n\colon 1 \to N$, so $sn\colon 1 \to N \to N$. Thus, $sn \in N$.

(ii) Suppose there exists $n \in N$ such that $sn = 0$. Then by lemma 3.3, we have $n = psn = p0 = 0$ so that $s0 = 0$. Then for any $t\colon X \to X$ and $x_0 \in X$ by axiom 3.1 there must exist an $x\colon N \to X$ such that the following commutes:



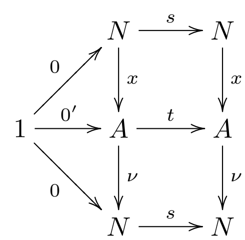
Since $s0 = 0$, $x_0 = x0 = xs0 = tx0 = tx_0$. By axiom 2.4, since this must hold for all $x_0 \in X$, we have that $t = X$ for any object X and any endomorphism $x \to X$. Thus, the only endomorphism for any object is the identity morphism. However, axiom 2.14 ensures the existence of non-identity endomorphisms. By contradiction, we see that 0 is not the successor of any $n \in N$.

(iii) Suppose for x, x' with codomain N, $sx = sx'$. Then $x = psx = psx' = x'$ by lemma 3.3, so s is injective.

(iv) Let $\nu : A \rightarrowtail N$. By axiom 2.7, there exists $\mu' : N \to A$ such that $\nu\mu\nu = \nu$. Define

$$t \equiv \mu s \nu \colon A \to A$$

For all $a \in A$, there exists a unique $n \in N$ such that $n = \nu a$, and for all $n \in \nu$ there exists a unique $a' \in A$ such that $sn = \nu a'$ by hypothesis. Thus, we have that for all $a \in \nu$ there exists a unique $a' \in A$ such that $\nu a' = s\nu a$. We see that $s\nu a = \nu a' = \nu\mu\nu a' = \nu\mu s\nu a$, and by axiom 2.12, $s\nu = \nu\mu s\nu = \nu t$, and thus the bottom square commutes below.

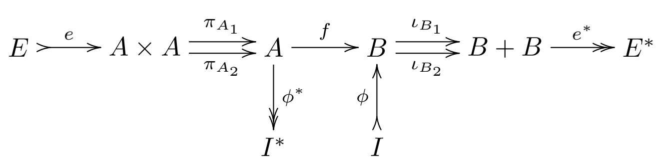


There exists a unique $0' \in A$ such that the bottom triangle commutes by hypothesis, and by axiom 3.1, there exists a unique $x \colon N \to A$ such that the upper half commutes. By the universal property of $(N, 0, s)$, we see that $\nu x = N$. Then $\nu x \nu = \nu$ and, since ν is monic, $x\nu = A$. Therefore, $A \cong N$. □

4. Functions and subsets.

We show that we can take the union and complement of any collection of subsets of a common object in $\mathscr{S}$.

Definition 4.1. Let e be the equalizer of $f\pi_{A_1}$ and $f\pi_{A_2}$, e^* the co-equalizer of $\iota_{B_1} f$ and $\iota_{B_2} f$, ϕ the equalizer of $e^*\iota_{B_1}$ and $e^*\iota_{B_2}$, ϕ^* the co-equalizer of $\pi_{A_1} e$ and $\pi_{A_2} e$ as below:



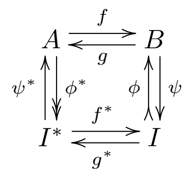
We call the object I the *image* of f and the object I^* the *co-image* of f.

Theorem 4.2. *The image and co-image of a function are isomorphic.*

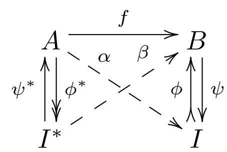
Proof. By axiom 2.7 and 2.9 there exist arrows $\psi^* \colon I^* \to A$, $\psi \colon B \to I$, and $g \colon B \to A$ such that $\phi^*\psi^* = I^*$ and $\psi\phi = I$. Define

$$f^* \equiv \psi f \psi^* \colon I^* \to I$$

$$g^* \equiv \phi^* g \phi \colon I \to I^*$$



By the universal properties of I and I^*, we see that there exist arrows α, β

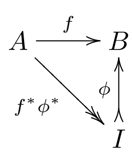


such that $f = \beta\phi^*$ and $f = \phi\alpha$. Thus we have $\phi\psi f = \phi\psi\phi\alpha = \phi\alpha = f$, and similarly $f\psi^*\phi^* = f$.

