THE GAUSS-BONNET THEOREM

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ABSTRACT. The Gauss Bonnet theorem links differential geometry with topology. The following expository piece presents a proof of this theorem, building up all of the necessary topological tools. Important applications of this theorem are discussed.

CONTENTS

1. INTRODUCTION

The Gauss Bonnet theorem bridges the gap between topology and differential geometry. Its importance lies in relating geometrical information of a surface to a purely topological characteristic, which has resulted in varied and powerful applications. Though this paper presents no original mathematics, it carefully works through the necessary tools for proving Gauss-Bonnet. Gauss first proved this theorem in 1827, for the case of a hyperbolic triangle. This theorem established a remarkable invariant relating curvature to the notion of angle within the surface. However, with the developments in topology in the 19th and 20th centuries, this theorem has become an invaluable piece of modern mathematics.

2. TOPOLOGICAL PRELIMINARIES

Definition 2.1. A **surface** S is a two dimensional sub-manifold of Euclidean space. The word **regular** ensures differentiability in a neighborhood of each point of the surface; however, we note that the full definition of this term includes other criteria which will be inconsequential in this paper.

Definition 2.2. A **triangulation** of a regular sufrace R is a finite collection J of triangles $\{T_j\}_{j=1}^n$ such that

$$\cup_{j=1}^n T_j = R$$

and the only possible intersection of T_i and T_j with i not equal to j is a common edge or a common vertex.

Date: 30 July 2010.

Proposition 2.3. *Every regular surface S admits a triangulation.*

Definition 2.4. A surface is **orientable** if all of the triangles in a given triangulation have compatible orientations (i.e., they preserve the notion of clockwise/counterclockwise as a path moves between triangles).

Definition 2.5. The **Euler characteristic** $\chi(S)$ of a surface S is defined as

$$\chi(S) = V - E + F,$$

where V, E, F are the numbers of vertices, edges, and faces of a given triangulation of S.

Proposition 2.6. *The Euler characteristic is a topological invariant.*

Remark 2.7. All surfaces (in two dimensions) are classified up to homeomorphism by their genus, which is related to the Euler Characteristic. Essentially, the number of holes in a surface classifies it topologically.

Definition 2.8. Let X be a space and x_0 be a point in X. A path in X that begins and ends at x_0 is a loop based at x_0. The **fundamental group** $\pi_1(X, x_0)$ is the set of path homotopy classes of loops based at x_0, with some operation, namely concatenation.

In order to prove the Jordan Curve Theorem, we list several lemmas without proof. However, proof can be found in Topology: A First Course, by James Munkres.

Proposition 2.9. *Let C be a simple closed curve in S^2. Then $S^2 - C$ is not connected. This is the* **Jordan Separation Theorem.**

Proposition 2.10. *Let X be the union of open sets U and V. Suppose*

$$U \cup V = A \cup A' \cup B$$

three disjoint open sets. Given

$$a \in A, a' \in A', b \in B$$

path-connected in U and V, then $\pi_1(X, a)$ is not infinite cyclic (i.e., not isomorphic to $(\mathbb{Z}, +, 0)$).

Proposition 2.11 (A nonseparation theorem)**.** *Let A be an arc in S^2. Then $S^2 - A$ is connected.*

Theorem 2.12 (The Jordan Curve Theorem)**.** *Let C be a simple closed curve in S^2. Then, $S^2 - C$ has two and only two components W_1 and W_2, of which C is the common boundary.*

Proof. First, we show that $S^2 - C$ has exactly two components. Using the proposition above, we write C as the union of two arcs C_1 and C_2 which intersect in exactly two points p and q. Let X be $S^2 - p - q$. Let

$$U = S^2 - C_1$$

and

$$V = S^2 - C_2.$$

We note that

$$X = U \cup V$$

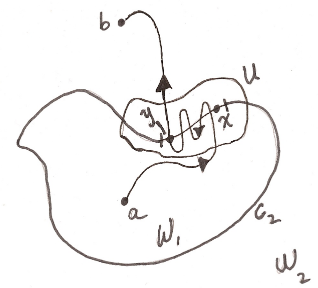

FIGURE 1. The terms of the proof of the Jordan Curve Theorem.

and

$$S^2 - C = U \cap V,$$

which we know has at least two components by proposition 2.10. Suppose, in order to find a contradiction, that $S^2 - C$ has more than two components. Let A and A' be two components, and call the union of the rest of the components B. Each of these sets is open because $S^2 - C$ is locally connected. Given

$$a \in A, a' \in A', b \in B,$$

by the nonseparation theorem above, U and V are path-connected, because no arc separates S^2. Thus $\pi_1(X, a)$ is not infinite cyclic. However, we recall that $X = S^2 - p - q$ is homeomorphic to the punctured plane, which has an infinite cyclic fundamental group, which is a contradiction.

