MORLEY'S CATEGORICITY THEOREM

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ABSTRACT. A theory is called κ-categorical, or categorical in power κ, if it has one model up to isomorphism of cardinality κ. Morley's Categoricity Theorem states that if a theory of first order logic is categorical in some uncountable power κ, then it is categorical in every uncountable power. We provide an elementary exposition of this theorem, by showing that a theory is categorical in some uncountable power if and only if it is ω-stable and has no Vaughtian pairs. Along the way, we will develop the theory of Vaughtian pairs, stable theories, and indiscernibles and provide a proof of Vaught's two-cardinal theorem.

CONTENTS

A theory is called κ-categorical, or categorical in power κ, if it has one model up to isomorphism of cardinality κ. Morley's Categoricity Theorem states that if a theory of first order logic is categorical in some uncountable power κ, then it is categorical in every uncountable power. This striking result still provides the impetus and motivation for various areas of contemporary research within model theory.

The categoricity theorem is remarkable for a number of reasons. The Löwenheim-Skolem theorem tells us that every theory in a countable language with an infinite model has a model of any infinite cardinality. It is counterintuitive that such a restrictive structural property as categoricity in an uncountable power holds as models get very large. Furthermore, many examples of theories categorical in every uncountable power were known before the categoricity theorem was discovered, but the proofs of these facts relied on properties of the theories themselves. For example, the categoricity in uncountable powers of the theory of algebraically closed fields of characteristic p (zero or prime) depends on facts about transcendence degree. The categoricity theorem manages to prove that theories categorical in some uncountable power must be categorical in *every* uncountable power using model-theoretic methods alone.

Examples of theories categorical in an uncountable power are somewhat limited. However, some natural examples of such theories include:

- algebraically closed fields of characteristic p (zero or prime)
- pure identity theory
- torsion-free divisible Abelian groups
- infinite Abelian groups in which every element has order p (prime)
- natural numbers with a successor function
- vector spaces over a countable field.

This paper aims to provide a fairly elementary exposition of Morley's categoricity theorem, closely following the presentation in Marker [2]. In section 1, we discuss the basic definitions and concepts of model theory and state a number of results which are used, but not proved, throughout the paper. In subsequent sections, we develop a wide array of tools to attack the central result. It is a fortunate fact that the road to the categoricity theorem is paved with model theoretic gold, so many results which are interesting in their own right will be proved in the body of the paper.

Finally, it is worth mentioning that Morley's categoricity theorem may be of historical interest to a UChicago audience. It was proved by Michael Morley in his 1961 PhD dissertation in the UChicago mathematics department under Saunders Mac Lane (together with Robert Vaught at Berkeley). One can trace the UChicago lineage of the theorem back even further, as the very notion of categoricity was first introduced by Oswald Veblen in *his* 1903 dissertation in the mathematics department.

1. Logical Preliminaries

We presume the reader is already familiar with the rudiments of first-order logic, including the compactness theorem and the notion of an $\mathcal{L}$-structure. In this section, we state some important logical results which are necessary for the rest of the paper and briefly state some definitions, but the presentation will be quick and terse. This allows us to develop the more complicated machinery required to prove the categoricity theorem in greater detail.

A *language* $\mathcal{L}$ is a set of function symbols, relation symbols, and a set of constant symbols. The $\mathcal{L}$-formulas are built up inductively. We begin with *terms*, which include constant symbols, variable symbols, and function symbols applied to terms. Then we build up to *atomic formulas*, which include formulas stating the equality of two terms and relations applied to terms. Then the full set of $\mathcal{L}$-formulas is the smallest set containing all of the atomic formulas closed under conjunction, negation, and quantification. To prove statements about all formulas, we induct on the complexity of formulas.

An *$\mathcal{L}$-structure* $\mathcal{M}$, also called a *model*, is an underlying set M together with interpretations in M of the function, relation, and constant symbols of the language $\mathcal{L}$. Throughout we use script letters $\mathcal{M}$, $\mathcal{N}$, etc to refer to structures and the corresponding capital Roman letter M, N to refer to the underlying sets. Although it requires some care to define satisfaction in a model, we say $\mathcal{M} \models \phi$ if the interpretations of the symbols in ϕ make a true statement about the elements of M. A *theory* is just a set of sentences in a language. We say $\mathcal{M} \models T$ if $\mathcal{M} \models \phi$ for every $\phi \in T$. Furthermore, we say T is *satisfiable* if it is possible to find a model that satisfies it. Additionally, we write $T \models \phi$ if for every $\mathcal{N}$ such that $\mathcal{N} \models T$, we have $\mathcal{N} \models \phi$. Finally, we write $Th(\mathcal{M})$, the *full theory of* $\mathcal{M}$, for the set of all sentences true in $\mathcal{M}$.

Throughout, we will always assume that T is a complete theory in a countable language.

Definition 1.1. An $\mathcal{L}$-*embedding* $\eta : \mathcal{M} \to \mathcal{N}$ is an injective map $\eta : M \to N$ that preserves the interpretation of all the symbols of $\mathcal{L}$. An *isomorphism* is a bijective $\mathcal{L}$-embedding. If $A \subseteq M$ and $B \subseteq N$, then a map $f : A \to B$ is a *partial embedding* when $f \cup \{(c^{\mathcal{M}}, c^{\mathcal{N}})|$ c is a constant in $\mathcal{L}\}$ is a bijection preserving all relations and functions of $\mathcal{L}$.

Definition 1.2. Suppose $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures. We say an $\mathcal{L}$-embedding $j : \mathcal{M} \to \mathcal{N}$ is an *elementary embedding* if

$$\mathcal{M} \models \phi(a_1, \ldots, a_n) \iff \mathcal{N} \models \phi(j(a_1), \ldots, j(a_n)),$$

for any formula ϕ of $\mathcal{L}$ and any $a_1, \ldots, a_n \in M$. This definition extends to partial embeddings in the obvious way: a partial elementary embedding is a partial embedding $f : A \to B$in which

$$\mathcal{M} \models \phi(c_1, \ldots, c_n) \iff \mathcal{N} \models \phi(f(c_1), \ldots, f(c_n)),$$

for $c_1, \ldots, c_n \in A$.

If the inclusion map $i : \mathcal{M} \to \mathcal{N}$ is an elementary embedding, we say that $\mathcal{M}$ is an *elementary substructure (or submodel)* of $\mathcal{N}$ or, interchangeably, we say that $\mathcal{N}$ is an *elementary extension* of $\mathcal{M}$.

The following theorem provides a useful criterion for determining when a substructure is an elementary substructure:

Theorem 1.3 (**Tarski-Vaught Test**). *Suppose that $\mathcal{M}$ is a substructure of $\mathcal{N}$. $\mathcal{M}$ is an elementary substructure if and only if, for any $\phi(v, \overline{w})$ and $\overline{a} \in M$, there is a $c \in M$ so that $\mathcal{N} \models \phi(c, \overline{a})$ whenever there is a $b \in N$ so that $\mathcal{N} \models \phi(b, \overline{a})$.*

The proof is an easy induction on the complexity of formulas.

We will also make extensive use of the following two theorems, which we state without proof:

Theorem 1.4 (**Compactness**). *A theory T is satisfiable if every finite subset of T is satisfiable.*

Theorem 1.5 (**Löwenheim-Skolem**). *Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure and $X \subseteq M$, there is an elementary submodel $\mathcal{N}$ of $\mathcal{M}$ such that $X \subseteq N$ and $|\mathcal{N}| \leq |X| + |\mathcal{L}| + \aleph_0$.*

Finally, we will use the fact that theories and languages have expansions that provide sufficient functions to witness existential sentences. An $\mathcal{L}$-theory has *built-in Skolem functions* if, given any $\mathcal{L}$-formula $\phi(v, \overline{w})$, there is a function f so that

$$T \models \forall \overline{w}((\exists v \phi(v, \overline{w})) \to \phi(f(\overline{w}), \overline{w})).$$

We will assume the following fact:

Theorem 1.6. *Given an $\mathcal{L}$-theory T, there is an expanded language $\mathcal{L}^*$ and expanded theory T^* so that T^* has built in Skolem functions. If $\mathcal{M} \models T$, then there is an expanded $\mathcal{L}^*$-structure $\mathcal{M}^*$ so that $\mathcal{M}^* \models T^*$. Additionally, $\mathcal{L}^*$ can be chosen so that $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$.*

We refer to $\mathcal{L}^*$ and T^*, as in the above theorem, as the *Skolem expansions* of $\mathcal{L}$ and T respectively. The function symbols are called *Skolem functions* and the terms built up with them are called *Skolem terms.*

We will use the following conventions throughout the paper: as mentioned before, $\mathcal{M}$, $\mathcal{N}$ will be used to denote $\mathcal{L}$-structures, and M, N their respective underlying sets. We write $\overline{a}$ to denote the n-tuple $(a_1, \ldots, a_n)$ and we write $\phi(\overline{v}, \overline{a})$ to denote a formula ϕ in free variables (variables not under the scope of a quantifier) $v_1, \ldots, v_k$ and parameters $\overline{a}$. We write $|A|$ to signify the cardinality of the set A. We identify a function f with its graph, so that it makes sense to take unions and intersections of functions, and write $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$ to denote the domain and range of f, respectively. Finally, we use $2^{<\omega}$ for the set of finite binary sequences, and given $\sigma, \tau \in 2^{<\omega}$, we let $\sigma \frown \tau$ be the concatenation of the two sequences, starting with σ.

2. Types and Topology

Given some $\mathcal{L}$-structure $\mathcal{M}$ and some $A \subset M$, the set of *parameters*, we can add a new constant symbol c_a to the language $\mathcal{L}$ for each $a \in A$, producing a new language $\mathcal{L}_A$. We can then turn $\mathcal{M}$ into an $\mathcal{L}_A$-structure by interpreting these new constant symbols by setting $c_a^{\mathcal{M}} = a$ for each $a \in A$. We call this expanded language the *enrichment of* $\mathcal{L}$ *by* A and we denote the resulting set of $\mathcal{L}_A$ formulas true in $\mathcal{M}$ by $Th_A(\mathcal{M})$. This construction allows us to define the notion of a type.

Definition 2.1. Let p be the set of $\mathcal{L}_A$ formulas in free variables $v_1, \ldots, v_n$. We say p is an *n-type* if $p \cup Th_A(\mathcal{M})$ is satisfiable. We say that p is *complete* if $\phi \in p$ or $\neg\phi \in p$ for all $\mathcal{L}_A$-formulas ϕ in free variables $v_1, \ldots, v_n$. We sometimes write $p(\overline{v})$ to emphasize the fact that the type p is in the free variables $\overline{v}$. We denote the set of all complete n-types by $S_n^{\mathcal{M}}(A)$.

The requirement that $p \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable means that a type in a given structure must be consistent with the structure's full theory. A type does not, however, have to be realized in that structure.

