THE GYSIN SEQUENCE AND THE HOPF INVARIANT

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Abstract. This paper will establish that there are only four sphere bundles over spheres that are in turn spheres. The first sections consist of introductions to fiber bundles, the basics of cohomology, and the Hopf invariant, while the rest of the paper establishes the Gysin sequence and uses it to prove the final theorem. Some prior knowledge of basic homotopy theory, homology, and CW complexes is assumed, and for the sake of brevity, some results on cohomology are stated rather than proved.

Contents

1. Basics of Fiber Bundles and Fibrations

A fiber bundle can be thought of as a twisted product: given a particular space (the *base space*), we replace every point with a copy of another space. This is called the *fiber* and the resulting object is called the *total space*. We want a nice projection $p : E \to B$ from the total space E to the base space B, and we want the total space to locally look like the product of the fiber and the base space.

Example 1.1. A simple example of a fiber bundle that is not a product is the Möbius band M, where the base space is S^1 and the fiber is the unit interval. Every point on the circle has an open neighborhood U over which M looks like $U \times [0, 1]$, but M is not homeomorphic to the cylinder $S^1 \times [0, 1]$.

Definition 1.2. A *fiber bundle* is a space E together with a projection $p : E \to B$ such that each $p^{-1}(b)$ is homeomorphic to some *fiber* F, and such that for every $b \in B$ there is a neighborhood U such that $p^{-1}(U)$ is homeomorphic to $U \times F$. This last condition is the requirement that a fiber bundle locally look like a product of F and B. Sometimes we will say that E is an F-bundle over B.

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Example 1.3. If a fiber bundle has a discrete fiber, then the definition of a fiber bundle becomes that of a covering space $p : E \to B$. One example of this is S^n as a two point bundle over $\mathbb{R}\mathrm{P}^n$ that comes from the two-sheeted covering of $\mathbb{R}\mathrm{P}^n$ by S^n.

We will usually write fiber bundles with fiber F, base space B, and total space E as a "short exact sequence of spaces" $F \xrightarrow{i} E \xrightarrow{p} B$, where i is an inclusion of the fiber into the total space. At the core of this paper are the *Hopf bundles*, in which all three spaces are spheres, and these are born of the following example.

Example 1.4 (Projective Spaces)**.** In the real projective case we have $S^0 \to S^n \to \mathbb{R}\mathrm{P}^n$ as above. The complex analogue of this is the bundle $S^1 \to S^{2n-1} \to \mathbb{C}\mathrm{P}^n$. Here S^{2n-1} is identified with the unit sphere in $\mathbb{C}^n$, and the projection is the map that sends a point to its equivalence class given by the relation $(v_0, \dots, v_n) \sim \lambda(v_0, \dots, c_n)$ for $\lambda \in S^1$.

From the examples above come the Hopf bundles. In all three, the total space will be the unit sphere in $\mathbb{C}^2$, $\mathbb{H}^2$, or $\mathbb{O}^2$, and the projection map will take (z_0, z_1) to $z_0 z_1^{-1}$.

Example 1.5. The previous example specializes in the case of $n = 1$ to the bundle $S^1 \to S^3 \to S^2 = \mathbb{C}\mathrm{P}^1$. The projection can be given explicitly by $(z_0 z_1) \mapsto z_0/z_1$.

Example 1.6. By replacing $\mathbb{C}$ with $\mathbb{H}$, we get another bundle $S^3 \to S^{4n-3} \to \mathbb{H}\mathrm{P}^n$, and again, taking $n = 1$ yields a sphere bundle $S^3 \to S^7 \to S^4 = \mathbb{H}\mathrm{P}^1$.

Example 1.7. We move to the Cayley octonions $\mathbb{O}$ to get another example. Again with the special case of $n = 1$, we get a Hopf bundle $S^7 \to S^{15} \to S^8$. However, $\mathbb{O}$ is not associative, and thus some extra work is necessary. We have that S^{15} is the unit sphere in $\mathbb{O}^2$, and the projection $p : S^{15} \to S^8$ is given by the map $(z_0, z_1) \mapsto z_0 z_1^{-1} \in \mathbb{O} \cup \infty = S^8$, but it is not immediately clear that this gives a fiber bundle with the desired fiber.

Let U_0 and U_∞ be the complements of 0 and ∞ in the one-point compactification of $\mathbb{O}$. We claim that the following pairs of maps $h_i : p^{-1}(U_i) \to U_i \times S^7$ and $g_i : U_i \times S^7 \to p^{-1}(U_i)$ are inverse homeomorphisms, and thus that we have a fiber bundle with correct fiber:

$$h_0(z_0, z_1) = (z_o z_1^{-1}, z_1/|z_1|) \quad \text{and} \quad g_0(z, w) = (zw, w)/|(zw, w)|$$
$$h_\infty(z_0, z_1) = (z_o z_1^{-1}, z_1/|z_1|) \quad \text{and} \quad g_\infty(z, w) = (w, z^{-1}w)/|(w, z^{-1}w)|$$

Assuming the fact that any subalgebra of $\mathbb{O}$ generated by two elements is associative, it is a straightforward calculation to check that g_i and h_i are inverse homeomorphisms. For more details we refer to [4] and [3].