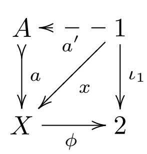
Then we have that $f^*g^*f^* = \psi(f\psi^*\phi^*)g(\phi\psi f)\psi^* = \psi fgf\psi^* = \psi f\psi^* = f^*$.

Now we have that $f\pi_{A_1}\langle\psi^*, \psi^*g^*f^*\rangle = f\psi^* = \phi\psi f\psi^* = \phi f^* = \phi f^*g^*f^* = f\psi^*g^*f^* = f\pi_{A_2}\langle\psi^*, \psi^*g^*f^*\rangle$. Since e is the equalizer of $f\pi_{A_1}$ and $f\pi_{A_2}$, there exists $\epsilon\colon I^* \times I^* \to E$ such that $e\epsilon = \langle\psi^*, \psi^*g^*f^*\rangle$. Then we have since ϕ^* is the co-equalizer of $\pi_{A_1}e$ and $\pi_{A_2}e$, $I^* = \phi^*\psi^* = \phi^*\pi_{A_1}\langle\psi^*, \psi^*g^*f^*\rangle = \phi^*\pi_{A_1}e\epsilon = \phi^*\pi_{A_2}e\epsilon = \phi^*\pi_{A_2}\langle\psi^*, \psi^*g^*f^*\rangle = \phi^*\psi^*g^*f^* = g^*f^*$. Similarly, we have that $f^*g^* = I$. $\square$

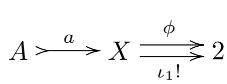
Remark 4.3. Given a function f, we see that f factors through its image I as below.



Definition 4.4. The *characteristic function* of a subset $A \subset X$ is a mapping $\phi\colon X \to 2$ such that $x \in A$ if and only if $\phi x = \iota_1$, that is, there exists a unique a' to make the upper triangle commute if and only if the bottom triangle commutes.



Remark 4.5. Given $\phi\colon X \to 2$ we see that the equalizer of ϕ and $\iota_1!$



is a subset of X such that for $x \in X$, $x \in A$ if and only if $\phi x = \iota_1!$. Thus, subsets and their characteristic functions are in bijective correspondence as we would expect.

Given $\phi\colon X \to 2$ we have an arrow $\tilde{\phi}\colon 1 \to 2^X$ given by the exponential adjoint. Conversely, given an arrow $\tilde{\phi}\colon 1 \to 2^X$ we have an arrow $\phi = \epsilon(\tilde{\phi}\times X)\colon X \to 2$ where $\epsilon\colon 2^X \times X \to 2$ is the evaluation for the exponential 2^X. Thus since characteristic functions of subsets of X and arrows $1 \to 2^X$ are in bijective correspondence so are subsets of X and arrows $1 \to 2^X$.

Definition 4.6. Given a collection of subsets of X, we say that U is their *union* if $x \in U$ if and only if x is a member of one of the subsets.

Theorem 4.7. *Given any collection of subsets of X, their union exists.*

Proof. Let the collection of subsets be indexed by an arrow $\alpha : I \to 2^X$ where I is an 'indexing' object in the sense that the elements $i, j, ... \in I$ induce elements $\alpha i, \alpha j, ... \in 2^X$. By the above remark, we see that $x \in A$ for some $A \subset X$ if and only if there is some $j \in I$ such that $\epsilon_{2^X}\langle \alpha j, x\rangle = \iota_1$. We show there is an object $U \subset X$ which is the union of the indexed subsets in the sense that $x \in U$ if and only if there exists $j \in I$ such that $\epsilon_{2^X}\langle \alpha j, x\rangle = \iota_1$.

Define U by

$$\begin{array}{ccccc}
E & \overset{e}{\rightarrowtail} & I \times X & \overset{\epsilon_{2^X}\langle \alpha, X\rangle}{\longrightarrow} & 2 \\
 & & & \underset{!}{\longrightarrow} 1 \underset{\iota_1}{\longrightarrow} & \\
\downarrow v & & \downarrow \pi_X & & \\
U & \overset{u}{\rightarrowtail} & X & &
\end{array}$$

where E is the equalizer of $\epsilon_{2^X}\langle \alpha, X\rangle$ and $\iota_1!$ and $uv = \pi_X e$ is the factorization of $\pi_X e$ through its image U.