Definition 2.2. We say an n-type p over A is *realized* by $\overline{a} \in M^n$ whenever $\mathcal{M} \models \phi(\overline{a})$ for every formula $\phi \in p$. If there is no $\overline{b} \in M^n$ so that p is realized, then we say that p is *omitted* in $\mathcal{M}$.

In order to clarify the notion of a type, we will consider some examples. Consider $\mathcal{M} = (\mathbb{Q}, <)$, the rational numbers with the usual ordering relation. If we let $\mathbb{N} \subset \mathbb{Q}$ be the set of parameters, one possible type is

$$p(x) = \{x > 1, x > 2, x > 3, \ldots\}.$$

If we take any finite $\delta \subset p$, we can find the formula in δ with the greatest parameter $x > n$. Interpreting x as $n+1$, then, satisfies every formula in δ, which, by compactness, implies that p is satisfiable. Thus, p is a 1-type.

Returning the example type $p(x)$ above, we can see that p is not realized in $\mathbb{Q}$. An easy way to find complete types realized in a model is to define them as follows. Let

$$s(v) = \{\phi(v) | \phi \text{ an } \mathcal{L}_A\text{-formula}, \mathcal{M} \models \phi\left(\frac{1}{2}\right)\}.$$

Since $\mathcal{M} \models \phi$ if and only if $\mathcal{M} \not\models \neg\phi$, s is a complete type, and it is obviously realized in $\mathcal{M}$ by $\frac{1}{2}$.

This method of generating complete types generalizes into one particular n-type (over A in a structure $\mathcal{M}$) of interest, called *the type of* $\overline{a}$ for any $\overline{a} \in M^n$, which is defined as the set of all formulas in n free variables satisfied by $\overline{a}$. Put more explicitly, we define the type of $\overline{a}$ as

$$\mathrm{tp}^{\mathcal{M}}(\overline{a}/A) = \{\phi(v_1, \dots, v_n) \in \mathcal{L}_A | \mathcal{M} \models \phi(a_1, \dots, a_n)\}.$$

Clearly, $tp^{\mathcal{M}}(\overline{a}/A)$ is realized by $\overline{a}$. When $\emptyset$ is the set of parameters, we just write $\mathrm{tp}^{\mathcal{M}}(\overline{a})$. In the lemma that follows, we show that every complete type can be realized in an elementary extension of the given model, where it will be *the* type of the element which realizes it.

Given an $\mathcal{L}$-theory T, we let $S_n(T)$ be the set of complete n-types so that $p \cup T$ is satisfiable. If T is complete and $\mathcal{M} \models T$, then clearly $S_n(T) = S_n^{\mathcal{M}}(\emptyset)$.

Even if a type is not realized in a some structure $\mathcal{M}$, we can find a realization of it in an elementary extension of $\mathcal{M}$.

Lemma 2.3. *If $\mathcal{M}$ is an $\mathcal{L}$-structure, $A \subseteq M$, and p is an n-type over A, then there is an elementary extension $\mathcal{N}$ so that p is realized in $\mathcal{N}$.*

Proof. Let $\Gamma = p \cup \{\phi(m_1, \dots, m_n) | \mathcal{M} \models \phi(m_1, \dots, m_n), \phi \text{ an } \mathcal{L}\text{-formula}\}$. We can use a straightforward compactness argument to show that Γ is satisfiable.

To show that any finite subset of Γ is satisfiable, it suffices to consider finite subsets Δ of the form

$$\phi(v_1, \dots, v_n, a_1, \dots, a_m) \wedge \psi(a_1, \dots, a_m, b_1, \dots, b_l),$$

with $a_1, \dots, a_m \in A$, $b_1, \dots, b_l \in M \setminus A$, $\phi(\overline{v}, \overline{a}) \in p$, and $\mathcal{M} \models \psi(\overline{a}, \overline{b})$, since any finite subset of Γ is satisfiable if and only if the conjunction of the conjunction of the formulas in the type and the conjunction of the formulas satisfied in $\mathcal{M}$ is satisfiable.

Since p is a type, we know there is $\mathcal{N}_0$ so that $\mathcal{N}_0 \models p \cup \mathrm{Th}_A(\mathcal{M})$. Furthermore, since $\exists \overline{w} \psi(\overline{a}, \overline{w}) \in \mathrm{Th}_A(\mathcal{M})$, we have

$$\mathcal{N}_0 \models \phi(\overline{v}, \overline{a}) \wedge \exists \overline{w} \psi(\overline{a}, \overline{w}).$$

Interpreting $b_1, \dots, b_l$ as the elements of N_0 that witness $\exists \overline{w} \psi(a_1, \dots, a_m . \overline{w})$, we have $\mathcal{N}_0 \models \Delta$, so Δ is satisfiable, which tells us that Γ is satisfiable.

Since Γ is satisfiable, let $\mathcal{N} \models \Gamma$. Because we have

$$\mathcal{N} \models \{\phi(m_1, \dots, m_n) | \mathcal{M} \models \phi(m_1, \dots, m_n), \phi \text{ an } \mathcal{L}\text{-formula}\},$$

we know that the map that sends $c^{\mathcal{M}} \mapsto c^{\mathcal{N}}$ is an elementary embedding. If $c_1, \dots, c_n \in N$ are the interpretations of the v_i above, then $\overline{c}$ is a realization of p. □

We can put a topology on $S_n^{\mathcal{M}}(A)$ as follows. Given any $\mathcal{L}_A$-formula ϕ, we set

$$[\phi] = \{p \in S_n^{\mathcal{M}}(A) | \phi \in p\},$$

i.e. the set of complete n-types that contain ϕ. The sets for each $\mathcal{L}_A$-formula form a basis of open sets. Accordingly, we can put $[\phi \wedge \psi] = [\phi] \cap [\psi]$ and $[\phi \vee \psi] = [\phi] \cup [\psi]$. It can be shown that this space is compact and totally disconnected.

Definition 2.4. A type $p \in S_n^{\mathcal{M}}(A)$ is isolated if $\{p\}$ is open in $S_n^{\mathcal{M}}(A)$.

Since a set is open only if it is a union of the basic open sets, we can characterize these types by saying a type $p \in S_n^{\mathcal{M}}(A)$ is isolated if $\{p\} = [\psi]$ for some $\mathcal{L}_A$ formula ψ. We can also define isolated types by saying a type p is isolated if and only if there is a formula $\phi(\overline{v}) \in p$ so that for all $\psi(\overline{v}) \in p$,

$$\mathrm{Th}_A(\mathcal{M}) \models \phi(\overline{v}) \to \psi(\overline{v}).$$

Although we omit the proof, each of these three definitions is equivalent. We will employ whichever is more convenient for completing the proof at hand.

Likewise, we say $p \in S_n(T)$ is isolated if and only if there is $\phi(\overline{v}) \in p$ so that for all $\psi(\overline{v}) \in p$,

$$T \models \phi(\overline{v}) \to \psi(\overline{v}),$$

in the same way.

Earlier, we stated a result saying that omitted types can be realized in elementary extensions of a given model. However, the following theorem tells us that, if a realized type is non-isolated, then there is an elementary submodel which omits it:

Theorem 2.5 (**Omitting Types**). *Let $\mathcal{L}$ be a countable language, T an $\mathcal{L}$-theory, and p a non-isolated n-type over $\emptyset$. Then there is a countable $\mathcal{M} \models T$ omitting p.*

The isolated types in a structure have a number of interesting and useful properties. The following lemmas tell us certain facts about isolated types which will help us prove important results in the subsequent sections.

Lemma 2.6. *Suppose $\phi(\overline{v})$ is an $\mathcal{L}_A$-formula so that $[\phi]$ contains no isolated types in $S_n^{\mathcal{M}}(A)$. Then, there is a formula $\psi(\overline{v})$ so that $[\phi \wedge \psi]$ and $[\phi \wedge \neg\psi]$ are both non-empty and do not contain an isolated type.*

Proof. It is first easy to check that there is a formula ψ so that $[\phi \wedge \psi]$ and $[\phi \wedge \neg\psi]$ cannot both be empty. We can pick any type $p \in [\phi] \subseteq S_n^{\mathcal{M}}(A)$. Since p is complete, either ψ or $\neg\psi$ is in p so at least one of $[\phi \wedge \psi] = [\phi] \cap [\psi]$ and $[\phi \wedge \neg\psi] = [\phi] \cap [\neg\psi]$ is non-empty.

Next, suppose that for all $\mathcal{L}_A$-formulas ψ, exactly one of $[\phi \wedge \psi]$ and $[\phi \wedge \neg\psi]$ is non-empty. Let $p' = \{\theta | [\phi \wedge \theta] \neq \emptyset\}$. We note p' is a complete type and

$$\mathrm{Th}_A(\mathcal{M}) \models \forall \overline{v}(\phi(\overline{v}) \to \theta(\overline{v})),$$

for all $\theta \in p'$. Therefore, ϕ isolates p', contradicting our assumption. This shows that there is a formula $\psi(\overline{v})$ so that $[\phi \wedge \psi]$ and $[\phi \wedge \neg\psi]$ are both non-empty. ☐

Lemma 2.7. *If $A \subseteq M$ and $(\overline{a}, \overline{b}) \in M^{m+n}$ realizes an isolated type in $S_{m+n}^{\mathcal{M}}(A)$, then $tp^{\mathcal{M}}(\overline{a}/A)$ is isolated.*

Proof. Let $\phi(\overline{v}, \overline{w})$ isolate $\mathrm{tp}^{\mathcal{M}}(\overline{a}, \overline{b}/A)$. We want to show that $\exists \overline{w} \phi(\overline{v}, \overline{w})$ isolates $\mathrm{tp}^{\mathcal{M}}(\overline{a}/A)$.

Let $\psi(\overline{v})$ be any $\mathcal{L}_A$-formula such that $\mathcal{M} \models \psi(\overline{a})$. We need to show that

$$\mathrm{Th}_A(\mathcal{M}) \models \exists \overline{w}(\phi(\overline{v}, \overline{w}) \to \psi(\overline{v})).$$

Suppose not. Then, there is a $\overline{c} \in M^m$ so that $\mathcal{M} \models \exists \overline{w}(\phi(\overline{c}, \overline{w}) \wedge \neg\psi(\overline{c}))$.

Pick $\overline{d} \in M^n$ so that $\mathcal{M} \models \phi(\overline{c}, \overline{d}) \wedge \neg\psi(\overline{c})$. Because $\phi(\overline{v}, \overline{w})$ isolates $\mathrm{tp}^{\mathcal{M}}(\overline{a}, \overline{b}/A)$, we have $\mathrm{Th}_A \models \phi(\overline{v}, \overline{w}) \to \psi(\overline{v})$.

