A RESULT ON REPRESENTATIONS OF HOMOLOGY MANIFOLDS BY FINITE SPACES

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ABSTRACT. We prove a result relating the Euler characteristic of a polyhedral closed homology manifold to the finite space associated with a triangulation of the manifold. We then give a new proof that polyhedral closed homology manifolds have Euler characteristic 0.

Contents

1. Introduction

Finite topological spaces provide a number of interesting connections between combinatorics and algebraic topology. In particular, a finite T_0 space X can be assigned a partial order and associated with a simplicial complex $\mathcal{K}(X)$ in a natural way.

In this spirit, this paper connects the concepts of links and pure complexes, both familiar topics in the theory of simplical complexes, to posets. Next, given a finite T_0 space X, we define the level $\ell_X(x)$ of a point $x \in X$. With this definition, we are able to prove the following theorem about the Euler characteristic of a finite T_0 space X when the geometric realization of $\mathcal{K}(X)$ is a closed homology manifold:

Theorem 4.2. Let X be a finite T_0 space. If $|\mathcal{K}(X)|$ is a closed homology manifold, then

$$\chi(X) = \sum_{x \in X} (-1)^{\ell_X(x)}.$$

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It is then a simple corollary that polyhedral odd-dimensional closed homology manifolds have Euler characteristic 0.

2. Preliminaries

We recall certain facts about finite spaces required to develop this paper.

2.1. **Finite spaces and posets.** Let X be a finite topological space; that is, X is a finite set with a topology. For each $x \in X$, let U_x be the intersection of all open sets containing x. (Note that since X is finite, U_x is open.) Then we can put a preorder on X (that is, we can establish a relation $\leq$ that is reflexive and transitive) by having $x \leq y$ if and only if $U_x \subset U_y$.

Conversely, let X be a preordered set. For each $x \in X$, let $U_x = \{y \in X \mid y \leq x\}$. Then $\{U_x\}_{x \in X}$ is a basis (in fact, a minimal basis) for a topology on X.

The above constructions are inverses in the sense that given a topology on a finite space X, if we take the associated preorder, the topology associated with that preorder is equal to the original topology, and vice versa. Furthermore, recall that a topological space X is T_0 if for all $x, y \in X$, there exists an open set U such that either $x \in U, y \notin U$ or $y \in U, x \notin U$. Then a finite space is T_0 if and only if its preorder is antisymmetric—in other words, a partial order. Henceforth, we will refer to finite T_0 spaces and finite posets interchangeably. In particular, we will discuss topological properties of posets and order properties of T_0 spaces.[1]

2.2. **Finite spaces and simplicial complexes.**

Definition 2.1. Let X be a finite poset. The *order complex* $\mathcal{K}(X)$ of X is the abstract simplicial complex whose vertex set is X, and whose simplices are the non-empty chains of X.

Henceforth, we will refer to vertices (respectively, chains) of a finite poset X and vertices (respectively, simplices) of $\mathcal{K}(X)$ as the same objects.

Definition 2.2. Let K be a finite simplicial complex. The *face poset* K_Δ [1] of K is the poset whose vertices are the simplices of K ordered by inclusion.

One interesting property of these constructions is that given a finite simplicial complex K, $\mathcal{K}(K_\Delta)$ is the first barycentric subdivision of K. Less obviously, given a finite space X, the geometric realization $|\mathcal{K}(X)|$ of $\mathcal{K}(X)$ is weak homotopy equivalent to X (see [2] for details). Given the above two facts, we can also conclude that given a finite simplicial complex K, its geometric realization $|K|$ is weak homotopy equivalent to K_Δ.

Finally, note that for every finite topological space X, there exists a space X^{OP} produced simply by reversing the direction of the preorder. If X is T_0, it is obvious that X^{OP} is T_0 and that $|\mathcal{K}(X)|$ and $|\mathcal{K}(X^{OP})|$ are homeomorphic, so X and X^{OP}

[1] Much of the literature on finite spaces refers to this construction as $\mathcal{X}(K)$. We opt for K_Δ to avoid confusion with the Euler characteristic χ.

are weak homotopy equivalent.[2]

3. Links and pure complexes and spaces

In this section, we elaborate on the relationship between finite T_0 spaces and abstract simplicial complexes.

3.1. **Links.** We begin with a standard definition.

Definition 3.1. Let K be an abstract simplicial complex, and let σ be a face in K. Then the *link* of σ in K is given by

$$lk_K(\sigma) = \{\tau \in K \mid \tau \cup \sigma \in K, \tau \cap \sigma = \emptyset\}.$$

In other words, the link consists of all faces of K whose union with σ is a face of K, and whose intersection with σ is empty. Note that a link is always itself a simplicial complex.

We now define an analogous term for finite T_0 spaces.