It is the goal of this paper to show that these are the only sphere bundles in which the base space and total space are both spheres. We end this section with a generalization of the concept of a fiber bundle.

Definition 1.8. A *fibration* is a map $p : E \to B$ that has the *homotopy lifting property* with respect to all spaces X. That is, given a lift $\tilde{f}_0 : X \to E$ of a map $f_0 : X \to B$ and a homotopy $f_t : X \to B$, there exists a homotopy $\tilde{f}_t : X \to B$ of $\tilde{f}_0$ that projects to the homotopy of maps into the base space.

We will use later without proof the fact that, for CW complexes, all fiber bundles are fibrations. One way to think of a fibration is a fiber bundle in which the fibers need not be homeomorphic, but only homotopy equivalent. This equivalence is evident by taking X to be the fiber over a basepoint b of B: let $y \in B$ be any point, and for some path $\gamma : I \to B$ from b to y let $f_t(x) = \gamma(t)$, and the homotopy lifting property gives a homotopy equivalence between the fibers over the two points.

2. Cohomology and Poincaré Duality

In this section, we quickly document some necessary definitions and results about cohomology. For a longer exposition we refer to [4] and [2]. We will assume some basic facts about homology groups, relative homology groups, and homology with coefficients in an arbitrary abelian group G.

Definition 2.1. Given a chain complex $C_* = C_*(X; G)$ associated to a space X with coefficients in G, the *cochain complex* $C^* = C^*(X; G)$ associated with the space X is given by the dual groups $C^n = \operatorname{Hom}(C_n, G)$, the group of homomorphisms from C_n to G, connected via the *coboundary maps* $\delta_n = \partial_n^* : C^n \to C^{n+1}$. Elements of C^n are *cochains*, elements of $\ker \delta_n$ *cocycles*, and elements of $\operatorname{im} \delta_{n+1}$ *coboundaries*.

A simple calculation verifies that $\delta^2 = 0$, and thus it is possible to take homology groups of the resulting complex.

Definition 2.2. Given a cochain complex $0 \to C^0(X; G) \to C^1(X; G) \to \cdots$, the associated cohomology groups are defined to be $H^n(X; G) = \ker \delta_n / \operatorname{im} \delta_{n+1}$.

Just as in homology, we can define relative cohomology groups $H^n(X, A; G)$, and many of the basic results from homology have cohomological equivalents. Here we record three for later use.

Proposition 2.3 (Long exact sequence of a pair)**.** *If $A \subset X$, then there is a long exact sequence in cohomology groups*

$$\cdots \longrightarrow H^n(X, A; G) \xrightarrow{j^*} H^n(X; G) \xrightarrow{i^*} H^n(A; G) \xrightarrow{\delta} H^{n+1}(X, A; G) \longrightarrow \cdots$$

where i^* and j^* are the maps induced by $i : C_n(A; G) \to C_n(X; G)$ and $j : C_n(X; G) \to C_n(X, A; G)$.

Proposition 2.4 (Excision)**.** *For $Z \subset A \subset X$, with the closure of Z in the interior of A, the inclusion $i : (X - Z, A - Z) \hookrightarrow (X, A)$ induces isomorphisms $i^* : H^n(X, A; G) \to H^n(X - Z, A - Z; G)$ for all n.*

Proposition 2.5 (Mayer-Vietoris)**.** *If X is the union of the interiors of A and B, then there is a long exact sequence*

$$\cdots \longrightarrow H^n(X) \xrightarrow{\Psi} H^n(A) \oplus H^n(B) \xrightarrow{\Phi} H^n(A \cap B) \longrightarrow H^{n+1}(X) \longrightarrow \cdots$$

where the maps Ψ and Φ are induced by inclusion. A relative version similar to that for homology holds as well.

A major advantage of cohomology over homology is the possibility of turning the collection of cohomology groups into a single graded ring via the *cup product*. To do this it is necessary to take cohomology groups with coefficients in a ring R, which will be assumed to be $\mathbb{Z}$ unless otherwise specified.

Definition 2.6. For cochains $\phi \in C^k(X;R), \psi \in C^l(X;R)$, the cup product $\phi \smile \psi$ is a cochain in $C^{k+l}(X;R)$ that takes the value on a $(k+l)$-simplex σ as follows:

$$(\phi \smile \psi)(\sigma) = \phi(\sigma|[v_0, \dots, v_k])\psi(\sigma|[v_k, \dots, v_{k+l}]).$$

Straightforward calculation gives an induced cup product on cohomology:

$$H^k(X;R) \times H^l(X;R) \xrightarrow{\smile} H^{k+l}(X;R).$$

This unifies the cohomology groups into a graded ring $H^*(X;R)$. The *relative cup product* can be defined similarly, since if a cochain vanishes on chains in A then so does its cup product with any other cochain.

$$H^k(X,A;R) \times H^l(X;R) \xrightarrow{\smile} H^{k+l}(X,A;R)$$
$$H^k(X,A;R) \times H^l(X,A;R) \xrightarrow{\smile} H^{k+l}(X,A;R)$$

We record here two facts about the cup product that will be necessary later.