This is a contradiction, since

$$\psi(\overline{v}) \in \mathrm{tp}^{\mathcal{M}}(\overline{a}/A) \subset \mathrm{tp}^{\mathcal{M}}(\overline{a}, \overline{b}/A),$$

which completes the proof. ☐

Lemma 2.8. *Suppose that $\mathcal{M} \models T$, $A \subseteq B \subseteq M$, and every $\bar{b} \in B^m$ realizes an isolated type in $S_m^{\mathcal{M}}(A)$. If $\bar{a} \in M^n$ realizes an isolated type in $S_n^{\mathcal{M}}(B)$, then $\bar{a}$ realizes an isolated type in $S_n^{\mathcal{M}}(A)$.*

Proof. Let $\phi(\overline{v}, \overline{w})$ be an $\mathcal{L}$-formula and suppose $\bar{b} \in B^m$ is selected so that $\phi(\overline{v}, \bar{b})$ isolates $\text{tp}^{\mathcal{M}}(\bar{a}/B)$. Let $\theta(\overline{w})$ be an $\mathcal{L}_A$-formula isolating $\text{tp}^{\mathcal{M}}(\bar{b}/A)$. We want to show that $\phi(\overline{v}, \overline{w}) \wedge \theta(\overline{w})$ isolates $\text{tp}^{\mathcal{M}}(\bar{a}/B)$.

Suppose $\mathcal{M} \models \psi(\bar{a}, \bar{b})$. Because $\phi(\overline{v}, \bar{b})$ isolates $\text{tp}^{\mathcal{M}}(\bar{a}/B)$, we have

$$\text{Th}_A(\mathcal{M}) \models \theta(\overline{w}) \to \psi(\overline{v}, \bar{b}).$$

Furthermore, because $\theta(\overline{w})$ isolates $\text{tp}^{\mathcal{M}}(\bar{b}/A)$,

$$\text{Th}_A(\mathcal{M}) \models \theta(\overline{w}) \models (\phi(\overline{v}, \overline{w}) \to \psi(\overline{v}, \overline{w})),$$

and, consequently,

$$\text{Th}_A(\mathcal{M}) \models (\theta(\overline{w}) \wedge \phi(\overline{v}, \overline{w})) \to \psi(\overline{v}, \overline{w}),$$

which is what we want.

Because $\text{tp}^{\mathcal{M}}(\bar{a}, \bar{b}/A)$ is isolated, we know that $\text{tp}^{\mathcal{M}}(\bar{a}/A)$ is isolated, by the previous lemma. □

3. Prime Models and Stable Theories

Definition 3.1. Suppose that $\mathcal{M} \models T$. We say $\mathcal{M}$ is a *prime model* of T if whenever $\mathcal{N} \models T$ there is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$. Furthermore, given some $A \subseteq M$, we say that $\mathcal{M}$ is *prime over* A if whenever $\mathcal{N} \models T$ and $f : A \to \mathcal{N}$ is partial elementary, there is an elementary $f^* : \mathcal{M} \to \mathcal{N}$ extending f.

Definition 3.2. We say that a theory T is *ω-stable* if, given any $\mathcal{M}$ such that $\mathcal{M} \models T$ and countable $A \subseteq M$, then $S_n^{\mathcal{M}}(A)$ is countable. Moreover, we say that T is *κ-stable* if, whenever $\mathcal{M} \models T$ and $A \subset M$ with $|A| = \kappa > \aleph_0$, we have $|S_n^{\mathcal{M}}(A)| = \kappa$.

Theorem 3.3. *Suppose that T is a complete theory in a countable language. If T is ω-stable, then for all structures $\mathcal{M}$ so that $\mathcal{M} \models T$ and all $A \subseteq M$, the isolated types in $S_n^{\mathcal{M}}(A)$ are dense.*

Proof. Suppose not. We know, then, there is an $\mathcal{L}_A$ formula ϕ so that $[\phi]$ does not contain any isolated types. By Lemma 2.6, we can find ψ so that $[\phi \wedge \psi]$ and $[\phi \wedge \neg\psi]$ are non-empty and do not contain isolated types. In particular, we can repeat this process arbitrarily to build a tree as follows:

- Let $\phi_\emptyset = \phi$, for any ϕ so that $[\phi]$ does not contain any isolated types.
- For each $\sigma \in 2^{<\omega}$, put $\phi_{\sigma\frown 0} = \phi_\sigma \wedge \psi$ and $\phi_{\sigma\frown 1} = \phi_\sigma \wedge \neg\psi$, where ψ is some formula chosen as in the lemma.

We observe that each $[\phi_\sigma]$ is non-empty and does not contain any isolated types and, futhermore, if $\sigma \subset \tau$, then $\phi_\tau \models \phi_\sigma$ and $\phi_{\sigma\frown i} \models \neg\phi_{\sigma\frown(1-i)}$. Notice that each branch of the tree encodes a consistent set of formulas - a type - and any two branches disagree about a formula and are hence pairwise inconsistent.

Let A_0 be the set of all parameters occurring in all ϕ_σ. Since this is a countable union of finite sets, A_0 is countable. Since there are $2^{\aleph_0}$ branches in the tree, we can conclude that $|S_n^{\mathcal{M}}(A_0)| = 2^{\aleph_0}$, which contradicts the assumption of ω-stability. Therefore, the isolated types are dense. □

Lemma 3.4. *Suppose $A \subseteq M$. If $f : A \to \mathcal{N}$ is a partial elementary embedding, then $S_n^{\mathcal{M}}(A)$ is homeomorphic to $S^{\mathcal{N}}(f(A))$.*

Proof. If $f : A \to \mathcal{N}$ is a partial elementary embedding, then we can show that f induces a homeomorphism so that $p \mapsto f(p)$, where we define

$$f(p) = \{\phi(\overline{v}, f(\overline{a})) | \phi(\overline{v}, \overline{a}) \in p\},$$

where $p \in S_n^{\mathcal{M}}(A)$.

First, we can show that $p \mapsto f(p)$ is surjective. Given $p \in S_n^{\mathcal{M}}(A)$, fix any finite $\Delta \subset f(p)$. We then have

$$\Delta = \{\phi_1(\overline{v}, f(\overline{a})), \ldots, \phi_m(\overline{v}, f(\overline{a}))\}.$$

Since $p \in S_n^{\mathcal{M}}(A)$, we know that $p \cup \mathrm{Th}_A(\mathcal{M})$ is consistent, so

$$\mathcal{M} \models \exists \overline{v} \bigwedge_{i=1}^{m} \phi_i(\overline{v}, \overline{a})$$

and, since f is a partial elementary embedding, we have

$$\mathcal{N} \models \exists \overline{v} \bigwedge_{i=1}^{m} \phi_i(\overline{v}, f(\overline{a})),$$

so, by compactness, $f(p) \cup \mathrm{Th}_{f(A)}(\mathcal{N})$. This shows that if $p \in S_n^{\mathcal{M}}(A)$, then $f(p) \in S_n^{\mathcal{N}}(f(A))$. A similar argument shows that $p \in S_n^{\mathcal{M}}(A)$ only if $f(p) \in S_n^{\mathcal{N}}(f(A))$, so the mapping $p \mapsto f(p)$ is a surjection.

To see that it must also be injective, suppose we have $p, q \in S_n^{\mathcal{M}}(A)$ with $p \neq q$. Since these are complete types, there must be some $\psi(\overline{v}, \overline{a}) \in p$ and $\neg\psi(\overline{v}, \overline{a}) \in q$. If $f(p) = f(q)$, then $\psi(\overline{v}, f(\overline{a})) \wedge \neg\psi(\overline{v}, f(\overline{a})) \in f(p) = f(q)$, which contradicts the fact that $f(p) \cup \mathrm{Th}_{f(A)}(\mathcal{N})$ is satisfiable. Therefore, $p \mapsto f(p)$ is a bijection.

Now to see that $p \mapsto f(p)$ is continuous, we simply note that if we have

$$[\phi(\overline{v}, f(\overline{a}))] = \{f(p) \in S_n^{\mathcal{N}}(f(A)) | \phi(\overline{v}, f(\overline{a})) \in f(p)\},$$

an open subset of $S_n^{\mathcal{N}}(f(A))$, then we get

$$f^{-1}([\phi(\overline{v}, f(\overline{a}))]) = \{p \in S_n^{\mathcal{M}}(A) | \phi(\overline{v}, \overline{a}) \in p\} = [\phi(\overline{v}, \overline{a})],$$

an open subset of $S_n^{\mathcal{M}}(A)$. This shows that the preimage of an open set is open, so $p \mapsto f(p)$ is continuous. It is clear that a similar argument shows that $p \mapsto f^{-1}(p)$ is continuous, so $S_n^{\mathcal{M}}(A)$ is homeomorphic to $S_n^{\mathcal{N}}(f(A))$ under the map induced by f. □

Theorem 3.5. *Given an ω-stable theory T, let $\mathcal{M} \models T$ and $A \subseteq M$. There is an elementary substructure of $\mathcal{M}$, $\mathcal{M}_0$, which is a prime model extension over A, and it can be selected so that every element of $\mathcal{M}_0$ realizes an isolated type over A.*

Proof. We'll build a sequence $(A_\alpha)_{\alpha \leq \delta}$ of nested subsets of M as follows:

- Set $A_0 = A$,
- For a limit ordinal α, we put $A_\alpha = \bigcup_{\beta < \alpha} A_\beta$,
- If no element $M \setminus A_\alpha$ realizes an isolated type, then we put $\delta = \alpha$. If not, we pick some a_α which realizes an isolated type over A_α and let $A_{\alpha+1} = A_\alpha \cup \{a_\alpha\}$.

We then let $\mathcal{M}_0$ be the substructure of $\mathcal{M}$ with underlying set A_δ.

We first want to show that $\mathcal{M}_0$ is indeed an elementary substructure of $\mathcal{M}$. We can check by applying Tarski-Vaught (Theorem 1.3). Suppose $\mathcal{M} \models \exists v\phi(\overline{v}, \overline{a})$, where $\overline{a} \in A_\delta$. Since the isolated types in $S^{\mathcal{M}}(A_\delta)$ are dense, by Theorem 3.3, there is a $b \in M$ so that $\mathcal{M} \models \phi(b, \overline{a})$ and $\text{tp}^{\mathcal{M}}(b/A_\delta)$ is isolated. But we chose δ so that $b \in A_\delta$ so $\mathcal{M}_0$ is an elementary substructure of $\mathcal{M}$.

Next, we want to show that $\mathcal{M}_0$ is a prime model extension over A. Suppose $\mathcal{N} \models T$ and $f : A \to \mathcal{N}$ is partial elementary. We can show by a quick induction argument that f can be extended into an elementary map $f^* : \mathcal{M} \to \mathcal{N}$. We want to build a sequence of functions

$$f = f_0 \subset \ldots \subset f_\alpha \subset \ldots \subset f_\delta,$$

where $f_\alpha : A_\alpha \to \mathcal{N}$ is elementary. In this sequence, if α is a limit ordinal, then we set

$$f_\alpha = \bigcup_{\beta<\alpha} f_\beta.$$

Given a partial elementary embedding $f_\alpha : A_\alpha \to \mathcal{N}$, suppose $\phi(v, \overline{a})$ isolates $\text{tp}^{\mathcal{M}_0}(a_\alpha/A_\alpha)$. Because f_α is partial elementary, by Lemma 3.4, we know that $\phi(v, f_\alpha(\overline{a}))$ isolates $f_\alpha(\text{tp}^{\mathcal{M}_0}(a_\alpha/A_\alpha))$ in $S_1^{\mathcal{N}}(f_\alpha(A))$. Additionally, we know that, since f_α is partial elementary, there is a $b \in N$ so that $\mathcal{N} \models \phi(b, f_\alpha(\overline{a}))$. Therefore, $f_{\alpha+1} = f_\alpha \cup \{(a_\alpha, \overline{b})\}$ is elementary.