Let $x \in U$. There exists $v' : U \to E$ such that $vv' = U$ by axiom 2.7, and x' such that $ux' = x$ by definition. Define

$$x'' \equiv v'x' \in E$$

$$j \equiv \pi_I e x'' \in I$$

Then $x = ux' = uvv'x' = uvx'' = \pi_X e x''$. Thus we have that $\epsilon\langle \alpha, X\rangle\langle j, x\rangle = \epsilon\langle \alpha, X\rangle\langle \pi_I e x'', \pi_X e x''\rangle = \epsilon\langle \alpha, X\rangle e x'' = \iota_1 ! e x'' = \iota_1$ since $!ex'' : 1 \to 1$ must be the identity of 1.

Conversely, if there exists $j \in I$ and $x \in X$ such that $\epsilon\langle \alpha, X\rangle\langle j, x\rangle = \iota_1 = \iota_1 ! e x''$, then by the universal property of E, there exists $x'' \in E$ such that $ex'' = \langle j, x\rangle$. Then we have that $x = \pi_X\langle j, x\rangle = \pi_X e x'' = u(vx'')$ and the arrow vx'', which is unique since u is monic, shows that $x \in U$. □

Definition 4.8. For two subsets $A, B \subset X$, we say that B is the *complement* of A if $X \cong A + B$.

Theorem 4.9. *The complement of any subset exists.*

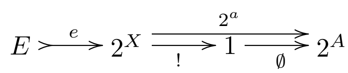
Proof. Let $a : A \rightarrowtail X$. We will construct the complement of A.

Define $2^a : 2^X \to 2^A$ as the exponential adjoint of $\epsilon_{2^X}\langle 2^X, a\rangle : 2^X \times A \to 2$, $\emptyset : 1 \to 2^A$ as the exponential adjoint of $\iota_0! : A \to 2$.

$$\begin{array}{ccc}
2^A \times A & \xrightarrow{\epsilon_{2^A}} & 2 \\
\uparrow \langle 2^a, A\rangle & & \uparrow \epsilon_{2^X} \\
2^X \times A & \xrightarrow{\langle 2^X, a\rangle} & 2^X \times X
\end{array}
\qquad
\begin{array}{ccc}
2^A \times A & \xrightarrow{\epsilon_{2^A}} & 2 \\
\uparrow \langle \emptyset, A\rangle & & \uparrow \iota_0 \\
A & \xrightarrow{!} & 1
\end{array}$$

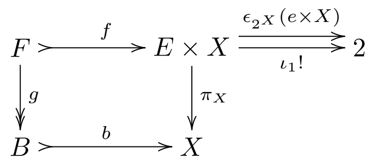
We may think of 2^a as analogous to a function taking subsets of X to their preimage in A, and $\emptyset$ as the subset of A which is empty.

Define $e : E \rightarrowtail 2^X$ to be the equalizer of $2^a, \emptyset! : 2^X \to 2^A$.



Here, E is analogous to a the set of subsets of X whose preimages under a are empty.

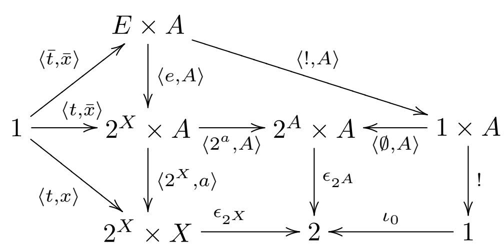
Define $b : B \rightarrowtail X$ to be the union of the collection of subsets indexed by $e : E \rightarrowtail 2^X$ by theorem 4.7.



We show that $b : B \rightarrowtail X$ is the complement of $a : A \rightarrowtail X$.

Now we show that for all $x \in a$ and $t \in e$ we have $\epsilon_{2^X}\langle t, x\rangle = \iota_0$, but for all $x \in b$, there exists $t \in e$ such that $\epsilon_{2^X}\langle t, x\rangle = \iota_1$. Thus, for all $x \in X$, we cannot have both $x \in a$ and $x \in b$.