Proposition 2.7. *For $\phi \in C^k(X,A;R)$ and $\psi \in C^l(X,A;R)$,*

$$\delta(\phi \smile \psi) = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi.$$

Proposition 2.8 (Skew-commutativity of cup product)**.** *If R is a commutative ring, then $\alpha \smile \beta = (-1)^{kl}\beta \smile \alpha$ for all $\alpha \in H^k(X,A;R)$ and $\beta \in H^l(X,A;R)$.*

Poincaré duality is a symmetry between the homology and cohomology of very nice spaces. By very nice we mean that our space M must be a *closed, orientable n-manifold.*

Definition 2.9. A closed n-manifold is a compact topological space in which each point has an open neighborhood homeomorphic to $\mathbb{R}^n$.

Definition 2.10. Let M be an n-manifold. An orientation at a point $x \in M$, called a *local orientation* of M at x, is a choice of generator of $H_n(M, M-x)$, which is infinite cyclic by excision. An *orientation* of M is a selection of local orientations at every point $x \in M$ that is locally consistent, i.e. such that every point x has an open neighborhood U homeomorphic to $\mathbb{R}^n$ such that for every $y \in U$ the orientation of M at y is the image of one generator of $H_n(M, M-U)$. A manifold is *orientable* if an orientation exists.

Example 2.11. The surface of genus g is orientable. To see this, observe that choosing a local orientation at a point on a surface amounts to choosing a generator of the first homology group of an annulus, i.e. $\mathbb{Z}$. It is clearly possible to assign "clockwise" to every point on the surface of genus g without any contradiction. However, the Mobius strip is not orientable, because any attempt to assign local orientations to points along the central circle will result in a point that must have both orientations.

Fact 2.12. Every manifold has a two sheeted covering space $M' = \{(x, \mu_x) \mid x \in M \text{ and } \mu_x \text{ is a local orientation at } x\}$. If we assume that M is connected, then M is orientable if and only if M' has two components. For example, if M is the Klein bottle then M' is the torus, and if M is the torus then M' is just the disjoint union of two tori.

We can generalize the idea of orientability to work with coefficients in any commutative ring R. Similarly to the previous definition, an *R-orientation* of M assigns to each $x \in M$ a unit $u \in H_n(M, M-x; R) = R$. If not mentioned, we assume R to be $\mathbb{Z}$. The following is a theorem we need to show the existence of a distinguished element of $H_n(M)$ called a *fundamental class.*

Definition 2.13. A *fundamental class* for M is an element $[M] \in H_n(M)$ whose image in $H_n(M, M-x)$ is a generator for all x.

Theorem 2.14. *Let M be a closed and connected n-manifold. If M is R-orientable, then then the map $H_n(M; R) \to H_n(M, M-x; R) = R$ is an isomorphism for all x. If it is not R-orientable, then the map is injective with image $\{r \mid 2r = 0\}$. Furthermore, $H_i(M; R) = 0$ for $i > n$.*

The theorem above then states that if M is an R-orientable, closed, connected n-manifold, then M has a fundamental class. Before Poincaré duality, we need to define the *cap product.*

Definition 2.15. For $k \geq l$ the *cap product* is the map $\frown: C_k(X; R) \times C^l(X; R) \to C_{k-l}(X; R)$ defined by

$$\sigma \frown \phi = \phi(\sigma|[v_0, \ldots, v_l])\, \sigma|[v_l, \ldots, v_k].$$

Similarly to the cup product, this induces a map on homology and cohomology groups.

Theorem 2.16 (Poincaré Duality)**.** *Let M be a closed R-orientable n-manifold with fundamental class $[M] \in H_n(M; R)$. Then the map $D : H^k(M; R) \to H_{n-k}(M; R)$ given by $\alpha \mapsto [M] \frown \alpha$ is an isomorphism for all k.*

Proof. We omit the somewhat lengthy proof here and refer the reader to [4] or [5]. □

As an example of the usefulness of Poincaré duality, we include here a calculation of the cohomology ring of projective spaces (straightforward calculation is possible, but much more tedious). Before this we cite a corollary of Poincaré duality.

Corollary 2.17. *If M is as above, then for each $a \in H^k(M; \mathbb{Z})$ of infinite order that is not a multiple of another element, there is a $b \in H^{n-k}(M; \mathbb{Z})$ such that $a \smile b$ is a generator of $H^n(M; \mathbb{Z})$. With coefficients in a field, the result holds for any $a \neq 0$.*

Definition 2.18. The *dimension* of $\alpha \in H^*(X; R)$, denoted by $|\alpha|$, is the integer n such that $\alpha \in H^n(X; R)$.