Definition 3.2. Let X be a finite T_0 space, and let C be a non-empty chain of X. Then the *link* of C in X is given by

$$lk_X(C) = \{x \in X \backslash C \mid C \cup \{x\} \text{ is a chain}\}.$$

We can easily see that these correspond in the expected manner.

Proposition 3.3. *Let X be a finite T_0 space, and let C be a chain in X. Then $\mathcal{K}(lk_X(C)) = lk_{\mathcal{K}(X)}(C)$.*

Proof. If D is a chain in $lk_X(C)$, then $D \cup C$ is a chain in X and $D \cap C = \emptyset$. Conversely, if v is a vertex in $lk_{\mathcal{K}(X)}(C)$, then $v \notin C$ and $C \cup \{v\}$ is a chain in X. □

Finally, we introduce a related concept for individual vertices.

Definition 3.4. Let X be a finite T_0 space, and let $x \in X$. The *lower link* of x in X is given by

$$\hat{U}_x^X = \{y \in X \mid y < x\}.$$

The *upper link* of x in X is given by

$$\hat{F}_x^X = \{y \in X \mid y > x\}.$$

When it is clear from context where the lower or upper link comes from, we write simply $\hat{U}_x$ and $\hat{F}_x$.

We define lower and upper links for $\mathcal{K}(X)$ in the expected manner.

Note that $\hat{U}_x \cup \hat{F}_x = lk_X(\{x\})$. Furthermore, we can extend x "upwards" into a chain C such that $\hat{U}_x = lk_X(C)$, and similarly "downwards" into a chain D such that $\hat{F}_x = lk_X(D)$. Finally, note that

$$\hat{U}_x^X = \hat{F}_x^{X^{OP}}, \tag{3.5}$$

and similarly

$$\hat{F}_x^X = \hat{U}_x^{X^{OP}}. \tag{3.6}$$

3.2. **Pure complexes and spaces.** Again, we begin with a standard definition and an analogy to finite T_0 spaces.

Definition 3.7. An abstract simplicial complex K is *pure* if all maximal faces have the same dimension.

Definition 3.8. A finite T_0 space is *pure* if all maximal chains have the same cardinality.

It is obvious that a finite T_0 space X is pure if and only if $\mathcal{K}(X)$ is pure.

Definition 3.9. Let X be a finite T_0 space, and let $\mathcal{C}(X)$ be the set of non-empty chains of X. For $C \in \mathcal{C}(X)$, the *height* of C is given by

$$ht(C) = \#C - 1. \tag{3.10}$$

The height of X is given by

$$ht(X) = \max_{C \in \mathcal{C}(X)} \{ht(C)\}. \tag{3.11}$$

Note that the height of a chain is equal to the dimension of its corresponding simplex.

Definition 3.12. Let X be a finite T_0 space, and let $x \in X$. The *level* of x in X is given by

$$\ell_X(x) = ht(\hat{U}_x^X) + 1.$$

Equivalently, the level of x is the maximum height of all chains in X with x as its greatest element.

We conclude this section with the following proposition.

Proposition 3.13. *If X is a pure finite T_0 space, then for all $x \in X$,*

$$\ell_X(x) = ht(X) - \ell_{X^{OP}}(x).$$

Proof. Let $x \in X$, and let C be a maximal chain in X containing x. Since X is pure, $ht(C) = ht(X)$. Let $C_{\leq} = \{y \in C \mid y \leq x\}$, and $C_{\geq} = \{y \in C \mid y \geq x\}$. Then $ht(C) = ht(C_{\leq}) + ht(C_{\geq})$. We know that $ht(C_{\leq}) = \ell_X(x)$ (otherwise, there would be some maximal chain longer than C), and by 3.6, $ht(C_{\geq}) = \ell_{X^{OP}}(x)$ for the same reasons. Our desired result immediately follows. $\square$

4. The Euler characteristic of finite spaces

Given a topological space X, let $\chi(X)$ be the Euler characteristic of X. If K is a finite simplicial complex, it is clear that

$$\chi(|K|) = \sum_{\sigma \in K} (-1)^{dim(\sigma)}.$$

Let X be a finite T_0 space. Since $|\mathcal{K}(X)|$ and X are weak homotopy equivalent, their homology groups are isomorphic, and hence they have the same Euler characteristic. Let $\mathcal{C}(X)$ be the set of non-empty chains of X. The definition of $\mathcal{K}$ allows us to conclude [3]

$$\chi(X) = \sum_{C \in \mathcal{C}(X)} (-1)^{ht(C)}.$$

We can relate the Euler characteristic of a finite T_0 space X to the Euler characteristics of lower links in X with the following proposition.