For all $x \in a$ and $t \in e$, there exist unique $\bar{x}$ and $\bar{t}$ such that $a\bar{x} = x$ and $e\bar{t} = t$ by definition, and we see that the two leftmost triangles below commute. By the definition of 2^a and $\emptyset$, the two sqaures below commute, and by the definition of E, the rightmost triangle commutes.

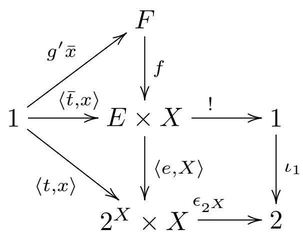


From the diagram, we see that $\epsilon_{2^X}\langle t, x\rangle = \epsilon_{2^X}\langle 2^X, a\rangle\langle t, \bar{x}\rangle = \epsilon_{2^X}\langle 2^X, a\rangle\langle e, A\rangle\langle \bar{t}, \bar{x}\rangle = \epsilon_{2^A}\langle 2^a, A\rangle\langle e, A\rangle\langle \bar{t}, \bar{x}\rangle = \epsilon_{2^A}\langle \emptyset, A\rangle\langle !, A\rangle\langle \bar{t}, \bar{x}\rangle = \iota_0!\langle !, A\rangle\langle \bar{t}, \bar{x}\rangle = \iota_0$ since $!\langle !, A\rangle\langle \bar{t}, \bar{x}\rangle$ must be the identity of 1.

Now for all $x \in b$ there exists a unique $\bar{x}$ such that $b\bar{x} = x$. Let $g' : B \to F$ be such that $gg' = B$ by axiom 2.7. Define

$$t \equiv e\pi_E f g'\bar{x} \in e$$

where $\bar{t} = \pi_E f g'\bar{x}$ is the unique arrow such that $e\bar{t} = t$ since e is monic. We have $x = b\bar{x} = bgg'\bar{x} = \pi_X f g'\bar{x}$ which implies $\langle t, x\rangle = \langle e\pi_E f g'\bar{x}, \pi_X f g'\bar{x}\rangle = \langle e, X\rangle f g'\bar{x}$ so that the following diagram commutes.

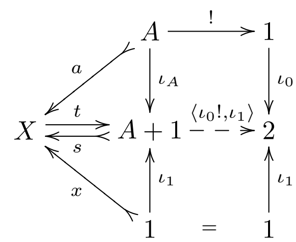


We see that $\epsilon_{2^X}\langle t, x\rangle = \iota_1 ! f g' \bar{x} = \iota_1$.

Therefore we see that there is no $x \in X$ such that both $x \in a$ and $x \in b$.

Now we show that if $x \notin a$, then $x \in b$ so for all $x \in X$, we have that $x \in a$ or $x \in b$.

Let $x \in X$, and suppose $x \notin a$. Let $s : A + 1 \rightarrowtail X$ be arrow induced by a and x and $t : X \to A + 1$ the arrow guaranteed by axiom 2.7 such that $ts = A + 1$ as below.



We see that $ta = ts\iota_A = \iota_A$ and $tx = ts\iota_1 = \iota_1$. Define

$$\tilde{\phi} \equiv \langle \iota_0 !, \iota_1\rangle t$$

Then $\tilde{\phi} a = \iota_0 !$ and $\tilde{\phi} x = \iota_1$. Let $\phi \in 2^X$ be the exponential adjoint of $\tilde{\phi}$. Since $\tilde{\phi} a = \iota_0 !$, by exponential adjointness we see that $2^a \phi = \emptyset = \emptyset ! \phi$. Thus, there is a unique $\psi : 1 \to E$ such that $e\psi = \phi$. Now we see that $\epsilon_{2^X}\langle e, X\rangle\langle \psi, x\rangle = \epsilon_{2^X}\langle \phi, x\rangle = \tilde{\phi} x = \iota_0$. Thus, there is a $\rho : 1 \to F$ such that $f\rho = \langle \psi, x\rangle$. We see that $b(g\rho) = \pi_X\langle \psi, x\rangle = x$ and that $g\rho$ is unique in this regard since b is monic. Thus, $x \in b$. Therefore, for every $x \in X$ either $x \in A$ or $x \in B$.