Proposition 2.19. *For real projective space, $H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$, where $|\alpha| = 1$. For complex projective space, $H^*(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$, where $|\alpha| = 2$.*

Proof. We prove the result for real projective space with coefficients in $\mathbb{Z}_2$. The inclusion $\mathbb{R}\mathrm{P}^{n-1} \hookrightarrow \mathbb{R}\mathrm{P}^n$ induces an isomorphism on cohomology up to dimension $n-1$, so by induction on n we get that $H^i(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$ is generated by α^i, where α is a generator in dimension 1. The corollary implies the existence of an integer k such that $\alpha \smile k\alpha^{n-1} = k\alpha^n$ generates $H^i(\mathbb{R}\mathrm{P}^n)$, and thus k can only be ± 1. Therefore, $H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$. The complex case is proved with a similar argument, but with coefficients in $\mathbb{Z}$. □

Remark 2.20. A similar result holds via a similar proof for quaternionic projective space. In octonionic projective space, a similar result holds for $\mathbb{O}\mathrm{P}^2$, in which the attaching map is given by the Hopf map in Example 1.7. However, the lack of associativity prevents defining an octonionic analogue of $\mathbb{R}\mathrm{P}^n$ for $n \geq 2$.

3. The Hopf Invariant and Some Examples

The Hopf invariant is a useful cohomological variant of the degree of a map $f : S^n \to S^n$. We will see that it only makes sense with respect to cohomology because we need the existence of the cup product.

Definition 3.1. The *degree* of a map $f : S^n \to S^n$ is the integer d such that the induced map $f_* : H_n(S^n) \to H_n(S^n)$ is multiplication by d.

The degree of such a map is a homotopy invariant and can be used to distinguish between homotopy classes of maps from S^n to S^n. So what if we want a homotopy invariant of maps $f : S^m \to S^n$ even if $m \neq n$? Begin with the following construction:

If we have a map $f : S^m \to S^n$ with $m \geq n$, then let C_f be the CW complex formed by attaching an $(m+1)$-cell to S^n via f. This is called the *mapping cone* of f, the quotient of the mapping cylinder of f by S^m. It is easy to show that the homotopy type of this complex depends only on the homotopy type of f. If $m = n$, we can determine the homotopy type of f via homology:

Fact 3.2. If d is the degree of $f : S^n \to S^n$, then $H_n(C_f) = \mathbb{Z}_{|d|}$.

This determines precisely the homotopy type of f. However, if $m > n$, we have $H_i(C_f) = \mathbb{Z}$ for $i = 0$, n, and $m + 1$ (and 0 otherwise), regardless of the homotopy type of f. This is also the case for $H^*(C_f)$, but now we can use the cup product to glean information about f if $m = 2n-1$. In this case, n and $2n$ are the two nonzero dimensions with nontrivial cohomology. If we choose generators $a \in H^n(C_f)$ and $b \in H^{2n}(C_f)$, then the Hopf invariant comes from the multiplicative structure on $H^*(C_f)$, specifically the relation between a and b.

Definition 3.3. The *Hopf invariant* of a map $f : S^{2n-1} \to S^n$ is the integer $H(f)$ such that $a^2 = a \smile a = H(f)b$.

This is defined only up to sign, but can be forced to be positive via composing f with a reflection of S^{2n-1} (which has degree -1). The Hopf invariant is a homotopy invariant because if $f \simeq g$ then $C_f \simeq C_g$ and thus, up to sign, the generator $b_f \in H^{2n}(C_f)$ corresponds to the generator $b_g \in H^{2n}(C_g)$.

Example 3.4. Let f be a constant map. Then C_f is $S^n \vee S^{2n}$, so what is $H(f)$? Since C_f retracts to just S^n, we can deduce that $a^2 = 0$ and thus $H(f) = 0$.

Example 3.5. If n is odd, by the skew-commutativity of cup product we get that $a^2 = -a^2$ and thus again $H(f)$ is 0.

So what are some maps with nonzero Hopf invariant? The first example comes from the lowest dimensional Hopf bundle $S^1 \to S^3 \to S^2$, where the second map is the attaching map for the 4-cell of $\mathbb{C}\mathrm{P}^2$. We see from the fact that $H^*(\mathbb{C}\mathrm{P}^2; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^3)$ that we must have $H(f) = 1$. From the higher Hopf bundles, we also get maps of Hopf invariant 1 from the attaching maps of the 8-cell and 16-cell of $\mathbb{H}\mathrm{P}^2$ and $\mathbb{O}\mathrm{P}^2$, respectively.

Theorem 3.6. *Maps $f : S^{2n-1} \to S^n$ of Hopf invariant 1 only exist when $n = 2, 4, 8$, as in the three examples above.*

For the proof of this result we refer the reader to [1] for the original proof and to [5] for a more modern treatment. This theorem can be used to prove the following facts:

Corollary 3.7. *$\mathbb{R}^n$ can only be given the structure of a division algebra if $n = 1, 2, 4, 8$.*

Corollary 3.8. *S^n is an H-space only if $n = 0, 1, 3, 7$.*

Corollary 3.9. *The only fiber bundles for which the fiber, base space, and total space are all spheres are $S^0 \to S^1 \to S^1$, $S^1 \to S^3 \to S^2$, $S^3 \to S^7 \to S^4$, and $S^7 \to S^{15} \to S^8$.*

In this paper we will prove Corollary 3.9.

4. The Leray-Hirsch Theorem

The Leray-Hirsch theorem gives the conditions for the cohomology ring of a fiber bundle to "look like" that of a product bundle. Throughout this section, let all cohomology groups be with coefficients in some commutative ring R unless otherwise specified.