Proposition 4.1. *Let X be a finite T_0 space. Then*
$$\chi(X) = \sum_{x\in X}(1-\chi(\hat{U}_x)).$$

Proof. Proof by induction on the cardinality $\#X$ of X. The case $\#X = 0$ is trivial. Assume our hypothesis is true for $\#X = k$. Let $\#X = k+1$, and let $x_0 \in X$ be a maximal point. Since $x_0 \notin \hat{U}_y$ for all $y \neq x_0$, we have
$$\chi(X\backslash\{x_0\}) = \sum_{y\in X\backslash\{x_0\}} (1-\chi(\hat{U}^X_y)).$$
by our hypothesis. Furthermore,
$$\begin{aligned}\chi(X) &= \sum_{C\in\mathcal{C}(X)} (-1)^{ht(C)} \\ &= \sum_{\substack{C\in\mathcal{C}(X),\\ x_0\in C}} (-1)^{ht(C)} + \sum_{\substack{D\in\mathcal{C}(X),\\ x_0\notin D}} (-1)^{ht(D)} \\ &= \sum_{\substack{C\in\mathcal{C}(X),\\ x_0\in C}} (-1)^{ht(C)} + \chi(X\backslash\{x_0\}).\end{aligned}$$

Clearly, if $x_0 \in C \subset X$, then $C \in \mathcal{C}(X)$ if and only if $C = \{x_0\}$ or $C\backslash\{x_0\} \in \mathcal{C}(\hat{U}_{x_0})$. Hence,
$$\begin{aligned}\sum_{\substack{C\in\mathcal{C}(X),\\ x_0\in C}} (-1)^{ht(C)} &= 1 - \sum_{C\in\mathcal{C}(\hat{U}_{x_0})} (-1)^{ht(C)} \\ &= 1-\chi(\hat{U}_{x_0}).\end{aligned}$$
Our induction immediately follows. ∎

Of course, this proof can be altered slightly to provide an analogous result for upper links.

We now reach the main result of this paper.

Theorem 4.2. *Let X be a finite T_0 space. If $|\mathcal{K}(X)|$ is a closed homology manifold, then*
$$\chi(X) = \sum_{x\in X}(-1)^{\ell_X(x)}.$$

Proof. Recall that a compact polyhedron M is a *closed homology manifold* if its underlying simplicial complex K is such that for any simplex σ of K, the homology groups of $|lk_K(\sigma)|$ are isomorphic to the homology groups of $S^{dim(M)-dim(\sigma)-1}$. [2] Note that the polyhedron condition implies that K is pure.

[2] Note the similarity between this definition and piecewise-linear triangulations of a manifold, in which the link of a simplex is homeomorphic to a sphere of appropriate dimension.

For $x \in X$, let C be a maximal chain in X containing x, and let $C_{\geq} = \{y \in C \mid y \geq x\}$. Since $\mathcal{K}(X)$ is pure, X is pure, so $ht(C_{\geq}) = ht(\hat{F}^X_x) + 1$. Furthermore, $lk_X(C_{\geq}) = \hat{U}^X_x$. Hence, by 3.6, 3.12, and 3.13,

$$\begin{aligned}\chi(\hat{U}^X_x) &= \chi(S^{ht(X)-ht(C_{\geq})-1}) \\ &= 1 + (-1)^{ht(X)-ht(\hat{F}^X_x)} \\ &= 1 + (-1)^{ht(X)-ht(\hat{U}^{X^{OP}}_x)} \\ &= 1 + (-1)^{ht(X)-\ell_{X^{OP}}(x)+1} \\ &= 1 + (-1)^{\ell_X(x)+1}\end{aligned}$$

Our result follows from the above proposition.

□

With this result, we can now provide another proof of a well-known fact.

Corollary 4.3. *All odd-dimensional polyhedral closed homology manifolds have Euler characteristic 0.*

Proof. Let M be an odd-dimensional polyhedral homology manifold with underlying complex K. Then K_Δ is a finite T_0 space such that $\mathcal{K}(K_\Delta)$ is a triangulation of M, so

$$\chi(X) = \sum_{x \in K_\Delta} (-1)^{\ell_{K_\Delta}(x)}. \tag{4.4}$$

But $(K_\Delta)^{OP}$ is also a finite T_0 space such that $\mathcal{K}((K_\Delta)^{OP})$ is a triangulation of M, so

$$\chi(X) = \sum_{x \in (K_\Delta)^{OP}} (-1)^{\ell_{(K_\Delta)^{OP}}(x)}. \tag{4.5}$$

Since $\ell_{K_\Delta}(x) = ht_{K_\Delta}(x) - \ell_{(K_\Delta)^{OP}}(x)$, and since $ht(K_\Delta)$ is odd, $\ell_{K_\Delta}(x)$ and $\ell_{(K_\Delta)^{OP}}(x)$ have different parities. Hence we conclude that $\chi(X) = -\chi(X) = 0$, and thus that $\chi(M) = 0$. □

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