Since for all $x \in X$, $x \in a$ or $x \in b$ but not both, by lemma 2.14, we see that X is the sum of A and B. Therefore, B is the complement of A by definition. □

5. The axiom of choice.

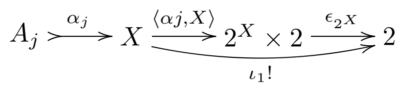
We show that axiom 2.7 is stronger than the Axiom of Choice. In brief, axiom 2.7 ensures that both epimorphisms and monomorphisms spilt, but the Axiom of Choice only implies that the former split. In the following, we do not assume axiom 2.7.

Theorem 5.1. *For any $f : A \to B$, if A has elements, then there exists $g : B \to A$ such that $f = fgf$ if and only if both of the following are true.*

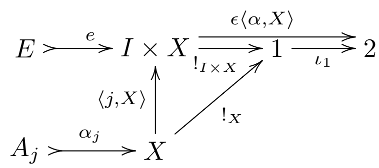
(i) Given $\alpha : I \to 2^X$, there exists $f : I \to X$ such that for all $j \in I$ $fj \in \alpha_j$ where α_j is the equalizer of $\iota_1 !$ and $\epsilon_{2^X}\langle \alpha j, X\rangle$.

(ii) The complement of any subset exists.

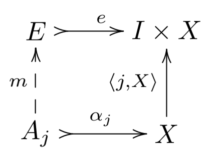
Remark 5.2. Statement (i) is the statement of the usual Axiom of Choice we will use in this proof. In usual set theory, we would say that given an index function, α, of subsets of X, there exists an index function, f, of elements of X such that $f(j) \in \alpha(j)$ for all $j \in I$. However, in this category, αj is only an element of 2^X and so we must construct the subset of X represented by αj as below.



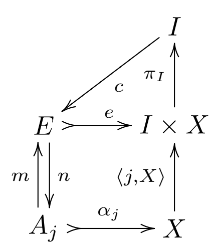
Proof. We show that axiom 2.7 implies statement (i). Given a collection of subsets indexed by an arrow $\alpha\colon I \to 2^X$ and any $j \in I$, let $e\colon E \rightarrowtail I \times X$ be the equalizer of $\epsilon\langle \alpha, X\rangle$ and $\iota_1 !_{I\times X}$, and let $\alpha_j\colon A_j \rightarrowtail X$ be the equalizer of $\epsilon\langle \alpha, X\rangle\langle j, X\rangle$ and $\iota_1 !_X$.



We have that $\epsilon\langle \alpha, X\rangle\langle j, X\rangle\alpha_j = \iota_1 !_X \alpha_j = \iota_1 !_{I\times X}\langle j, X\rangle\alpha_j$, and thus by the universal property of E, there is a unique arrow $m\colon A_j \to E$ such that the following commutes.



Let $n\colon E \to A_j$ be such that $mnm = m$ by axiom 2.7. Then we have that $mn = A_j$ and $nm = E$ by the universal properties of E and A_j, so that n is the inverse of m. Let $c\colon I \to E$ such that $(\pi_I e)c(\pi_I e) = \pi_I e$ as ensured by axiom 2.7.



Define

$$f \equiv \alpha_j nc\colon I \to X$$

Then we see that $\alpha_j(ncj) = fj$, and that ncj is the unique such arrow since α_j is monic. Thus ncj shows that $fj \in \alpha_j$.

Thus we have shown that axiom 2.7 implies (i), and theorem 4.9 uses axiom 2.7 to show (ii). Therefore axiom 2.7 implies (i) and (ii).

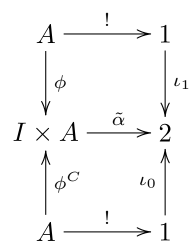
Now we show that statements (i) and (ii) imply axiom 2.7, that is, given $f : A \to B$, we show there exists $g : B \to A$ such that $fgf = f$. In summary, we first construct an arrow $\bar{g}$ from the image of A under f back to A, and then construct

$g : B \to A$ by applying $\bar{g}$ to the image of A and sending everything else to some selected point in A.