This shows that $f_\delta : \mathcal{M}_0 \to \mathcal{N}$ is elementary, so $\mathcal{M}_0$ is a prime model extension over A. It follows immediately from lemmas 2.6 and 2.7 that every element in $\mathcal{M}_0$ realizes an isolated type over A. □

Definition 3.6. Let κ be an infinite cardinal. We say $\mathcal{M} \models T$ is *κ-homogeneous* if, whenever $A \subset M$ with $|A| < \kappa$, $f : A \to M$ is a partial elementary map and if $a \in M$, then there is $f^* \supseteq f$ so that $f^* : A \cup \{a\} \to M$ is partial elementary. Often, we simply say $\mathcal{M}$ is *homogeneous* when it is $|M|$-homogeneous.

Theorem 3.7. *Let T be a complete theory in a countable language. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are countable homogeneous models of T and $\mathcal{M}$ and $\mathcal{N}$ realize the same types in $S_n(T)$ for $n \geq 1$. Then $\mathcal{M} \cong \mathcal{N}$.*

Proof. For this proof, we construct an isomorphism $f : \mathcal{M} \to \mathcal{N}$ by building a sequence of partial elementary maps with finite domain and letting f be their union. Since M and N are countable, let $M = \{a_n | n \in \mathbb{N}\}$ and $N = \{b_n | n \in \mathbb{N}\}$. In our construction, we will make sure $a_i \in \text{dom}(f_{2i+1})$ and $b_i \in \text{ran}(f_{2i+2})$, so when we pass to their union, we have $\text{dom}(f) = M$ and $\text{ran}(f) = N$ with $f : \mathcal{M} \to \mathcal{N}$ an isomorphism.

We start by setting $f_0 = \emptyset$. Because T is complete, f_0 is partial elementary. For the inductive step, we assume f_s is a partial elementary embedding, with $\overline{a} = \text{dom}(f_s)$ and $f_s(\overline{a}) = \overline{b}$.

For $s + 1 = 2i + 1$, we let $p = \text{tp}^{\mathcal{M}}(\overline{a}, a_i)$. Because $\mathcal{M}$ and $\mathcal{N}$ realize the same types, we can find $\overline{c}, d \in N$ so that $\text{tp}^{\mathcal{N}}(\overline{c}, d) = p$. We have $\text{tp}^{\mathcal{N}}(\overline{c}) = \text{tp}^{\mathcal{M}}(\overline{a})$ by choice of $\overline{c}$, and $\text{tp}^{\mathcal{M}}(\overline{a}) = \text{tp}^{\mathcal{N}}(\overline{b})$ because f_s is partial elementary. Therefore $\text{tp}^{\mathcal{N}}(\overline{c}) = \text{tp}^{\mathcal{N}}(\overline{b})$. Because $\mathcal{N}$ is homogeneous, there is an $e \in N$ so that $\text{tp}^{\mathcal{N}}(\overline{b}, e) = \text{tp}^{\mathcal{N}}(\overline{c}, d) = p$. Thus, we have $f_{s+1} = f_s \cup \{(a_i, e)\}$, a partial elementary embedding with $a_i \in \text{dom}(f_{s+1})$.

For $s+1 = 2_i + 2$, we can repeat the above steps to find $\overline{c}, d \in M$ so that $\mathrm{tp}^{\mathcal{M}}(\overline{c}, d) = \mathrm{tp}^{\mathcal{N}}(\overline{b}, b_i)$. Because $\mathcal{M}$ is homogeneous, there is $e \in M$ so that $\mathrm{tp}^{\mathcal{M}}(\overline{c}, d) = \mathrm{tp}^{\mathcal{M}}(\overline{a}, e)$. We then have $f_{s+1} = f_s \cup \{(e, b_i)\}$, with $b_i \in \mathrm{ran}(f_{s+1})$.

If we set

$$f = \bigcup_{i<\omega} f_i,$$

we clearly obtain an isomorphism between $\mathcal{M}$ and $\mathcal{N}$, which is what we want. □

We underwent this digression on homogeneous models in order to develop a tool to show that two prime models are isomorphic. This comes up naturally in a discussion of ω-stable theories, since models of ω-stable theories have elementary prime model extensions.

Theorem 3.8. *Let T be a complete theory in a countable language. If $\mathcal{M}$ and $\mathcal{N}$ are prime models of T, then $\mathcal{M} \cong \mathcal{N}$.*

Proof. First, we can show that if $\mathcal{M}$ is a prime model of T, then $\mathrm{tp}^{\mathcal{M}}(\overline{a})$ is isolated for all $\overline{a} \in M^n$. Suppose $j : \mathcal{M} \to \mathcal{N}$ is an elementary embedding. If $\overline{a} \in M^n$ realizes $p \in S_n(T)$, then $j(\overline{a})$ must do so, as well. But if $p \in S_n(T)$ is non-isolated, then there is an $\mathcal{N}$ so that $\mathcal{N}$ omits p, by the omitting types theorem (Theorem 2.5). Therefore, $\mathcal{M}$ cannot realize a non-isolated type. Since $\mathrm{tp}^{\mathcal{M}}(\overline{a})$ is realized in $\mathcal{M}$ for all $\overline{a} \in M^n$, we know that $\mathrm{tp}^{\mathcal{M}}(\overline{a})$ is isolated.

Next, we can show that if $\mathcal{M}$ is a prime model, then it is $\aleph_0$-homogeneous. Suppose $\overline{a} \mapsto \overline{b}$ is elementary and $c \in M$. Let $\phi(\overline{v}, w)$ isolate $\mathrm{tp}^{\mathcal{M}}(\overline{a}, c)$. Because $\mathcal{M} \models \exists w \phi(\overline{a}, w)$ and $\overline{a} \mapsto \overline{b}$ is elementary, we know $\mathcal{M} \models \exists w \phi(\overline{b}, w)$. Suppose $\mathcal{M} \models \phi(\overline{b}, d)$. Since $\phi(\overline{v}, w)$ isolates a type $\mathrm{tp}^{\mathcal{M}}(\overline{a}, c) = \mathrm{tp}^{\mathcal{M}}(\overline{b}, d)$, so $\overline{a}, c \mapsto \overline{b}, d$ is elementary. This shows that $\mathcal{M}$ is homogeneous.

Finally, if we have $\mathcal{M}$, $\mathcal{N}$ prime models of T, we know that they realize the same types and are countable homogeneous models, so by the previous theorem, $\mathcal{M} \cong \mathcal{N}$. □

4. Vaughtian Pairs

Given some formula ϕ in n free variables and an $\mathcal{L}$-structure $\mathcal{M}$, let $\phi(\mathcal{M}) = \{\overline{x} \in M^n | \mathcal{M} \models \phi(\overline{x})\}$. This is called the set defined by ϕ in $\mathcal{M}$.

Definition 4.1. Let κ and λ be cardinals with $\kappa > \lambda \geq \aleph_0$. An $\mathcal{L}$-theory T has a (κ, λ)-*model* if there is an $\mathcal{L}$-structure $\mathcal{M}$ so that $\mathcal{M} \models T$ and $|M| = \kappa$ and an $\mathcal{L}$-formula $\phi(\overline{v})$ with $|\phi(\mathcal{M})| = \lambda$.

Definition 4.2. A *Vaughtian pair* is a pair of models of T, $(\mathcal{N}, \mathcal{M})$, for which the following hold:

(1) there is an elementary embedding $j : \mathcal{M} \to \mathcal{N}$
(2) $M \neq N$
(3) there is an $\mathcal{L}_M$-formula ϕ such that $\phi(\mathcal{M})$ is infinite and $\phi(\mathcal{M}) = \phi(\mathcal{N})$

Lemma 4.3. *If T has a (κ, λ)-model where $\kappa > \lambda \geq \aleph_0$, then there is $(\mathcal{N}, \mathcal{M})$, a Vaughtian pair of models of T.*

The notions of Vaughtian pairs and (κ, λ)-models naturally arise in a discussion of categoricity because their presence is a sure indicator that the theory is not categorical. If $\mathcal{M}$ is a (κ, λ)-model with $\phi(\mathcal{M}) = \lambda$, then one can add κ-many constants to the language and add an axiom schema to the theory asserting that

ϕ holds on all of these constants. Since λ is infinite, this will be finitely-satisfiable. Thus, compactness guarantees that there is a model $\mathcal{N}$ of the theory where the set $\phi(\mathcal{N})$ has cardinality κ. This shows that there are two non-isomorphic models of size κ, since any isomorphism $\mathcal{M} \to \mathcal{N}$ would have to bijectively map $\phi(\mathcal{M})$ onto $\phi(\mathcal{N})$, which is impossible since these definable sets have different cardinalities.

Proof. Let $\mathcal{N}$ be a (κ, λ)-model and let $X = \phi(\mathcal{N})$. Suppose that $|X| = \lambda$. Since $X \subseteq N$, by Löwenheim-Skolem, there is an elementary submodel $\mathcal{M}$ of $\mathcal{N}$ so that $X \subseteq M$ and $|M| = \lambda$. But, since $X \subseteq M$, we have $\phi(\mathcal{M}) = \phi(\mathcal{N})$. Therefore, the pair $(\mathcal{N}, \mathcal{M})$ forms a Vaughtian pair. □

Observation 4.4. Although $(\mathcal{N}, \mathcal{M})$ is a *pair* of $\mathcal{L}$-structures, it is often useful to consider them as a single structure in a suitable language, given by the following construction.

Let $\mathcal{L}' = \mathcal{L} \cup \{U\}$, where U is a unary relation symbol. If $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, we can regard $(\mathcal{N}, \mathcal{M})$ as a single $\mathcal{L}'$-structure by interpreting $U^{(\mathcal{N},\mathcal{M})} = M$. Given $\phi(v_1, \ldots, v_n)$, an atomic $\mathcal{L}$-formula, we can define the restriction of ϕ to U by letting ϕ^U be $U(v_1) \wedge \ldots \wedge U(v_n) \wedge \phi$. We can extend this definition to all formulas by letting ϕ^U be $\neg\psi^U$ whenever ϕ is $\neg\psi$ and letting ϕ^U be $\psi^U \wedge \theta^U$ whenever ϕ is $\psi \wedge \theta$. Furthermore, if ϕ is $\exists v \psi$, we can let ϕ^U be $\exists v U(v) \wedge \psi^U$.