Next, we show that C is the common boundary of W_1 and W_2, which are the regions that we wish to show are separated by the curve. We note that W_1 and W_2 must be open because neither can contain a limit point of the other (because S^2 is locally connected). Therefore, since S^2 is the disjoint union of W_1, W_2, and C, it follows that

$$\overline{W_1} - W_1,$$

as well as

$$\overline{W_2} - W_2,$$

must be contained in C. Now, we show the converse, namely, that given a point,

$$x \in C,$$

we know,

$$x \in \overline{W_1} - W_1.$$

Thus, let U be a neighborhood of x. Then, using the fact that C is homeomorphic to S^1, we break it up into two curves C_1 and C_2 such that C_1 is entirely contained within U. We let a and b be points of W_1 and W_2, respectively. Again, using the nonseparation theorem, we know that we can find a path in $S^2 - C_2$, call it α, that connects a and b, because C_2 does not separate S^2. Therefore, $\alpha(I)$ must contain a point y such that

$$y \in \overline{W_1} - W_1, y \in C.$$

We know that such a y must exist because, otherwise, $\alpha(I)$, which we know is a connected set, would lie in the disjoint union of open sets W_1 and $S^2 - \overline{W_1}$ but also intersect both of them. Because we know that

$$C \supset \overline{W_1} - W_1,$$

it follows that

$$y \in C.$$

Now, as we know that α does not intersect C_2 the point y must be on C_1, and thusly

$$y \in U.$$

Therefore, as we wished to show, U intersects $\overline{W_1} - W_1$ in the point y, and thusly we have shown that C is the common boundary of W_1 and W_2. □

Definition 2.13. We say that f and g, continuous maps from X to Y are **homotopic** if there exists a map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$. This can be thought of as a continuous deformation of one map into the other.

Definition 2.14. A **convex map** is a map such that for any line connecting two points of the map, every point on that line is contained in the map.

Definition 2.15. If $\alpha : [a, b] \to \mathbb{R}^2$ is a convex map, then the **rotation index** of α is

$$\int_a^b \kappa(s)ds = \theta(b) - \theta(a) = 2\pi I,$$

where $\kappa(s)$ is the Gaussian curvature as defined in 3.4.

Remark 2.16. We note that the rotation index is homotopy invariant over α.

Definition 2.17. $\Psi : T \to S^1$ is the **secant map** from the triangular region

$$T = \{(t_1, t_2) \in [0, 1] \times [0, l]; 0 \leq t_1 \leq t_2 \leq l\},$$

onto the 1-sphere:

$$\Psi(x) = \begin{cases} \frac{\alpha(t_2)-\alpha(t_1)}{|\alpha(t_2)-\alpha(t_1)|} & t_1 \neq t_2, (t_1, t_2) \in T \setminus \{0, l\} \\ \frac{\alpha'(t)}{|\alpha'(t)|} & t_1 = t_2 \\ \frac{-\alpha'(0)}{|\alpha(0)|} & (t_1, t_2) = (0, l) \end{cases}.$$

Below, we prove the Theorem of Turning Tangents. Roughly, this theorem states that a loop on a surface turns 2π radians. This allows us to index the rotation of the boundary of a region of a surface, which is a key piece of the proof of the local Gauss-Bonnet theorem.

Theorem 2.18 (Turning Tangents)**.** $\beta : [0, l] \to \mathbb{R}^2$ *is a simple, closed, regular, plane curve, then the rotation index of* β *is* ± 1.

Proof. We assume the Jordan Curve Theorem as proven above. Now, we take some line b and position it such that it is tangent to β at the point p. We note that due to this choice of b, the curve lies entirely on one side of the line. Then, we choose a parametrization $\alpha \in [0, l$ such that

$$\alpha(0) = p.$$

We observe the secant map Ψ is a continuous function, which is trivially verified. Now, we let A, B, C denote the vertices of T at $(0,0),(0,l),(l,l)$, respectively. We notice that the side AC is simply the map of tangents to α into S^1, the degree of which is the rotation of α. Next, we recognize that the tangent map must be homotopic to the restriction of Ψ to the other two sides of the triangle AB and BC. Thus, it suffices to show that the rotation index of Ψ on

$$AB \cup BC = \pm 1.$$

To do so, we assume that we have an orientation such that the angle from $\alpha'(0)$ to $-\alpha'(0)$ is π. This implies that Ψ on AB, i.e.,

$$[0,0] \to [0,l]$$

covers half of S^1 in the positive direction, and Ψ on BC covers the other half. Since the rotation index is clearly 1 for S^1, we're done. We note we can reverse the assumed orientation to obtain the opposite index. Therefore the rotation index of β is ± 1. □

3. Local Gauss-Bonnet Theorem

Remark 3.1. We will now prove the local case of Gauss-Bonnet. However, in order to do this, we first need to define some key resources from differential geometry. These notions of curvature tell us roughly what a surface looks like both locally and globally. The theorem tells us that there is a remarkable invariance on surfaces that is balanced by the total curvature and geodesic curvature. Gauss proved this by considering the hyperbolic triangle.

