FINITE SPACES AND QUILLEN'S CONJECTURE

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Abstract. We introduce finite spaces and explain their relations to sets with reflexive and transitive relations and to simplicial complexes. We introduce Quillen's conjecture: If $\mathscr{A}_p(G)$ is weakly contractible then G has a nontrivial normal p-subgroup. We use Quillen's method to show this is true for solvable groups.

Contents

1. Introduction

We work through several concepts in the relation between finite spaces, relations on sets, and simplicial complexes, following the notes of J.P. May. We then introduce Quillen's conjecture and the final sections are focused on his proof of the conjecture for solvable groups.

2. Reflexive and Transitive Relations on Sets and Finite Spaces

We begin by establishing a bijective correspondence between reflexive and transitive relations on sets and finite spaces and we state some of its homotopy properties.

Definition 2.1. Let X be a finite space. For $x \in X$, define U_x to be the intersection of open sets that contain x. Define a relation on the set X by $x \leq y$ if $x \in U_y$ or, equivalently, $U_x \subset U_y$. Write $x < y$ if the inclusion is proper.

Note this is a partial order on a T_0 space.

Lemma 2.2. *The set of open sets U_x is the unique minimal basis for X.*

Proof. Let $\mathscr{U}$ be the topology on X. For each $x \in U \in \mathscr{U}$ we have $x \in U_x \subset U$, thus the set of open sets U_x is a basis. If $\mathscr{C}$ is another basis, there is a $C \in \mathscr{C}$ such that $x \in C \subset U_x$. This implies $C = U_x$, so that $U_x \in \mathscr{C}$ for all $x \in X$. □

We then get the following lemmas

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Lemma 2.3. *A finite set X with the reflexive and transitive relations $\leq$ determines a topology with basis the set of all sets $U_x = \{y|y \leq x\}$.*

Lemma 2.4. *For a finite set X, the topologies on X are in bijective correspondence with the reflexive and transitive $\leq$ on X. The topology corresponding to $\leq$ is T_0 if and only if the relation $\leq$ is a partial order.*

We then discuss homotopy properties of such spaces.

Lemma 2.5. *A function f: $X \longrightarrow Y$ is continuous if and only if $x \leq y$ in X implies $f(x) \leq f(y)$ in Y.*

Proof. Let f be continuous and suppose that $x \leq y$. Then $x \in U_y \subset f^{-1}(U_{f(y)})$ and so $f(x) \in U_{f(y)}$, thus $f(x) \leq f(y)$. Conversely, let V be open in Y. If $f(y) \in V$, then $U_{f(y)} \subset V$. If $x \in U_y$, then $x \leq y$ and thus $f(x) \leq f(y)$ and $f(x) \in U_{f(y)} \subset V$ so that $x \in f^{-1}(V)$. Thus $f^{-1}(V)$ is the union of these U_y and so is open. □

A nice result is that every finite space is homotopy equivalent to a T_0 space, thus when studying homotopy types of spaces we can restrict our attention to T_0 spaces.

Theorem 2.6. *Let X be a finite space. There is a quotient T_0 space X_0 such that the quotient map $q_X : X \longrightarrow X_0$ is a homotopy equivalence. For a map $f : X \longrightarrow Y$ of finite spaces, there is a unique map $f_0 : X_0 \longrightarrow Y_0$ such that $q_Y \circ f = f_0 \circ q_X$.*

Proof. Define $x \sim y$ if $U_x = U_y$, or equivalently, if $x \leq y$ and $y \leq x$. Let X_0 be the set of equivalence classes and let $q = q_X$ send x to its equivalence class $[x]$. Give X_0 the quotient topology. The relation $\leq$ on X induces a relation $\leq$ on X_0. Since X_0 is finite, we have the open set $U_{q(x)}$ for $x \in X$. Note $q^{-1}(q(U_x)) = U_x$ since if q(y) = q(z) where $z \in U_x$, then $y \in U_y = U_z \subset U_x$. Therefore $q(U_x)$ is open, hence contains $U_{q(x)}$. Conversely, $U_x \subset q^{-1}(U_{q(x)})$ by continuity and thus $q(U_x) \subset U_{q(x)}$.We get $q(U_x) = U_{q(x)}$. Then $[x] \leq [y]$ if and only if $x \leq y$. For $q(x) \leq q(y)$ implies $q(x) \in U_{q(y)} = q(U_y)$, thus $q(x) = q(z)$ for some $z \in U_y$ and $U_x = U_z \subset U_y$, so that $x \leq y$. Conversely, if $x \leq y$, then $U_x \subset U_y$ so $U_{q(x)} \subset U_{q(y)}$ giving $q(x) \leq q(y)$. It follows that $\leq$ is antisymmetric on X_0 so that X_0 is a T_0 space.