Theorem 4.1 (Leray-Hirsch)**.** *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fiber bundle such that the following two conditions hold:*

(1) $H^n(F)$ is a finitely generated free R-module for all n.
(2) There exist $c_k \in H^*(E)$ such that $i^*(c_k)$ form a basis for $H^n(F)$ for any inclusion of F into E.

Then the map $\Phi : H^*(B) \otimes H^*(F) \to H^*(E)$ is an isomorphism of R-modules, where $\sum_{jk} b_j \otimes i^*(c_k) \mapsto \sum_{jk} p^*(b_j) \smile c_k$. That is, $H^*(E)$ is a free $H^*(B)$-module.

Warning 4.2. This theorem does not assert that the map Φ is an isomorphism of *rings*.

Proof. We start by proving the result for a finite-dimensional CW complex B, and proceed by induction. If B is 0-dimensional, then it is discrete and the result is clear. Now, let B have dimension n and remove a single point from each n-cell of B to get B', which deformation retracts to the $(n-1)$-skeleton of B. Now we let E' be the corresponding subspace of E (i.e. $p^{-1}(B')$). We get the following commutative diagram, where all the horizontal maps are from the long exact sequences of the pairs (B, B') and (E, E'):

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H^*(B,B')\otimes H^*(F) & \longrightarrow & H^*(B)\otimes H^*(F) & \longrightarrow & H^*(B')\otimes H^*(F) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \Phi''} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi'} & & \\
\cdots & \longrightarrow & H^*(E,E') & \longrightarrow & H^*(E) & \longrightarrow & H^*(E') & \longrightarrow & \cdots
\end{array}$$

Note that Φ'' is defined similarly to the map in theorem 4.1, but instead using the relative cup product. The top row is exact because tensoring by a free module preserves exactness of the long exact sequence in cohomology. The bottom row is clearly exact, as it is the long exact sequence of the pair (E, E'). The commutativity of the two squares shown follows from the natural definition of Φ. Let us consider the third square (not shown in the diagram above). Start with $b\otimes i^*(c_j) \in H^*(B')\otimes H^*(F)$. Going over and down, we get first $\delta b \otimes i^*(c_j)$, then $p^*(\delta b) \smile c_j$. Going down and over, we get $p^*(b) \smile c_j$, then $\delta(p^*(b) \smile c_j) = \delta p^*(b) \smile c_j$ because $\delta c_j = 0$ (this is because c_j originates in $H^*(E)$ and the bottom row is exact). Finally, $p^*(\delta b) \smile c_j = \delta p^*(b) \smile c_j$, and this completes the proof of commutativity.

We use the following quick lemma to show that B' being homotopy equivalent to the $(n-1)$-skeleton of B implies that the inclusion of $p^{-1}(B^{n-1})$ into $p^{-1}(B')$ is a (weak) homotopy equivalence. Recall that a space X is *k-connected* if $\pi_i(X) = 0$ for all $i \leq k$.

Lemma 4.3. *Given a fiber bundle $p : E \to B$ with a subspace A of B such that (B, A) is k-connected, then $(E, p^{-1}(A))$ is also k-connected.*

Proof. Start with a map $g : (D^i, \partial D^i) \to (E, p^{-1}(A))$, for $i \leq k$. Then by hypothesis there is a homotopy $f_t : (D^i, \partial D^i) \to (B, A)$ of $f_0 = pg$ to some map f_1 with image completely in A. The homotopy lifting property of the fiber bundle gives a homotopy g_t to some map g_1 with image completely in $p^{-1}(A)$. □

The lemma gives the desired weak homotopy equivalence because (B', B^{n-1}) is k-connected for arbitrary k, and thus so is $(E', p^{-1}(B^{n-1}))$. We now return to complete the proof of Theorem 4.1.

Recall that B' was obtained by removing a point x_i from each n-cell, and that $E' = p^{-1}(B')$. By the fiber bundle property each such x_i has an open neighborhood U_i over which the bundle looks like the product bundle. Let U be the union of the U_i and let $U' = B' \cap U$. Excision gives us two isomorphisms: $H^*(B, B') \cong H^*(U, U')$ and $H^*(E, E') \cong H^*(p^{-1}(U), p^{-1}(U'))$. Therefore, to show that Φ'' in the diagram above is an isomorphism, it suffices to show that $\Phi'' : H^*(U, U') \otimes H^*(F) \to H^*(U \times F, U' \times F)$ is an isomorphism. This result is obtained by the following inductive argument:

Begin by considering the diagram above, but with (U, U') replacing (B, B') and $(p^{-1}(U), p^{-1}(U'))$ replacing (E, E'). Let us assume the result for dimension $i \leq n-1$ and note that U and U' deformation retract to 0-dimensional and $(n-1)$-dimensional complexes, respectively. Then the five lemma gives us the isomorphism $\Phi'' : H^*(U, U') \otimes H^*(F) \to H^*(U \times F, U' \times F)$, and this in turn shows that Φ'' in the diagram is an isomorphism. Note that it is the lemma above that allows us to consider the lower dimensional retracts and apply induction. Now that we have established that Φ'' is an isomorphism, we return the original diagram and apply lemma 4.3 and the induction hypothesis to see that Φ' is an isomorphism, so that we may now use the five-lemma to show that Φ is an isomorphism as well.