Although we omit the proof, if $\mathcal{M}$ is an elementary submodel of $\mathcal{N}$ and $\overline{a} \in M^n$, then

$$(\mathcal{N}, \mathcal{M}) \models \phi^U(\overline{a}) \iff \mathcal{M} \models \phi(\overline{a}).$$

This construction will be very helpful in the proof of the following lemma.

Lemma 4.5. *Suppose T is a theory in a countable language. If $(\mathcal{N}, \mathcal{M})$ is a Vaughtian pair for T, then there is a Vaughtian pair $(\mathcal{N}_0, \mathcal{M}_0)$ where $\mathcal{N}_0$ is countable.*

Proof. Since $(\mathcal{N}, \mathcal{M})$ is a Vaughtian pair, we may find some $\phi \in \mathcal{L}_M$, where $\mathcal{L}_M$ is the enrichment of $\mathcal{L}$ by M, so that $\phi(\mathcal{M})$ is infinite and $\phi(\mathcal{M}) = \phi(\mathcal{N})$. Let Y be the set of parameters of M contained in ϕ. Since Y must be finite, we may apply Löwenheim-Skolem to get $(\mathcal{N}_0, \mathcal{M}_0)$, a countable $\mathcal{L}'$-structure so that $Y \subseteq M$ and $(\mathcal{N}_0, \mathcal{M}_0)$ is an elementary substructure of $(\mathcal{N}, \mathcal{M})$.

Because $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, for any formula $\psi(v_1, \ldots, v_n)$, we have

$$(\mathcal{N}, \mathcal{M}) \models \forall \overline{v} \left(\left(\bigwedge_{i=1}^{k} U(v_i) \wedge \psi_i \right) \to \psi^U(\overline{v}) \right).$$

Because $(\mathcal{N}_0, \mathcal{M}_0)$ is an elementary substructure of $(\mathcal{N}, \mathcal{M})$, the above formula is also satisfied in $(\mathcal{N}_0, \mathcal{M}_0)$. Therefore, $\mathcal{N}_0$ is an elementary substructure of $\mathcal{M}_0$.

Since the pair $(\mathcal{N}, \mathcal{M})$ forms a Vaughtian pair, there is an $\mathcal{L}_M$ formula ϕ so that $\phi(\mathcal{M}) = \phi(\mathcal{N})$ and $\phi(\mathcal{M})$ is infinite. Consequently, we may find such a ϕ with infinitely many realizations in M and none in $N \setminus M$. For each k, then, we know the following three sentences hold in $(\mathcal{N}, \mathcal{M})$:

$$\exists \overline{v}_1 \ldots \exists \overline{v}_k \left(\bigwedge_{i<j} \overline{v}_i \neq \overline{v}_j \wedge \bigwedge_{i=1}^{k} \phi(v_i) \right)$$

$$\exists x \neg U(x)$$

$$\forall \overline{v} \left(\phi(\overline{v}) \to \wedge U(v_i) \right),$$

since these reflect the defining conditions of a Vaughtian pair. But these sentences also hold in $(\mathcal{N}_0, \mathcal{M}_0)$, so $(\mathcal{N}_0, \mathcal{M}_0)$ is a Vaughtian pair. □

Lemma 4.6. *Suppose that $\mathcal{M}_0$ is an elementary substructure of $\mathcal{N}_0$, where each is a countable model of T. If $\overline{a} \in M_0^n$ and $p \in S_n(\overline{a})$ is realized in $\mathcal{N}_0$, then there is a pair of models $(\mathcal{N}', \mathcal{M}')$, which, when considered as a single $\mathcal{L}'$structure, is an elementary extension of $(\mathcal{N}, \mathcal{M})$, so that p is realized in $\mathcal{M}'$.*

Proof. Let the $1-$type $\Gamma(\overline{v})$ be defined by

$$\Gamma(\overline{v}) = \{\phi^U(\overline{v}, \overline{a}) | \phi(\overline{v}, \overline{a}) \in p\} \cup \{\phi(m_1, \dots, m_n) | (\mathcal{N}_0, \mathcal{M}_0) \models \phi(m_1, \dots, m_n)\}.$$

If $\phi_1, \dots, \phi_m \in p$, then we have

$$\mathcal{N}_0 \models \exists \overline{v} \left(\bigwedge_{i=1}^m \phi_i(\overline{v}, \overline{a}) \right)$$

so we know

$$\mathcal{M}_0 \models \exists \overline{v} \left(\bigwedge_{i=1}^m \phi_i(\overline{v}, \overline{a}) \right) \text{ and } (\mathcal{N}_0, \mathcal{M}_0) \models \exists \overline{v} \left(\bigwedge_{i=1}^m \phi_i^U(\overline{v}, \overline{a}) \right).$$

This shows that $\Gamma(\overline{v})$ is satisfiable, by compactness. We,then, know that there exists $(\mathcal{N}', \mathcal{M}')$, a countable elementary extension of $(\mathcal{N}_0, \mathcal{M}_0)$ realizing $\Gamma(\overline{v})$, which is what we want. □

Lemma 4.7. *Suppose that $\mathcal{M}_0$ is an elementary substructure of $\mathcal{N}_0$, where each is a countable model of T. If $\overline{b} \in N_0$ and $p \in S_n(\overline{b})$, then there is a pair of models $(\mathcal{N}', \mathcal{M}')$, which, when considered as a single $\mathcal{L}'$ − structure, is an elementary extension of $(\mathcal{N}, \mathcal{M})$ so that p is realized in $\mathcal{M}'$.*

Proof. Let $\Gamma(\overline{v})$ be defined as in the previous lemma. If $\phi_1, \dots, \phi_m \in p$, then we have

$$\mathcal{N}_0 \models \exists \overline{v} \left(\bigwedge_{i=1}^m \phi_i(\overline{v}, \overline{b}) \right),$$

so we can find a countable elementary extension of $(\mathcal{N}_0, \mathcal{M}_0)$ realizing p. □

Lemma 4.8. *Suppose that $\mathcal{M}_0$ is an elementary substructure of $\mathcal{N}_0$, where each is a countable model of T. Then we can find $\mathcal{N}$ and $\mathcal{M}$ so that the pair $(\mathcal{N}, \mathcal{M})$, when considered as a single $\mathcal{L}'$-structure, is an elementary extension of the pair $(\mathcal{N}_0, \mathcal{M}_0)$ and each is a countable, homogeneous model that realizes the same types in $S_n(T)$.*

Proof. Using the two lemmas above, we can build a chain of pairs of models $(\mathcal{N}_\alpha, \mathcal{M}_\alpha)$ so that $(\mathcal{N}_{\alpha+1}, \mathcal{M}_{\alpha+1})$ is an elementary extension of $(\mathcal{N}_\alpha, \mathcal{M}_\alpha)_{\alpha<\omega}$ for each α and each of the following conditions hold:

- if $p \in S_n(T)$ is realized in $\mathcal{N}_{3i}$, then p is realized in $\mathcal{N}_{3i+1}$,
- If $\overline{a}, \overline{b}, c \in M_{3i+1}$ and $\text{tp}^{\mathcal{M}_{3i+1}}(\overline{a}) = \text{tp}^{\mathcal{M}_{3i+1}}(\overline{b})$, then there is a $d \in M_{3i+2}$ so that $\text{tp}^{\mathcal{M}_{3i+2}}(\overline{a}, c) = \text{tp}^{\mathcal{M}_{3i+2}}(\overline{b}, d)$,
- if $\overline{a}, \overline{b}, c \in N_{3i+2}$ and $\text{tp}^{\mathcal{N}_{3i+2}}(\overline{a}) = \text{tp}^{\mathcal{N}_{3i+2}}(\overline{b})$, then there is a $d \in N_{3i+3}$ so that $\text{tp}^{\mathcal{N}_{3i+3}}(\overline{a}, c) = \text{tp}^{\mathcal{N}_{3i+3}}(\overline{b}, d)$.

Lemma 4.6 proves that the first two conditions can be met, and the Lemma 4.7 tells us that the third can be satisfied.

Let

$$(\mathcal{N}, \mathcal{M}) = \bigcup_{\alpha<\omega} (\mathcal{N}_\alpha, \mathcal{M}_\alpha).$$

Since this is a countable union of countable models, $(\mathcal{N}, \mathcal{M})$ is a countable Vaughtian pair. By the first condition, $\mathcal{M}$ and $\mathcal{N}$ realize the same types. By the second and third condition, $\mathcal{M}$ and $\mathcal{N}$ are homogeneous and, therefore, isomorphic, by Theorem 3.7. □

The previous results will allow us to prove the following theorem, which shows that the presence of a (κ, λ)-model can always be witnessed by a model of size $\aleph_1$ and a countable definable set.

Theorem 4.9 (**Vaught's Two-Cardinal Theorem**). *Suppose T is in a countable language. If T has a (κ, λ)-model where $\kappa > \lambda \geq \aleph_0$, then T has an $(\aleph_1, \aleph_0)$-model.*

Proof. Suppose T has a (κ, λ)-model. By earlier results, we can find $(\mathcal{N}, \mathcal{M})$, a countable Vaughtian pair so that $\mathcal{M}$ and $\mathcal{N}$ are homogeneous models realizing the same types. Let $\phi(\overline{v})$ be an $\mathcal{L}_M$-formula with infinitely many realizations in M and none in $N \setminus M$.

We build an elementary chain $(\mathcal{N}_\alpha)_{\alpha<\omega_1}$ so that for each α we have isomorphisms $\mathcal{N}_\alpha \cong \mathcal{N}$ and $(\mathcal{N}_{\alpha+1}, \mathcal{N}_\alpha) \cong (\mathcal{N}, \mathcal{M})$, and with the property that $\mathcal{N}_{\alpha+1} \setminus \mathcal{N}_\alpha$ contains no elements satisfying ϕ.

To this end, let $\mathcal{N}_0 = \mathcal{N}$. If α is a limit ordinal, we set

$$\mathcal{N}_\alpha = \bigcup_{\beta<\alpha} \mathcal{N}_\beta.$$

Because $\mathcal{N}_\alpha$ is a union of models isomorphic to $\mathcal{N}$, we know $\mathcal{N}_\alpha$ is homogeneous and realizes the same types as $\mathcal{N}$. Consequently, there is an isomorphism $\mathcal{N}_\alpha \cong \mathcal{N}$ by Theorem 4.7.

Given any $\mathcal{N}_\alpha$ isomorphic to $\mathcal{N}$, since there is an isomorphism $\mathcal{N} \cong \mathcal{M}$, there is an elementary extension $\mathcal{N}_{\alpha+1}$ of $\mathcal{N}_\alpha$ so that $(\mathcal{N}, \mathcal{M})$ is isomorphic to $(\mathcal{N}_{\alpha+1}, \mathcal{N}_\alpha)$.