We now show q is a homotopy equivalence. Let $f : X_0 \longrightarrow X$ be any function such that $q \circ f = id$. That is, we choose a point from each equivalence class. Thus f preserves $\leq$ and is therefore continuous. Let $g = f \circ q$. Note g is obtained by choosing one x_u with $U_{x_u} = U$ for each U in the minimal basis for X and then letting $g(x) = x_u$ if $U_x = U$. Thus $U_{g(x)} = U_x$ and $g(x) \in U_x$, so that $g \leq id$. Thus $g \simeq id$.

A map $f : X \longrightarrow Y$ is a function that preserves $\leq$, and it follows that it induces a unique function $f_0 : X_0 \longrightarrow Y_0$ such that $q_Y \circ f = f_0 \circ q_X$. So f_0 preserves $\leq$ and so is continuous. □

3. Simplicial Complexes

Here we show a way to associate a simplicial complex to a finite space and we state an useful method of showing homotopy equivalency between two simplicial maps.

Definition 3.1. An abstract simplicial complex K is a set $V = V(K)$, whose elements are vertices with a set K of finite subsets of V, whose elements are called simplicies, such that every vertex is an element of some simplex and every subset of a simplex is a simplex.

Then for a partially ordered set X, we may associate to it an abstract simplicial complex $\mathscr{K}(X)$ whose vertices are the elements of X and whose simplicies are the nonempty finite totally ordered subsets, also called chains, of X.

Definition 3.2. A n-simplex σ spanned by a set $\{v_0, \ldots, v_n\}$ of geometrically independent points of $\mathbb{R}^N$ is the set of all points $x = \Sigma t_i v_i$ where $0 \leq t_i \leq 1$ and $\Sigma t_i = 1$. The points v_i are the vertices of σ. A simplex spanned by a subset of the vertices is a face of σ.

Definition 3.3. A geometric simplicial complex K is a set of simplices in some $\mathbb{R}^N$ such that every face of a simplex in K is a simplex in K and the intersection of two simplices in K is a simplex in K. The set of vertices of K is the union of the sets of vertices of its simplexes. The geometric realization $|K|$ is the union of the simplices of K, each regarded as a subspace of $\mathbb{R}^N$ with the topology whose closed sets are the sets that intersect each simplex in a closed subset.

Definition 3.4. A map $g : K \longrightarrow L$ of simplicial complexes is a function from the vertex set $V(K)$ to the vertex set $V(L)$ such that for each subset S of $V(K)$ that spans a simplex, the set $g(S)$ is the set of vertices of a simplex of L. Then g induces a continuous simplicial map $|g| : |K| \longrightarrow |L|$ that sends $\Sigma t_i v_i$ to $\Sigma t_i g(v_i)$. If g is a bijection on vertices and simplices, then it is an isomorphism in which case $|g|$ is a homeomorphism.

Abstract and geometric simplicial complexes may be used interchangably. The abstract simplicial complex aK determined by a geometric simplicial complex K has vertex set the union of the vertex sets of the simplicies of K and its simplices are the subsets that span a simplex of K. The geometric simplicial complex gK determined by an abstract simplicial complex has vertices the standard basis elements of $\mathbb{R}^N$ where N is the number of points in the vertex set $V(K)$ and the geometric simplices are spanned by the images of the bijection between the abstract vertex set and the geometric vertex set.

For ease of writing, for a poset X we write the geometric realization $|\mathscr{K}(X)|$ as $|X|$. We then define the homology groups of X to be those of $|X|$. In particular, we call X contractible if $|X|$ is contractible. We state without proof a nice method of showing homotopy equivalence between two simplicial maps.

Theorem 3.5. *If $f, g : X \longrightarrow Y$ are maps of posets such that $f(x) \leq g(x)$ for all $x \in X$, then $|f|$ and $|g|$ are homotopic.*

4. The Partially Ordered Sets $\mathscr{S}_p(G)$ and $\mathscr{A}_p(G)$

Let p be a prime number.