Now suppose that B is an infinite dimensional CW complex. The pair (B, B^n) is n-connected, and thus by lemma 4.3, so is $(E, p^{-1}(B^n))$. Thus, in the following diagram, the maps f and g are isomorphisms for dimension $i \leq n$. The Φ on the right is an isomorphism by the finite dimensional case; therefore, so is the Φ on the left, for arbitrary n, and this gives the result for an arbitrary CW complex B.

$$\begin{array}{ccc} H^i(B) \otimes H^i(F) & \xrightarrow{f} & H^i(B^n) \otimes H^i(F) \\ \downarrow \Phi & & \downarrow \Phi \\ H^i(E) & \xrightarrow{g} & H^i(p^{-1}(B^n)) \end{array}$$

In this paper only the CW result is required, since the topic at hand is sphere bundles over spheres. In the general case of a fiber bundle $p : E \to B$, the result is obtained from the CW case by first taking a *CW approximation* $f : A \to B$, then forming the *pullback bundle* $f^*(E)$, then showing that the result for $p : E \to B$ follows from the approximation. □

5. The Gysin Sequence

The cohomology of sphere bundles can be examined with relatively elementary methods via the *Gysin sequence.* Here we follow [4] and establish the result for all disk bundles; for a simpler treatment restricted to manifolds we refer the reader to [2].

Theorem 5.1 (Gysin Sequence)**.** *Suppose that E is an $(n-1)$-sphere bundle over B. Then there is a long exact sequence:*

$$\cdots \longrightarrow H^{i-n}(B) \xrightarrow{\smile e} H^i(B) \xrightarrow{p^*} H^i(E) \longrightarrow H^{i-n+1}(B) \longrightarrow \cdots$$

where e is a particular Euler class *in $H^n(B)$, to be defined later.*

If the bundle in question is a product $E = S^{n-1} \times B$, then there is a *section*, a map $s : B \to E$ such that $ps = \mathrm{id}_B$. Then the injectivity of p^* and exactness implies that the Gysin sequence breaks into split short exact sequences:

$$0 \longrightarrow H^i(B) \xrightarrow{p^*} H^i(E) \longrightarrow H^{i-n+1}(B) \longrightarrow 0.$$

This is true as long as a section exists, regardless of whether or not the bundle is a product bundle. Before the proof, we give an example.

Example 5.2. Let T be the unit tangent bundle to S^n, i.e. the fiber bundle $S^{n-1} \to T \xrightarrow{p} S^n$, where the fiber over every point is the space of unit vectors tangent to S^n canonically embedded in $\mathbb{R}^{n+1}$. This is sometimes called the *real Stiefel manifold* $V_{n+1,2}$, or the space of 2-frames in $\mathbb{R}^{n+1}$. A section of this bundle is the same thing as a field of unit tangent vectors on S^n, which exists if and only if n is odd. Therefore, the Gysin sequence tells us that T has the same cohomology groups as $S^{n-1} \times S^n$ when n is odd.

The route followed here begins with a definition and a relative version of the Leray-Hirsch theorem.

Definition 5.3. If $p : E \to B$ is a fiber bundle and E' is a subspace of E, then (E, E') is a *fiber bundle pair* if there is a subspace F' of the fiber F such that E' is an F'-bundle over B, and the local trivializations for E' are given by restrictions from E.

Theorem 5.4 (Relative Leray-Hirsch)**.** *Given a fiber bundle pair $(F, F') \to (E, E') \to B$ such that $H^*(F, F')$ is a free R-module, finitely generated in each dimension, and classes c_j in $H^*(E, E')$ that restrict to a basis for $H^*(F, F')$, then $H^*(E, E')$ is a free $H^*(B)$-module with basis $\{c_j\}$.*

Proof. This result follows from the absolute version of the Leray-Hirsch theorem, and we refer the reader to [4] for a complete proof. □

Remark 5.5. The scalar multiplication that gives $H^*(E, E')$ the $H^*(B)$-module structure is now the *relative* cup product $H^*(E) \otimes H^*(E, E') \to H^*(E, E')$, where for $b \in H^*(B)$ and $c \in H^*(E, E')$ we have $bc = p^*(b) \smile c$.

Definition 5.6. Consider the special case in which $F = D^n$ and $F' = S^{n-1}$. If there is an element $\tau \in H^n(E, E')$ that restricts to a generator of $H^n(D^n, S^{n-1})$ on each fiber, then it is called a *Thom class*.

Clearly $H^n(D^n, S^{n-1})$ is a free and finitely generated R-module, since it is isomorphic to R. Therefore, if there is a Thom class in $H^n(E, E')$ then the conditions of the relative Leray-Hirsch theorem are satisfied, and so there is an isomorphism between $H^*(B)$ and $H^*(E, E')$:

Corollary 5.7. *If there exists a Thom class τ for E, where E is an n-disk bundle over B, then for every i there is an isomorphism $\Phi : H^i(B) \to H^{i+n}(E, E')$ given by $b \mapsto p^*(b) \smile \tau$.*

Now begins the task of establishing the existence of a Thom class for orientable disk bundles, where a disk bundle is orientable if the fibers of the associated sphere bundle can be oriented in a continuous and consistent way.