In this case, we have $\mathcal{N}_{\alpha+1} \cong \mathcal{N}$. Let

$$\mathcal{N}^* = \bigcup_{\alpha<\omega_1} \mathcal{N}_\alpha.$$

Then we have $|N^*| = \aleph_1$ and, if $\mathcal{N}^* \models \phi(\overline{a})$, then $\overline{a} \in M$, so $\mathcal{N}^*$ is an $(\aleph_1, \aleph_0)$-model. □

Lemma 4.10. *Suppose that T is ω-stable, $\mathcal{M} \models T$, and $|M| \geq \aleph_1$. There is an $\mathcal{L}_M$-formula $\phi(v)$ so that $|[\phi(v)]| \geq \aleph_1$ and for all $\mathcal{L}_M$-formulas $\psi(v)$, either $|[\phi(v) \wedge \psi(v)]| \leq \aleph_0$ or $[\phi(v) \wedge \psi(v)]| \leq \aleph_0$.*

Proof. If ϕ is the formula $v = v$, then clearly ϕ holds for every element in M so we have $|[\phi]| = |M| \geq \aleph_1$.

To see that for every $\mathcal{L}_M$ formula $\psi(v)$ either $[\phi(v) \wedge \psi(v)]$ or $[\phi(v) \wedge \neg\psi(v)]$ is countable, we will do a proof by contradiction. Suppose that for any $\mathcal{L}_M$ formula $\theta(v)$ with $|[\theta(v)]| \geq \aleph_1$, there is a formula $\psi(v)$ so that both $[\theta(v) \wedge \psi(v)]$ and $[\theta(v) \wedge \neg\psi(v)]$ are uncountable. To generate a contradiction, we will utilize the ω-stability of T and build a tree.

Let $\theta_\emptyset$ be ϕ as above, the formula $v = v$. By our supposition, for any $\sigma \in 2^{<\omega}$, we can find a $\psi(v)$ so that $[\theta_\sigma(v) \wedge \psi(v)]$ and $[\theta_\sigma \wedge \neg\psi(v)]$ are both uncountable. Setting $\theta_{\sigma\frown 0}(v) = \theta_\sigma(v) \wedge \psi(v)$ and $\theta_{\sigma\frown 1}(v) = \theta_\sigma(v) \wedge \neg\psi(v)$ for such a ψ, we obtain a tree $(\theta_\sigma)_{\sigma\in 2^{<\omega}}$ so that $|[\theta_\sigma]| \geq \aleph_1$ and $[\theta_{\sigma\frown 0}] \cap [\theta_{\sigma\frown 1}] = \emptyset$.

Letting A be countable set of parameters occurring in all of the θ_σ, we can count branches to determine that $|S_1^{\mathcal{M}}(A)| = 2^{\aleph_0}$, which contradicts the ω-stability of T. This completes the proof. □

Lemma 4.11. *Suppose T is ω-stable, $\mathcal{M} \models T$ and $|M| \geq \aleph_1$. There is a proper elementary extension $\mathcal{N}$ of $\mathcal{M}$ so that if $\Gamma(\overline{w})$ is a countable type over M realized in $\mathcal{N}$, then $\Gamma(\overline{w})$ is realized in $\mathcal{M}$.*

Proof. As in the previous lemma, let $\phi(v)$ be the formula $v = v$. We can define a type

$$p = \{\psi(v) | \psi(v) \text{an } \mathcal{L}_M\text{-formula and } |[\phi(v) \wedge \psi(v)]| \geq \aleph_1\}.$$

If $\psi_1, \ldots, \psi_m \in p$, then, by the previous lemma, $|[\phi(v) \wedge \neg\psi_i(v)]| \leq \aleph_0$ for $i = 1, \ldots, m$. Since we know

$$\bigcup_{i=1}^{m} [\phi(v) \wedge \neg\psi_i(v)] = [\phi(v) \wedge \bigvee_{i=1}^{m} \neg\psi_i(v)],$$

and a finite union of countable sets is countable, we have

$$|[\phi(v) \wedge \bigvee_{i=1}^{m} \neg\psi_i(v)]| \leq \aleph_0.$$

Using DeMorgan's law and Lemma 4.10 once again, this shows that $\bigwedge_{i=1}^{m} \psi_i(v) \in p$ and p is finitely satisfiable. Since $[\phi(v)]$ is uncountable, for each $\mathcal{L}_M$-formula $\psi(v)$, exactly one of $\psi(v)$ or $\neg\psi(v)$ is in p, so p is a complete type over M.

Let $\mathcal{M}'$ be an elementary extension of $\mathcal{M}$ containing c, a realization of p. By Theorem 3.5, there is an elementary substructure $\mathcal{N}$ of $\mathcal{M}'$ which is prime over $M \cup \{c\}$ such that every $\overline{a} \in N$ realizes an isolated type over $M \cup \{c\}$.

Let $\Gamma(\overline{w})$ be a countable type over $\mathcal{M}$ realized by $\overline{b} \in N$. There is an $\mathcal{L}_M$-formula $\theta(\overline{w}, v)$ so that $\theta(\overline{w}, c)$ isolates $\text{tp}^{\mathcal{M}}(\overline{b}/M \cup \{c\})$. Note that $\exists\overline{w}\theta(\overline{w}, v) \in p$ and also, since θ isolates a type, we have

$$\forall\overline{w}(\theta(\overline{w}, v) \to \gamma(\overline{w})) \in p,$$

for all $\gamma(\overline{w}) \in \Gamma$. Let

$$\Delta = \{\exists\overline{w}\theta(\overline{w}, v)\} \cup \{\forall\overline{w}(\theta(\overline{w}, v) \to \gamma(\overline{w})) | \gamma \in \Gamma\}.$$

We then know that $\Delta \subset p$ is countable and, if c' realizes Δ, then $\exists\overline{w}\theta(\overline{w}, c')$, and if $\theta(\overline{b}', c')$, then $\overline{b}'$ realizes Γ.

Since Δ is countable, we can write $\Delta = \{\delta_n(v) | n \in \mathbb{N}\}$. By the construction of p, we have $|\{x \in M | \phi(x)\}| \geq \aleph_1$ and, additionally,

$$|\{x \in M | \phi(x) \wedge \neg(\delta_0(x) \wedge \ldots \wedge \delta_n(x))\}| \leq \aleph_0,$$

for all $n \in \mathbb{N}$. By a similar argument as above, we can then conclude

$$|\{x \in M | \phi(x) \text{ and } x \text{ realizes } \Delta\}| \geq \aleph_1.$$

Let $c' \in M$ realize Δ and choose $\overline{b}'$ so that $\mathcal{M} \models \theta(\overline{b}', c')$. In this case, $\overline{b}'$ is a realization of Γ in $\mathcal{M}$. □

Theorem 4.12. *Suppose that T is ω-stable and there is an $(\aleph_1, \aleph_0)$-model of T. If $\kappa > \aleph_1$, then there is a $(\kappa, \aleph_0)$-model of T.*

Proof. Let $\mathcal{M} \models T$ with $|M| \geq \aleph_1$ and pick ϕ so that $|\phi(\mathcal{M})| = \aleph_0$. Furthermore, let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$ as in Lemma 4.11.

The type $\Gamma(v)$ defined as

$$\Gamma(v) = \{\phi(v)\} \cup \{v \neq m | m \in M \text{ and } \mathcal{M} \models \phi(m)\}$$

is a countable type omitted in $\mathcal{M}$ and hence in $\mathcal{N}$. Therefore, $\phi(\mathcal{N}) = \phi(\mathcal{M})$.

We can utilize this method to build a chain $(\mathcal{M}_\alpha)_{\alpha<\kappa}$ in the following manner: let $\mathcal{M}_0 = \mathcal{M}$ and let $\mathcal{M}_{\alpha+1}$ be an elementary extension of $\mathcal{M}_\alpha$ so that $\mathcal{M}_{\alpha+1} \neq \mathcal{M}_\alpha$ and $\phi(\mathcal{M}_{\alpha+1}) = \phi(\mathcal{M}_\alpha)$.

Let $\mathcal{N} = \bigcup_{\alpha<\kappa} \mathcal{M}_\alpha$. then $\mathcal{N}$ is a $(\kappa, \aleph_0)$-model of T. □

5. Order indiscernibles

In this section, we utilize Vaught's Two-Cardinal Theorem and related results to show that a theory categorical in some uncountable power must be ω-stable and can have no Vaughtian pairs. Before we can prove that, however, we need to introduce the notion of order indiscernibles.

Definition 5.1. Let $(I, <)$ be an ordered set and let $(x_i)_{i\in I}$ be a sequence of distinct elements of M. We say $(x_i)_{i\in I}$ is a sequence of *order indiscernibles* if whenever $i_1 < i_2 < \ldots < i_m$ and $j_1 < j_2 < \ldots < j_m$ are two increasing sequences from I, then we have

$$\mathcal{M} \models \phi(x_{i_1}, \ldots, x_{i_m}) \leftrightarrow \phi(x_{j_1}, \ldots, x_{j_m}).$$

The *order type* of a set of indiscernibles $(I, <)$ is the ordinal α so that $(I, <)$ is isomorphic to $(\alpha, \in)$.

Order indiscernibles are sequences that can not be detected as different by formulas, so long as any two sequences are ordered the same. They are, for this reason, "indiscernible." We omit the argument, but indiscernible sequences exist in some model of any given theory by compactness and Ramsey's theorem.

Definition 5.2. Suppose $\mathcal{M} \models T^*$ and $X \subseteq M$. Let $\mathcal{H}(X)$ be the $\mathcal{L}^*$-substructure of $\mathcal{M}$ generated by X. We call $\mathcal{H}(X)$ the *Skolem hull* of X.

Theorem 5.3. *Let $\mathcal{L}$ be countable and T be an $\mathcal{L}$-theory with infinite models. For all $\kappa \geq \aleph_0$, there is a model $\mathcal{M}$ so that $\mathcal{M} \models T^*$ with $|M| = \kappa$ and if $A \subseteq M$, then $\mathcal{M}$ realizes at most $|A| + \aleph_0$ types in $S_n^{\mathcal{M}}(A)$.*

Proof. The strategy of this proof is to define equivalence classes of elements that satisfy the same formulas, and then count them to determine the maximum number of realized types. Throughout, we'll work only with 1-types, although the proof will generalize in an obvious way. Let $\mathcal{L}^*$ and T^* be as above. Let $\mathcal{M}$ such that $\mathcal{M} \models T$ be the Skolem hull of a sequence of order indiscernibles I of order type $(\kappa, <)$, so that $|M| = \kappa$.

Given $A \subseteq M$, for each $a \in A$, there is a term t_a and sequence from I so that $\overline{x}_a$ so that $a = t_a(\overline{x}_a)$. Let $X = \{x \in I | x \text{ occurs in some } \overline{x}_a\}$. Then we have $|X| \leq |A| + \aleph_0$.