Definition 4.1. A p-torus is a finite abelian group A whose elements have order 1 or p. The rank of A, denoted $r_p(A)$, is the dimension of A considered as a vector space over $\mathbb{F}_p$.

Definition 4.2. Given a group G, the poset $\mathscr{A}_p(G)$ is the set of all nonidentity p-tori in G and the poset $\mathscr{S}_p(G)$ is the set of all p-subgroups in G, both ordered by inclusion.

Lemma 4.3. *If P is a nontrivial p-group, then $\mathscr{A}_p(P)$ is contractible.*

Proof. Let B be the subgroup of the center of P consiting of the elements of order 1 or p, then $B > 1$ since $P > 1$. So in any subgroup A of P we have $A \subset AB \supset B$ for all A. Let f be the map from $\mathscr{A}_p(P) \longrightarrow \mathscr{A}_p(P)$ that sends A to AB and c_B be the constant map that sends A to B. Then we have $id \leq f \geq c_B$ and so we have $id \simeq f \simeq c_B$. □

Proposition 4.4. *The inclusion $i : \mathscr{A}_p(G) \longrightarrow \mathscr{S}_p(G)$ is a weak homotopy equivalence.*

Proof. We have the open cover of $\mathscr{S}_p(G)$ given by U_P where P is a nontrivial finite p-group. Then $i^{-1}(U_P)$ is the poset of p-tori of G that are contained in P, and this is the contractible space $\mathscr{A}_p(P)$. Since weak homotopy equivalence is a local notion, we get the desired proposition. □

Recall this implies that $|\mathscr{A}_p(G)|$ and $|\mathscr{S}_p(G)|$ are homotopy equivalent spaces.

Proposition 4.5. *If G has a nontrivial normal p-subgroup, then $\mathscr{S}_p(G)$ is contractible. Conversely, if either $\mathscr{A}_p(G)$ or $\mathscr{S}_p(G)$ is contractible, then G has a nontrivial normal p-subgroup.*

Proof. Suppose G has a normal p-subgroup P. Then for any p-subgroup Q of G, we have $Q \subset QP \supset P$, hence $id \leq f \geq c_p$ hence $\mathscr{S}_p(G)$ is contractible. Conversely, suppose $\mathscr{S}_p(G)$ is contractible, then it is G-contractible to a G-fixed point P, which means P is a normal p-subgroup. Similarly for $\mathscr{A}_p(G)$. □

We then come to Quillen's conjecture.

Conjecture 4.6. *If G is finite and $|\mathscr{A}_p(G)|$ is contractible, then G has a nontrivial normal p-subgroup.*

Proposition 4.7. *The above conjecture holds if $r_p(G) \leq 2$.*

Proof. We ignore the case $r_p(G) = 0$ since then $\mathscr{A}_p(G)$ is empty. If $r_p(G) = 1$ then $\mathscr{A}_p(G)$ is zero dimensional and weakly contractible, hence it is a single point fixed under G, so G has a nontrivial normal p-torus. If $r_p(G) = 2$ then $|\mathscr{A}_p(G)|$ is one-dimensional and contractible, hence it is a tree and so has a fixed point under G, thus G has a nontrivial normal p-torus. □

5. Cohen-Macaulay Posets

Here we introduce the concept of Cohen-Macaulay posets which gives us the tools necessary to prove the conjecture for solvable groups.

Definition 5.1. A link of a simplex σ in a simplicial complex is the subcomplex consiting of those simplices disjoint from σ whose union with σ is a simplex.

So the link of the simplex $\sigma = \{x_0 < ... < x_p\}$ in $|X|$ is the simplicial complex associated to the subposet of X consisting of elements not in σ which can be adjoined to σ to form a chain. We denote this subposet by Link(σ, X).

Definition 5.2. If x is an element of a poset X, we let $X_{>x}$ be the subposet consisting of $x' > x$. Similarly for $X_{\geq x}$, $X_{<x}$, $X_{\leq x}$. The height of x, denoted $h(x)$ is the dimension of the poset $X_{\leq x}$, that is, the supremum of the lengths of chains having x as largest element.