Theorem 5.8. *An orientable disk bundle $D^n \to E \xrightarrow{p} B$ has a Thom class with $\mathbb{Z}$ coefficients.*

Proof. As in the Leray-Hirsch theorem, the non-CW case is covered by considering a pullback bundle over a CW approximation for B (but this result is unnecessary at present). So now assume that B is a CW complex. For each x in B, let (D^n_x, S^{n-1}_x) be the fiber over that point. Suppose first that B has finite dimension k.

Claim: the restriction $H^i(E, E') \to H^i(D^n_x, S^{n-1}_x)$ is an isomorphism for all $i \le n$ and for all x in the base space B. We will prove this claim after making use of it.

If γ is a path from x to y in B, then it determines a homotopy equivalence L_γ from the fiber over y to the fiber over x. Then a choice of generator for $H^n(D^n_x, S^{n-1}_x) = \mathbb{Z}$ determines a choice of generator for $H^n(D^n_y, S^{n-1}_y)$ via composition with L^*_γ. If the claim is true, then this gives a preferred isomorphism between $H^n(E, E')$ and $\mathbb{Z}$ which restricts to the chosen isomorphism for each fiber. This in turn gives that a generator of $H^n(E, E')$ is a Thom class, proving the theorem when B is a finite-dimensional CW complex.

Proof of Claim: Let B' be B with a point removed from each k-cell (i.e. B' deformation retracts to B^{k-1}). Let V be a union of open neighborhoods of the deleted points, so $B' \cup V = B$. For a subspace $A \subset B$, let E_A and E'_A be the disk and sphere bundles that project to A, respectively. We have the following Mayer-Vietoris sequence:

$$\cdots \to H^n(E, E') \to H^n(E_{B'}, E'_{B'}) \oplus H^n(E_V, E'_V) \xrightarrow{f} H^n(E_{B' \cap V}, E'_{B' \cap V}) \to \cdots$$

Now, $B' \cap V$ deformation retracts onto a disjoint union of $(k-1)$-spheres. By Lemma 4.3, we can replace $E_{B' \cap V}$ with the part of E over this union. This implies that the first map is injective because the preceding term in the sequence is 0 by induction on k (assuming the claim for $(k-1)$-dimensional complexes gives us that $H^{n-1}(E_{\sqcup S^{k-1}}, E'_{\sqcup S^{k-1}}) = 0$). Then $H^n(E, E') \cong \ker f$ by exactness.

Using Lemma 4.3 (to replace our spaces with more convenient deformation retracts) and induction on k again, we get that each of $H^n(E_{B'}, E'_{B'})$, $H^n(E_V, E'_V)$, and $H^n(E_{B'\cap V}, E'_{B'\cap V})$ is a product of $\mathbb{Z}$s; there is one factor of $\mathbb{Z}$ for each path component of the spaces, and the projection onto any given $\mathbb{Z}$ factor is given by restriction to any fiber in that component. The kernel of f consists of pairs (a, b) that restrict to the same thing in $H^n(E_{B'\cap V}, E'_{B'\cap V})$. Now, we can assume B to be connected, and thus we can get from any component of B' to any component of V via an alternating sequence of components of B' and V such that each component has nonempty intersection with the next. This means that all the coordinates of a and b must be equal, and thus the kernel of f is a copy of $\mathbb{Z}$, and a restriction to any particular fiber is the desired isomorphism $H^n(E, E') \cong \mathbb{Z}$.

To show the claim holds for $i < n$, it suffices to show for $i < n$ that $H^i(E, E') = 0$. For this, we look earlier in the Mayer-Vietoris sequence, where the terms to either side of $H^i(E, E')$ are 0. To see this, first observe that the preceding term, $H^{i-1}(E_{B'\cap V}, E'_{B'\cap V})$, is 0 by induction on k because $B' \cap V$ deformation retracts to a $(k-1)$-dimensional complex and we can apply Lemma 4.3. Then notice that the subsequent term, $H^i(E_{B'}, E'_{B'}) \oplus H^i(E_V, E'_V)$, is also 0 for similar reasons (B' and V deformation retract to complexes of dimension $k-1$ and 0, respectively). This completes the proof of the claim for a finite-dimensional CW complex B. □

The infinite-dimensional case is reduced to the finite case in a manner similar to that in the Leray-Hirsch theorem. □

Now the necessary results for the Gysin sequence have been established:

Proof of Theorem 5.1. We will show that the following sequence is exact:

$$\cdots \longrightarrow H^{i-n}(B) \xrightarrow{\smile e} H^i(B) \xrightarrow{p^*} H^i(E) \longrightarrow H^{i-n+1}(B) \longrightarrow \cdots$$