Given X as above, we can create an equivalence relation on sequences as follows: if $y_1 < \ldots < y_n$ and $z_1 < \ldots < z_n$, say $\overline{y} \sim_X \overline{z}$ if, for all $x \in X$,

(1) $y_i < x$ if and only if $z_i < x$
(2) $y_i = x$ if and only if $z_i = x$,

where $i = 1, \dots, n$. This equivalence relation considers all sequences equivalent if they are indiscernible - i.e. in the same position with respect to the ordering.

Now we want to show that $\overline{y} \sim_X \overline{z}$ and t is a Skolem term, then $t(\overline{y})$ and $t(\overline{z})$ realize the same type in $S_1^{\mathcal{M}}(A)$.

Let $a_1, \dots, a_m \in A$. Because $\overline{y}$ and $\overline{z}$ are in the same position in the ordering with respect to X, by indiscernability, we have

$$\begin{aligned}\mathcal{M} \models \phi(t(\overline{y}), a_1, \dots, a_m) &\iff \mathcal{M} \models \phi(t(\overline{y}), t_{a_1}(\overline{x}_{a_1}), \dots, t_{a_m}(\overline{x}_{a_m}))\\ &\iff \mathcal{M} \models \phi(t(\overline{z}), t_{a_1}(\overline{x}_{a_1}), \dots, t_{a_m}(\overline{x}_{a_m}))\\ &\iff \mathcal{M} \models \phi(t(\overline{z}), a_1, \dots, a_m).\end{aligned}$$

Next, we can show that $|I^n/\sim_X| \leq |A| + \aleph_0$. For $y \in I\backslash X$, let $C_y = \{x \in X | x < y\}$. We call each C_y a *cut*. Accordingly, we note $\overline{y} \sim_X \overline{z}$ if and only if

- if $y_i \in X$, then $y_i = z_i$
- if $y_i \notin X$, then $z_i \notin X$ and $C_y = C_z = \emptyset$,

for $i = 1, \dots, n$. Because I is well-ordered, $C_y = C_z$ if and only if $C_y = C_z = \emptyset$ or

$$\inf\{i \in I | i > C_y\} = \inf\{i \in I | i > C_z\}.$$

It is clear that there can be at most $|X|+1$ possible cuts C_y. Therefore, $|I^n/\sim_X| \leq |A| + \aleph_0$ and $\mathcal{M}$ realizes at most $|A| + \aleph_0$ types over A. □

Theorem 5.4. *Let T be a complete theory in a countable language $\mathcal{L}$ with infinite models. If T is κ-categorical, for some $\kappa \geq \aleph_1$, then T is ω-stable and has no Vaughtian pairs.*

Proof. We will start by showing that T must be ω-stable. Supposing that it is not ω-stable, we know that there is a countable model $\mathcal{M} \models T$ and $A \subseteq M$ so that $|S_n^{\mathcal{M}}(A)| > \aleph_0$. We know that we can find $\mathcal{N}_0$, an elementary extension of $\mathcal{M}$, which has cardinality κ. By Theorem 5.4, we can find a model $\mathcal{N}_1$ of cardinality κ so that $N_1 \models T$ and for all $B \subseteq M$, if B is countable, then $\mathcal{N}_1$ realizes at most countably many types over B. Clearly, $\mathcal{N}_0 \ncong N_1$, which contradicts the assumption that T is κ-categorical.

Next, we know from Theorem 4.12 that, since T is ω-stable, if it has a Vaughtian pair of models, then there is an $(\aleph_1, \aleph_0)$-model and, consequently, a $(\kappa, \aleph_0)$ model. Because we can find a model of T of cardinality κ where every infinite definable set has cardinality κ, this is a contradiction. □

6. Strong Minimality

Two canonical examples of uncountably categorical theories are algebraically closed fields and vector spaces over a countable field. In this section, we make use of the features of each of these structures to handle the case of an arbitrary uncountably categorical theory. In particular, we show that uncountably categorical theories admit a notion of dimension which allows one to characterize models up to isomorphism. Along the way, we introduce a general notion of closure which mimics that of algebraic closure in the case of algebraically closed fields.

Given an $\mathcal{L}$-structure $\mathcal{M}$, we say that $X \subseteq M^n$ is *definable* if and only if there is an $\mathcal{L}$-formula $\phi(\overline{v}, \overline{w})$ and $\overline{b} \in M^m$ so that $X = \{\overline{a} \in M^n | \mathcal{M} \models \phi(\overline{a}, \overline{b})\}$. In this case, we say $\phi(\overline{v}, \overline{b})$ *defines* X. If $A \subseteq M$, we say that X is *A-definable* or *definable over A* if there is an $\mathcal{L}$-formula $\psi(\overline{v}, \overline{w})$ and $\overline{b} \in A^l$ so that ψ defines X.

Definition 6.1. If $\mathcal{M}$ is an $\mathcal{L}$-structure and $D \subseteq M^n$ is an infinite definable set, we say D is *minimal* in $\mathcal{M}$ if, for any definable $Y \subseteq D$, either Y is finite or $D \setminus Y$ is finite. If $\phi(\overline{v}, \overline{a})$ defines such a D, then $\phi(\overline{v}, \overline{a})$ is also said to be minimal. Such definable sets and formulas are *strongly minimal* if they are minimal in any elementary extension of $\mathcal{M}$.

Definition 6.2. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. We say $b \in M$ is *algebraic* over A if there is an $\mathcal{L}$-formula $\phi(v, \overline{w})$ and $\overline{a} \in A$ so that $\mathcal{M} \models \phi(b, \overline{a})$ and

$$\phi(\mathcal{M}, \overline{a}) = \{y \in M | \mathcal{M} \models \phi(y, \overline{a})\}$$

is finite.

It is usual to define an algebraic closure operation by setting

$$\mathrm{acl}(A) = \{x | x \text{ is algebraic over A}\}.$$

We, however, will need to restrict this operation to strongly minimal sets. If $D \subseteq M$ is strongly minimal, then given $A \subseteq D$, we set

$$\mathrm{acl}_D(A) = \{b \in D | b \text{ is algebraic over A}\}.$$

We note that this algebraic closure operation has the following property, called the *exchange principle*: if $D \subset M$ is strongly minimal, $A \subseteq D$, and $a, b \in D$, then $b \in \mathrm{acl}(A \cup \{a\})$ whenever $a \in \mathrm{acl}(A \cup \{b\}) \setminus \mathrm{acl}(A)$.

Definition 6.3. Let $\mathcal{M} \models T$ and $D \subseteq M$ be a strongly minimal subset. We say $A \subseteq D$ is *independent* if $a \notin \mathrm{acl}(A \setminus \{a\})$ for all $a \in A$. If $C \subset D$, we say A is *independent over* C if $a \notin \mathrm{acl}(C \cup (A \setminus \{a\}))$ for all $a \in A$.

Definition 6.4. A set A is a *basis* for $Y \subseteq D$ if $A \subseteq Y$ is independent and $\mathrm{acl}(A) = \mathrm{acl}(Y)$. The *dimension* of Y is the cardinality of a basis for Y, denoted by $\dim(Y)$.

If A and B are bases for $Y \subseteq D$, then it is easy to check that $|A| = |B|$, so the notion of dimension is indeed well-defined.

The idea of the above definitions is that we can abstract from familiar structures - polynomials and their roots, in the case of the algebraic closure operation, and vector spaces, in the case of bases and dimension - to the much more general setting provided by first-order formulas.

Lemma 6.5. *Suppose that $\mathcal{M}, \mathcal{N} \models T$ and $\phi(v)$ is a strongly minimal formula with parameters from $A \subseteq M_0$ where $\mathcal{M}_0 \models T$ and $\mathcal{M}_0$ is an elementary substructure of both $\mathcal{M}$ and $\mathcal{N}$. If $a_1, \ldots, a_n \in \phi(\mathcal{M})$ are independent over A and $b_1, \ldots, b_n \in \phi(\mathcal{N})$ are independent over A, then $tp^{\mathcal{M}}(\overline{a}/A) = tp^{\mathcal{N}}(\bar{b}/A)$.*

Proof. We prove this by induction. For the $n = 1$ case, suppose we have $a \in \phi(\mathcal{M}) \setminus \mathrm{acl}(A)$ and $b \in \phi(\mathcal{N}) \setminus \mathrm{acl}(A)$. Let $\psi(v)$ be a formula with parameters from A and suppose $\mathcal{M} \models \psi(a)$. Because $a \notin \mathrm{acl}(A)$, we know $\psi(\mathcal{M})$ is infinite. Because ϕ is strongly minimal, we know $\phi(\mathcal{M}) \setminus \psi(\mathcal{M})$ is finite. Therefore, there is an n so that

$$\mathcal{M} \models \exists x_1 \ldots \exists x_n \left[\bigwedge_{i<j} x_i \neq x_j \wedge \bigwedge_{i=1}^{n} \phi(x_i) \wedge \neg\psi(x_i) \right] \wedge \forall y \left[\phi(y) \wedge \neg\psi(y) \rightarrow \bigvee_{i=1}^{n} y = x_i \right],$$

that is, the cardinality of the set defined by $\phi(x) \wedge \neg\psi(x)$ has cardinality n.

Furthermore, because $\mathcal{M}_0$ is an elementary substructure of $\mathcal{M}$ and $\mathcal{N}$, and $b \notin \operatorname{acl}(A)$, we have $\mathcal{N} \models \psi(b)$. Therefore, we have $\operatorname{tp}^{\mathcal{M}}(a/A) = \operatorname{tp}^{\mathcal{N}}(b/A)$.

Assume that it holds for an arbitrary n and consider the $n+1$ case. Suppose we have $a_1, \ldots, a_{n+1} \in \phi(\mathcal{M})$ and $b_1, \ldots, b_{n+1} \in \phi(\mathcal{N})$ independent sequences over A. Let $\overline{a} = (a_1, \ldots, a_n)$ and $\overline{b} = (b_1, \ldots, b_n)$. By the inductive assumption, we have $\operatorname{tp}^{\mathcal{M}}(\overline{a}/A) = \operatorname{tp}^{\mathcal{N}}(\overline{b}/A)$. Let $\psi(\overline{w}, v)$ be a formula with parameters from A so that $\mathcal{M} \models \psi(\overline{a}, a_{n+1})$. Because $a_{n+1} \notin \operatorname{acl}(A, \overline{a})$, we have $\phi(\mathcal{M}) \cap \psi(\overline{a}, \mathcal{M})$ is infinite and $\phi(\mathcal{M}) \setminus \psi(\overline{a}, \mathcal{M})$ is finite. There is, then, an n so that

$$\mathcal{M} \models |\{v | \phi(v) \wedge \neg\psi(\overline{a}, v)\}| = n,$$

where the cardinality of the given definable set is expressed by a first-order sentence as above.