Definition 5.3. Given a map $f : X \longrightarrow Y$ of posets and an element y of Y, define the subposet f/y to be the set $\{x \in X | f(x) \leq y\}$

Definition 5.4. Let K be a finite-dimensional simplicial complex and let d be its dimension. K is d-spherical if it is $(d-1)$ connected, that is, it has homotopy type of a wedge of d-spheres. K is Cohen-Macaulay (CM) if it is spherical and if the link of each p-simplex in K is $(d-p-1)$-spherical. A poset will be called spherical or CM when the associated simplicial complex has this property.

Thus a poset is CM if and only if the following conditions are satisfied:
X is n-spherical
$X_{>x}$ is $(n - h(x) - 1)$-spherical
$X_{<x}$ is $(h(x) - 1)$-spherical
$X_{>x'} \cap X_{<x}$ is $(h(x) - h(x') - 2)$-spherical when $x' < x$

We state a theorem whose proof is beyond the scope of this paper. The proof may be found in Quillen's paper.

Theorem 5.5. *Let $f : X \longrightarrow Y$ be a map of posets. Assume Y is n-spherical and for each y in Y that $Y_{>y}$ is $(n-h(y)-1)$-spherical and f—y is h(y)-spherical. Then X is n-spherical. Moreover, there is a canonical filtration*

$$0 = F_{n+1} \subset F_n \subset ... \subset F_{-1} = \tilde{H}_n(X)$$

and isomorphisms

$$F_{-1}/F_0 \backsimeq \tilde{H}_n(Y), \tag{5.6}$$

$$F_q/F_{q+1} \backsimeq \bigoplus \tilde{H}_{n-q-1}(Y_{>y}) \otimes \tilde{H}_q(f/y)$$

for $0 \leq q \leq n$, where the direct sum is taken over all elements of height q in Y.

Corollary 5.7. *Let $f : X \longrightarrow Y$ be a map of posets which is strictly increasing, that is, $x' < x \Rightarrow f(x') < f(x)$. Assume Y is CM of dimension n and that f/y is CM of dimension $h(y)$ for all y in Y. Then X is CM of dimension n.*

Proof. Since $Y_{>y}$ is CM of dimension $n - h(y) - 1$, the preceding theorem implies that X is n-spherical. Fix an element x in X and put $y = f(x)$. Because f/y is closed in X, the height of an element f/y is the same as its height in X. Then the subposets $X_{<x}$ and $X_{>x'} \cap X_{<x}$ of f/y are spherical of dimension $h(x) - 1$ and $h(x)-h(x')-2$ as f/y is assumed to be CM. We need to show $X_{>x}$ is $(n-h(x)-1)$-spherical. Because f is strictly increasing, it induces a map $f' : X_{>x} \longrightarrow Y_{>y}$ and $h(x) = h(y)$ as x is a maximal element of f/y. Then again by the preceding theorem applied to f', we have $Y_{>y}$ is CM of dimension $n' = n - h(y) - 1$, and if $y' \in Y_{>y}$, we have that

$$f'/y' = \{x' \in X_{>x} | f(x') \leq y'\} = (f/y')_{>x}$$

is CM of dimension $h(y') - h(x) - 1 = h(y') - h(y) - 1$ the height of y' in $Y_{>y}$ so we see $X_{>x}$ is spherical of dimension $n' = n - h(x) - 1$. □

Now we may begin to get to the meat of Quillen's paper, but first we state two lemma.

Lemma 5.8. *If $\mathscr{A}_p(G_1)$ and $\mathscr{A}_p(G_2)$ are CM, then $\mathscr{A}_p(G_1 \times G_2)$ is CM.*

Proof. Let $G = G_1 \times G_2$ let $pr_i : G \longrightarrow G_i$ be the projections, and let B be a p-torus of G. We show $\mathscr{A}_p(G)_{>B}$ is spherical of dimension $r_p(G) - r_p(B) - 1$. This implies that $\mathscr{A}_p(G)$ is CM. Now, for any p-torus A in G, we have $A \subset pr_1(A) \times pr_2(A)$, so any maximal p-torus in G is of the form $A_1 \times A_2$ with A_i a maximal p-torus of G_i. Thus $r_p(G) = r_p(G_1) + r_p(G_2)$ and dim $\mathscr{A}_p(G)_{>B} = r_p(G) - r_p(B) - 1$, so it suffices to show $\mathscr{A}_p(G)_{>B}$ is $(r_p(G) - r_p(B) - 2)$-connected.