Let $i : I \rightarrowtail B$ be the image of $f : A \to B$, and let $\bar{f} : A \to I$ be the arrow given by the universal property of I such that $i\bar{f} = f$. Define

$$\phi \equiv \langle \bar{f}, A\rangle\Delta : A \rightarrowtail I \times A$$

and $\phi^C : A \rightarrowtail I \times A$ as the complement of ϕ by statement (ii). Let $\tilde{\alpha}$ be the arrow given in the following diagram by the universal property of $I \times A$ as the coproduct of ϕ and ϕ^C.

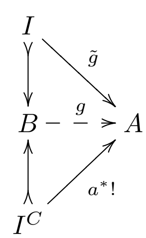


Then we have an arrow $\alpha : I \to 2^A$ by exponential adjointness and consequently an arrow $\bar{g} : I \to A$ by statement (i) such that for all $j \in I$, $\bar{g}j \in \alpha_j$, where $\alpha_j \subset A$ is the equalizer of $\iota_1!$ and $\tilde{\alpha}\langle j, A\rangle$. (Each such $\alpha_j \neq 0$ since, given $j \in J$, by exponential adjointness we have a subset arrows $1 \to 2^J \to 2^A$ by which we can find a suitable a by hypothesis.)

By axiom 2.11, $\langle j, \bar{g}j\rangle \in I \times A$ must be a member of either ϕ or ϕ^C. We have $\tilde{\alpha}\langle j, \bar{g}j\rangle = \tilde{\alpha}\langle j, A\rangle(\bar{g}j) = \iota_1!(\bar{g}j) = \iota_1$, so $\langle j, \bar{g}j\rangle$ must be a member of ϕ.

Thus, there exists $a \in A$ such that $\langle j, \bar{g}j\rangle = \phi a = \langle \bar{f}a, a\rangle$. We see $j = \bar{f}a$ and $\bar{g}j = a$. Hence $\bar{f}\bar{g}j = j$ for all $j \in I$. Therefore, by axiom 2.4, $\bar{f}\bar{g} = I$.

Let I^C be the complement of I. By statement (i), choose some $a^* \in A$. Define $g : B \to A$ by the universal property of B as a coproduct:



Since by the universal property of I, for all $a \in A$, $fa \in I$. Then we have $fa = i\bar{f}a = i\bar{f}\bar{g}\bar{f}a = fgfa$ by the universal property of I. Therefore, by axiom 2.4, $f = fgf$.

Therefore, statements (i) and (ii) imply axiom 2.7. □

6. Set

The above results suggest that a category characterized by the given axioms has properties similar to those of $\mathcal{S}et$. The following theorem shows that, in fact, a category characterized by these axioms is equivalent to $\mathcal{S}et$.

Theorem 6.1. *Let $\mathscr{C}$ be a locally small complete category which satisfies the axioms. Define a functor $F : \mathscr{C} \to \mathcal{S}et$ by*

$$FC = \mathscr{C}(1, C)$$

$$(Ff)c = fc$$

for all objects C in $\mathscr{C}$, arrows $f : C \to D$, and $c \in C$. Then F is an equivalence of $\mathscr{C}$ and $\mathcal{S}et$.

Proof. Given $g : FC \to FD$, construct $f : C \to D$ as follows. By lemma 2.14, C and D are each the sum of 1 over their respective elements. There exists $f : C \to D$ induced by the identity arrows $dom(c) \to dom(gc)$ such that $fc = gc$ for all $c \in C$. Then we have $(Ff)c = fc = gc$ for all $c \in C$ so $Ff = g$. Therefore, F is full.

Let $Ff = Ff'$. Then for all $c \in C$, $fc = (Ff)c = (Ff')c = f'c$ and thus $f = f'$ by axiom 2.4. Therefore, F is faithful.

For any set S, let C in $\mathscr{C}$ be the sum over the elements of S. Then by lemma 2.14, $\mathscr{C}(1, C)$ and S have the same cardinality, and thus are isomorphic. Therefore, F is essentially surjective.

Since F is full, faithful, and essentially surjective, it is an equivalence of $\mathscr{C}$ and $\mathcal{S}et$. $\square$

Acknowledgments. It is a pleasure to thank my mentor, Emily Riehl, for all of her time and patience in helping me learn the basics of category theory and work through this paper.

References

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