Let all coefficients be in $\mathbb{Z}$ and suppose that $S^{n-1} \to E \to B$ is an orientable sphere bundle. Then the mapping cylinder of p, M_p, is a D^n-bundle over B, and is still orientable. Therefore, there is a Thom class $\tau \in H^n(M_p, E)$. We define the *Euler class* to be $e \in H^n(B)$ given by $e = (p^*)^{-1} j^*(\tau)$, where p^* is an isomorphism because M_p deformation retracts onto B and where j^* is the map denoted below in the long exact sequence of the pair (M_p, E). Consider the following diagram, in which the top row is the long exact sequence in cohomology of the pair (M_p, E) and the bottom row is the desired sequence:

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H^i(M_p, E) & \xrightarrow{j^*} & H^i(M_p) & \xrightarrow{i^*} & H^i(E) & \xrightarrow{\delta} & H^{i+1}(M_p, E) & \longrightarrow \cdots \\
& \uparrow \Phi & & \uparrow p^* & & \| & & \uparrow \Phi & \\
\cdots \longrightarrow & H^{i-n}(B) & \xrightarrow{\smile e} & H^i(B) & \xrightarrow{p^*} & H^i(E) & \xrightarrow{\Phi^{-1}\delta} & H^{i-n+1}(B) & \longrightarrow \cdots
\end{array}$$

The maps Φ on either side are the Thom isomorphism. To prove the theorem, we now only need to show commutativity of this diagram. Commutativity of the rightmost and middle squares are clear, so it only remains to prove the commutativity of the leftmost square. Given $b \in H^{i-n}(B)$:

$$j^*\Phi(b) = j^*(p^*(b) \smile \tau) = p^*(b) \smile j^*(\tau) = p^*(b) \smile p^*(e) = p^*(b \smile e).$$

This completes the proof. □

6. An Application of the Gysin Sequence and Thom Isomorphism

We have already seen four examples of sphere bundles over spheres that are in turn spheres: $S^0 \to S^1 \to S^1$, $S^1 \to S^3 \to S^2$, $S^3 \to S^7 \to S^4$, and $S^7 \to S^{15} \to S^8$. As shown, these correspond to the four real division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$. Adams' theorem proves that $\mathbb{R}^n$ can only have the structure of a division algebra when $n = 1, 2, 4, 8$ and that maps $f : S^{2n-1} \to S^n$ of Hopf invariant 1 only exist when $n = 2, 4, 8$. We will use the latter fact (without proof) to show that we have found all the fiber bundles in which all three spaces are spheres.

Theorem 6.1 (Proof of Corollary 3.9)**.** *Given a fiber bundle $S^k \to S^m \xrightarrow{p} S^n$, then it must be the case that $k = n-1$ and $m = 2n-1$. Furthermore, the Hopf invariant of p must be ± 1.*

Proof. First, from the local triviality condition it is clear that m must be $k + n$. For $n = 1$ we observe that any bundle over S^1 will not be simply connected and thus the only possibility is $S^0 \to S^1 \to S^1 = \mathbb{R}\mathrm{P}^1$. Now suppose that $n > 1$, and consider the following part of the Gysin sequence:

$$\cdots \to H^i(S^{k+n}) \to H^{i-k}(S^n) \to H^{i+1}(S^n) \to H^{i+1}(S^{k+n}) \to \cdots$$

For $i = n - 1$, the first and last terms must be 0, so $H^{i-k}(S^n) \cong H^{i+1}(S^n) = \mathbb{Z}$, and thus $k = n - 1$. Now we know that we have a sphere bundle of the form $S^{n-1} \to S^{2n-1} \xrightarrow{p} S^n$ and want to find the Hopf invariant of p.

Consider the fiber bundle pair $(D^n, S^{n-1}) \to (M_p, S^{2n-1}) \xrightarrow{p} S^n$, where M_p is the mapping cylinder of p, which is a disk bundle over S^n. Let $\tau \in H^n(M_p, S^{2n-1})$ be a Thom class. Now $M_p/S^{2n-1} = C_p$, the mapping cone of p. Let α be a generator of $H^n(S^n) = \mathbb{Z}$. Then $p^*(\alpha)$ is a generator of $H^n(C_p) = \mathbb{Z}$. By the relative Leray-Hirsch theorem, τ is a generator of $H^n(C_p)$ as well and is thus $\pm p^*(\alpha)$. By the Thom isomorphism, we have $H^n(S^n) \cong H^{2n}(M_p, S^{2n-1})$ via the isomorphism that sends α to $p^*(\alpha) \smile \tau$, so $p^*(\alpha) \smile \tau = \pm(p^*(\alpha))^2$ is a generator of $H^{2n}(C_p)$ and thus the Hopf invariant of p is ± 1. □

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References

[1] J.F. Adams On the non-existence of elements of Hopf invariant one. *Ann. of Math.* 72. 1960.
[2] Glen E. Bredon. Topology and Geometry. Springer-Verlag GTM 139. 1993.
[3] H.-D. Ebbinghaus et. al. Numbers. Springer-Verlag GTM 123. 1991.
[4] Allen Hatcher. Algebraic Topology. Cambridge University Press. 2001.
[5] J. P. May. A Concise Course in Algebraic Topology. University of Chicago Press. 1999.
[6] Norman Steenrod. The Topology of Fibre Bundles. Princeton University Press. 1951.