Put $B_i = pr_i(B)$, $T_i = \mathscr{A}_p(G_i)_{>B_i}$, and let T be the subset of $\mathscr{A}_p(G)_{>B}$ consisting of p-tori of the form $A_1 \times A_2$ with $A_i \subset G_i$. Then T is isomorphic to the poset $CT_1 \times CT_2$ if $B < B_1 \times B_2$, and to $CT_1 \times CT_2 - (0, 0)$ if $B = B_1 \times B_2$. In the first case, T is contractible, in the second T is homeomorphic to the join $T_1 * T_2$ which is $(r_p(G)-r_p(B)-1)$-spherical because T_i is $(r_p(G)-r_p(B_i)-1)$-spherical by assumption. Thus T is $(r_p(G) - r_p(B) - 2)$-connected. The map $A \longmapsto pr_1(A) \times pr_2(A)$ from $\mathscr{A}_p(G)_{>B}$ to T is a homotopy equivalence so we are done. □

We state the following lemma without the proof which is of a group-theoretic nature.

Lemma 5.9. *Let A be a p-torus and let $P(A)$ be the set of hyerplanes in A, that is, subgroups B such that A/B is cyclic of order p. For each B in $P(A)$ there is a canonical A-module decomposition*

$$M^B = M^A \oplus M_{(B)}$$

where $M_{(B)}$ is the subgroup generated by $am - m$ with $a \in A$ and $m \in M^B$ and an A-module decomposition

$$M = M^A \oplus \bigoplus M_{(B)}$$

where the direct sum of $M_{(B)}$ is taken over all $B \in P(A)$.

Theorem 5.10. *Let G be a group having a chain of normal subgroups $H = H_0 \supset H_1 \supset ... \supset H_s = 1$ such that G/H is a p-torus of rank r and H_i/H_{i+1} is a uniquely p-divisible abelian group for $0 \leq i < s$. Then (i) $\mathscr{A}_p(G)$ is CM of dimension $r - 1$ (ii) if G contains no central elements of order p, then $\tilde{H}_{r-1}(\mathscr{A}_p(G)) \neq 0$.*

Proof. Several remarks first. Since H_i/H_{i+1} is a uniquely p-divisible abelian group, the extension G of G/H by H splits, that is, G is a semidirect product AH where A is a p-torus of rank r. Also note if the element ah of $G = AH$ is in the center and has order p, then $1 = (ah)^p = a^p h^p = h^p$ and since H_i/H_{i+1} has no elements of order p, H also has no elements of order p, we have $h = 1$. Thus if $Z_p(G)$ is the subgroup of central elements of order dividing p, we have that $Z_p(G)$ is the centralizer of H in A. So if G has no central elements of order p, as in (ii) above, then this is the same as saying A acts faithfully on H.

We prove the theorem by induction on s. Suppose $s > 1$, put $G' = G/H_s$ and let $\pi : G \longrightarrow G'$ be the canonical map. We have $A \simeq \pi(A)$ for any p-torus in G, hence π induces a map of posets $f : \mathscr{A}_p(G) \longrightarrow \mathscr{A}_p(G')$ to which we apply Corollary 5.7 with $n = r - 1$. By the induction hypothesis $\mathscr{A}_p(G')$ is CM of dimension $r - 1$ and we have

$$f/B = \{A \in \mathscr{A}_p(G) | \pi(A) \subset B\} = \mathscr{A}_p(\pi^{-1}B)$$

and $\pi^{-1}B$ is an extension of B by H_s. Since $s > 1$, the theorem holds for $\pi^{-1}B$ by induction. Hence f/B is CM of dimension $r_p(G) - 1$ which is the height of B in $\mathscr{A}_p(G')$. So from 5.7, we get (i) for G. Let $B = Z_p(G')$ and $t = r_p(B)$.