Because $\mathcal{M}_0$ is an elementary substructure of $\mathcal{M}$ and $\mathcal{N}$ and $\operatorname{tp}^{\mathcal{M}}(\overline{a}/A) = \operatorname{tp}^{\mathcal{N}}(\overline{b}/A)$, we know

$$\mathcal{N} \models |\{v | \phi(v) \wedge \neg\psi(\overline{b}, v)\}| = n.$$

Finally, because $b_{n+1} \notin \operatorname{acl}(A, \overline{b})$, we have $\mathcal{N} \models \psi(\overline{b}, b_{n+1})$. Therefore,

$$\operatorname{tp}^{\mathcal{M}}(\overline{a}, a_{n+1}/A) = \operatorname{tp}^{\mathcal{N}}(\overline{b}, b_{n+1}/A),$$

which is what we want. □

Theorem 6.6. *Suppose T is a strongly minimal theory. If $\mathcal{M}, \mathcal{N} \models T$, then $\mathcal{M} \cong \mathcal{N}$ if and only if dim(M) = dim(N). Furthermore, if $\phi(v)$ is a strongly minimal formula with parameters from A, where $A = \emptyset$ or $A \subseteq M_0$ for some $\mathcal{M}_0$, an elementary substructure of M and N, then there is a bijective partial elementary map $f : \phi(\mathcal{M}) \to \phi(\mathcal{N})$.*

Proof. Let B be a basis for $\phi(\mathcal{M})$ and C be a basis for $\phi(\mathcal{N})$. If $\mathcal{M} \cong \mathcal{N}$, then it is clear that $|B| = |C|$.

For the other direction, suppose that $\dim(M) = \dim(N)$. We can show that $\mathcal{M} \cong \mathcal{N}$ by a Zorn's lemma argument. In this case, $|B| = |C|$ so there is a bijection $f : B \to C$. By Lemma 6.5, B and C have the same type over A. This tells us that f is a partial elementary embedding, since B and C satisfy the same formulas.

Define I by

$$I = \{g : B' \to C' | B \subseteq B' \subseteq \phi(\mathcal{M}), C \subseteq C' \subseteq \phi(\mathcal{N}), f \text{ is partial elementary}\}$$

We may apply Zorn's lemma to get a maximal partial elementary map $g : B' \to C'$. Suppose $b \in \phi(\mathcal{M}) \setminus B'$. Because b is algebraic over B', there is a formula $\psi(v, \overline{d})$ which isolates $\operatorname{tp}^{\mathcal{M}}(b/B')$.

Because g is partial elementary, there is $c \in \phi(\mathcal{N})$ so that $\mathcal{N} \models \psi(c, g(d))$. Consequently, $\operatorname{tp}^{\mathcal{M}}(b/B') = \operatorname{tp}^{\mathcal{N}}(c/C')$ and g can be extended by setting $g(b) = c$. But this contradicts the maximality of g. Therefore $\phi(\mathcal{M}) = B'$.

A similar argument can be employed to show that $C' = \phi(\mathcal{N})$. □

Theorem 6.7. *Let T be an ω-stable theory. If $\mathcal{M} \models T$, then there is a minimal formula in $\mathcal{M}$.*

Proof. Suppose not. Then, as before, we can build a tree. Let $\phi_\emptyset$ be the formula $v = v$. Then $\phi_\emptyset$ is not minimal and $\phi_\emptyset(\mathcal{M})$ is infinite. If, given $\tau \in 2^{<\omega}$, ϕ_τ is a non-minimal formula such that $\phi_\tau(\mathcal{M})$ is infinite, then because it is not minimal, there is a formula ψ so that $(\phi_\tau \wedge \psi)(\mathcal{M})$ and $(\phi_\tau \wedge \neg\psi)(\mathcal{M})$ are both infinite. Let $\phi_{\tau\frown 0} = \phi_\tau \wedge \psi$ and $\phi_{\tau\frown 1} = \phi_\tau \wedge \neg\psi$.

Now, consider the resulting tree $(\phi_\sigma)_{\sigma\in 2^{<\omega}}$. If $\tau \subset \sigma$, then $\phi_\sigma \models \phi_\tau$. Furthermore, $\phi_{\sigma\frown i} \models \neg\phi_{\sigma\frown(1-i)}$, and for any $\sigma \in 2^{<\omega}$, $\phi_\sigma(\mathcal{M})$ is infinite.

Let A_0 be the set of parameters occurring in any of the ϕ_σ, which is clearly countable. Then $|S_1^{\mathcal{M}}(A_0)| = 2^{\aleph_0}$, contradicting the ω-stability of T. □

Lemma 6.8. *Suppose T is an $\mathcal{L}$-theory with no Vaughtian pairs. Let $\mathcal{M}$ be an $\mathcal{L}$-structure so that $\mathcal{M} \models T$ and let $\phi(\overline{v}, \overline{w})$ have parameters from M. There is $n \in \mathbb{N}$ so that if $\overline{a} \in M$ and $|\phi(\mathcal{M}, \overline{a})| > n$, then $\phi(\mathcal{M}, \overline{a})$ is infinite.*

Proof. Suppose not. Then for every $n \in \mathbb{N}$, there is an $\overline{a}_n \in M$ so that $\phi(\mathcal{M}, \overline{a}_n)$ is a finite set of at least size n. As before, let $\mathcal{L}' = \mathcal{L} \cup \{U\}$. Let $\Gamma(\overline{w}) \supset T$ be the $\mathcal{L}'$-type asserting (the first-order equivalent of):

- U defines a proper elementary submodel
- $\bigwedge_{i=1}^m U(w_i)$
- there are infinitely many elements $\overline{v}$ so that $\phi(\overline{v}, \overline{w})$
- $\phi(\overline{v}, \overline{w}) \to \bigwedge_{i=1}^k U(v_i)$.

Let $\mathcal{N}$ be a proper elementary extension of $\mathcal{M}$. Because $\phi(\mathcal{M}, \overline{a}_n)$ is finite and $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, we have

$$\phi(\mathcal{M}, \overline{a}_n) = \phi(\mathcal{N}, \overline{a}_n).$$

If $\Delta \subseteq \Gamma(\overline{w})$ is a finite subset, then by choosing a sufficiently large n, $\overline{a}_n$ realizes Δ in $(\mathcal{M}, \mathcal{N})$. This shows that Γ is satisfiable.

Suppose that $\overline{a}$ realizes $\Gamma(\overline{w})$ in $(\mathcal{N}', \mathcal{M}')$ where $\mathcal{M}' \models T$ and $\mathcal{N}'$ is a proper elementary extension of $\mathcal{M}'$. We then have $\phi(\mathcal{M}', \overline{a})$ infinite and

$$\phi(\mathcal{M}', \overline{a}) = \phi(\mathcal{N}', \overline{a}),$$

so $(\mathcal{M}', \mathcal{N}')$ forms a Vaughtian pair, a contradiction. □

Theorem 6.9. *If T has no Vaughtian pairs, then any minimal formula is strongly minimal.*

Proof. Let $\phi(\overline{v})$ be minimal over $\mathcal{M}$. Suppose that there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$, $\overline{b} \in N$, and an $\mathcal{L}$-formula $\psi(\overline{v}, \overline{w})$ so that $\psi(\mathcal{N}, \overline{b})$ is an infinite coinfinite subset of $\phi(\mathcal{N})$.

By the above lemma, there is a number $n \in \mathbb{N}$ so that for any $\mathcal{N}'$ which is an elementary extension of $\mathcal{M}$ and $\overline{a} \in N'$, $\psi(\mathcal{N}', \overline{a})$ is an infinite coinfinite subset of $\phi(\mathcal{N}')$ if and only if

$$|\psi(\mathcal{N}', \overline{a}) \cap \phi(\mathcal{N}')| > n.$$

and also

$$|\neg\psi(\mathcal{N}', \overline{a}) \cap \phi(\mathcal{N}')| > n.$$

However, we know that

$$\mathcal{M} \models \forall\overline{w}(|\psi(\mathcal{M}, \overline{w}) \cap \phi(\mathcal{M})| \leq n \wedge |\neg\psi(\mathcal{M}, \overline{w}) \cap \phi(\mathcal{M})| \leq n),$$

so the above formula is also true in $\mathcal{N}'$. This is a contradiction, so the theorem holds. □

7. The Categoricity Theorem

Lemma 7.1. *If T is an ω-stable theory with no Vaughtian pairs, $\mathcal{M} \models T$, and $X \subseteq M^n$ is infinite and definable, then $\mathcal{M}$ is prime over X and no proper elementary submodel of $\mathcal{M}$ contains X.*

Proof. Let $\phi(\overline{v})$ define X. If $\mathcal{N}$ is a proper elementary submodel of $\mathcal{M}$ containing X, then $X = \phi(\mathcal{M}) = \phi(\mathcal{N})$ and, since X is infinite, $(\mathcal{M}, \mathcal{N})$ form a Vaughtian pair. Therefore, since T has no Vaughtian pairs, no proper elementary submodel of $\mathcal{M}$ contains X.

Furthermore, theorem 3.5 tells us that, because T is ω-stable, there is an elementary submodel $\mathcal{N}$ of $\mathcal{M}$ which is prime over X. Since $\mathcal{N}$ cannot be proper, we know that $\mathcal{M}$ itself must be prime over X. □

Theorem 7.2. *Let T be a complete theory in a countable language with infinite models and suppose $\kappa \geq \aleph_1$. T is κ-categorical if and only if T is ω-stable and has no Vaughtian pairs.*

Proof. For one direction, suppose that T is κ-categorical. By theorem 5.4, T is ω-stable and has no Vaughtian pairs.

For the other direction, suppose T is ω-stable and has no Vaughtian pairs. Suppose $\mathcal{M}$ and $\mathcal{N}$ are models of T, each of cardinality $\kappa \geq \aleph_1$. We want to show that $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.

By Theorem 3.5, T has a prime model $\mathcal{M}_0$ and additionally, by Theorem 6.7, there is a strongly minimal formula $\phi(v)$ with parameters from $\mathcal{M}_0$. Consider $\mathcal{M}$ and $\mathcal{N}$ as elementary extensions of $\mathcal{M}_0$, so that $\dim(\phi(\mathcal{M})) = \dim(\phi(\mathcal{N})) = \kappa$. By Theorem 6.6, there is a partial elementary bijection $f : \phi(\mathcal{M}) \to \phi(\mathcal{N})$. By Lemma 7.1, $\mathcal{M}$ is prime over $\phi(\mathcal{M})$ so f can be extended to $f' : \mathcal{M} \to \mathcal{N}$. But we know $\mathcal{N}$ has no proper elementary submodels containing $\phi(\mathcal{N})$, so f' is surjective and, consequently, an isomorphism.

This shows that T is κ-categorical. □

Notice that in the previous theorem, if T is ω-stable and has no Vaughtian pairs, T is κ-categorical for *any given* κ. Consequently, we get the immediate corollary:

Corollary 7.3 (**Morley's Categoricity Theorem**). *Let T be a complete theory in a countable language with infinite models. T is κ-categorical for some uncountable κ if and only if T is λ-categorical for every uncountable λ.*

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