Assuming $Z_p(G) = 1$, we prove
(a) $Z_p(\pi^{-1}) = 1$
(b) $\tilde{H}_{r-t-1}(\mathscr{A}_p(G')_{>B}) \neq 0$

Put $A = Z_p(\pi^{-1}B)$. Since H_s is abelian and B is central in G', conjugation gives a homomorphism from A to the group T of automorphisms of G inducing the identity on G' and on H_s. The group T is a uniquely p-divisble abelian group so this homomorphism is trivial, thus $A \subset Z_p(G) = 1$, giving (a). Recall B is the subgroup of A centralizing H'. So in the semidirect product group $G'/B = (A/B)H'$ there are no central elements of order p because A/B acts faithfully on H'. But $\mathscr{A}_p(G')_{>B} = \mathscr{A}_p(G'/B)$ and the theorem holds for the group G'/B by induction so (ii) for G'/B gives (b). Then applying (ii) to $\pi^{-1}B$, (a) implies $\tilde{H}_{t-1}(f/B) \neq 0$. Then the second part of Theorem 5.5 combined with (b) gives $\tilde{H}_{r-1}(\mathscr{A}_p(G)) \neq 0$ proving (ii) for G.

Now for the case $s = 1$, G is then a semidirect product AH with A a p-torus of rank r and H a uniquely p-divisble abelian normal subgroup. Using the decomposition from Lemma 5.9 there is a chain of A-submodules $H = H_0 \subset ... \subset H_s = 1$ such that for each $i = 0, ..., s-1$ the quotient H_i/H_{i+1} is a uniquely p-divisble A-module on which either A acts trivially or $H_i/H_{i+1} > 1$ and there is a hyperplane A_0 in A such that A/A_0 acts freely on the complement of the identity in this quotient. We argue by induction on s but the induction step is identical to the one just given so we check the case $s = 1$.

There are two possibilities
(1) The group A acts trivially on H, so $G = AxH$ and $\mathscr{A}_p(G) = \mathscr{A}_p(A)$ is CM of dimension $r-1$, because $\mathscr{A}_p(A)_{>B}$ is contractible if $B < A$ and empty if $B = A$. Thus (i) holds. If G has no central elements of order p, then $r = 0$, so $\mathscr{A}_p(G)$ is empty and $\tilde{H}_{-1}(\mathscr{A}_p(G)) = \mathbb{Z}$, proving (ii).
(2) The group $H > 1$ and A/A_0 acts freely on $H - 1$ for some hyperplane A_0. If we choose a cyclic subgroup A_1 in A complementary to A_0, then $G = (A_1H)xA_0$. Since all p-tori in A_1H have rank ≤ 1, $\mathscr{A}_p(A_1H)$ is CM of dimension $r-1$ by Lemma 5.2 proving (i). If G has no central elements of order p, then we have $A_0 = 1$ and $\mathscr{A}_p(G)$ is zero dimensional with more than one element, because A is not normal in G. So $\tilde{H}_0(\mathscr{A}_p(G)) \neq 0$, proving (ii). □

Theorem 5.11. *Let G be a finite solvable group having no nontrivial normal p-subgroup. If G has a maximal p-torus of rank r, then $\tilde{H}_{r-1}(\mathscr{A}_p(G)) \neq 0$.*

Proof. Let A be a maximal p-torus in G of rank r, and let H be the largest normal p'-subgroup of G. By the Hall-Higman Lemma 1.2.3, our assumption implies $C_G(H) \subset H$ so A acts faithfully on H, and so $H_{r-1}(Y) \neq 0$ by our previous theorem, where $Y = \mathscr{A}_p(AH)$. Put $X = \mathscr{A}_p(G)$ and let Z be the closed subset of X consisting of all p-tori contained in some p-torus of G not contained in AH. We show that for any $B \in Y \cap Z$ has rank $< r$. So we have $x^{-1}Bx \subset A$ as A is a Sylow p-subgroup of AH, and we have $B < B_1$ for some p-torus B_1 not in AH. So $x^{-1}Bx < A$ by maximality of A. So $Y \cap Z$ has dimension $< r-1$. Since $|X| = |Y| \cup |Z|$, $|Y| \cap |Z| = |Y \cap Z|$ we have a Mayer-Vietoris sequence

$$\tilde{H}_{r-1}(Y \cap Z) \to \tilde{H}_{r-1}(Y) \oplus \tilde{H}_{r-1}(Z) \to \tilde{H}_{r-1}(X)$$

where $\tilde{H}_{r-1}(Y \cap Z) = 0$ and $\tilde{H}_{r-1}(Y) \neq 0$ which shows that $\tilde{H}_{r-1}(X) \neq 0$. □

This shows Quillen's conjecture is true for solvable groups.

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