Definition 3.2. A **simple** region R of a surface S is a region such that R is homeomorphic to the disk.

Definition 3.3. Given a parametrization of a surface, $\mathbf{x} : U \to S$, we have the following quantities

$$E = \langle x_u, x_u \rangle, F = \langle x_u, x_v \rangle, G = \langle x_v, x_v \rangle.$$

These are the coefficients of the first fundamental form of a surface.

Definition 3.4. Given a parametrization of a surface,

$$\mathbf{x} : U \to S,$$

we call that parametrization **orthogonal** if

$$F = 0.$$

Definition 3.5. The **geodesic curvature** k_g of a curve is a measure of the amount of deviance of the curve from the shortest arc between two points on a surface.

Definition 3.6. The **Gaussian curvature** κ of a surface is an intrinsic measure of the curvature of a surface at a point. It is calculated by considering the maximal and minimal curvatures on the surface at a point. Formally, these values are multiplied to give κ.

Theorem 3.7 (Gauss-Bonnet, Local)**.** *Let $R \subset \mathbf{x}(U)$ be a simple region of S with orthogonal parametrization, and choose $\alpha : I \to S$ such that $\alpha(I) = \partial R$. Assume that α is positively oriented and parametrized piecewise by arc-length s_i. Let $\{\theta_i\}_{i=0}^{k}$ be the external angles of α at the vertices $\{\alpha(s_i)\}_{i=0}^{k}$, then*

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_{g(s)} ds + \iint_R \kappa d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi.$$

Proof. We first let $u = u(s)$ and $v = v(s)$ be the expression of α in the parametrization $\mathbf{x}$. We recall that

$$k_{g(s)} = \frac{1}{2\sqrt{EG}}(G_u \frac{dv}{ds} - E_v \frac{du}{ds}) + \frac{d\varphi}{ds},$$

where we denote the differentiable function that measures the positive angle from $\mathbf{x}$ to $\alpha'(s)$ in $[s_i, s_{i+1}]$ as $\varphi(s_i)$. We now integrate the above expression, adding up the values for each $[s_i, s_{i+1}]$:

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_{g(s)} = \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{1}{2\sqrt{EG}} \left(G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right) ds + \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{d\varphi_i}{ds} ds,$$

now, using the Gauss-Green theorem in the uv-plane on the right hand side of the above equation, we obtain the expression:

$$\iint_{\mathbf{x}^{-1}(R)} \frac{E_v}{2\sqrt{EG}_v} + \frac{G_u}{2\sqrt{EG}_u} dudv + \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{d\varphi_i}{ds} ds.$$

We note that by the Gauss Formula,

$$-\iint_{\mathbf{x}^{-1}(R)} \kappa \sqrt{EG} dudv = -\iint_R \kappa d\sigma,$$

also, recalling the Theorem of Turning Tangents, we know that,

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{d\varphi_i}{ds} ds = \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \varphi_i(s_i + 1) - \varphi_i(s_i) = \pm 2\pi - \sum_{i=0}^{k} \theta_i,$$

which we get because the theorem does not account for the discontinuities along the curve at the theta values. As we have assumed a positive orientation, we have,

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_{g(s)} ds + \iint_R \kappa d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi,$$

we note that we can obtain the opposite sign by assuming the opposite orientation, and we thus we have proven the local case of the Gauss-Bonnet Theorem. □

4. Global Gauss-Bonnet Theorem

Remark 4.1. We have proven the local case of this theorem, and the global theorem tells us similar information. We prove this generalization by using the local theorem in each triangular region of our triangulation for the given surface. This theorem leads to a series very deep corollaries.

Theorem 4.2. *Let $R \subset S$ be a regular region of an oriented surface. Let ∂R be made up by closed, piecewise, simple, regular curves*

$$C_1 \dots C_n,$$

then

$$\sum_{i=0}^{k} \int_{C_i} k_{g(s)} ds + \iint_R \kappa d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi\chi(R).$$

Proof. Let J denote a triangulation of R such that each triangle T_j is contained in a neighborhood of orthogonal parametrizations compatible with the orientation of S. We note that such a triangulation exists by the proposition proven above. Now, we simply apply the local Gauss-Bonnet theorem to each T_j of the above triangulation, and we have:

$$\sum_{i=0}^{k} \int_{C_i} k_{g(s)} ds + \iint_R \kappa d\sigma + \sum_{j,k=1}^{F,3} \theta_{jk} = 2\pi F,$$

where the indexing of each theta accounts for each external angle of the triangles in J. We note that F here is the number of faces in our triangulation. We denote the interior angles of the triangles by $\varphi_{jk} = \pi - \theta_{jk}$. We calculate, in general, that,

$$\sum_{j,k=1} \theta_{jk} = 3\pi F - \sum_{j,k=1} \varphi_{jk}.$$

Now, we introduce notation to assist in counting the vertices and edges of our triangulation, so, the number of external/internal edge and vertices are V_e, E_e and V_i, E_i, respectively. Since the C_i are closed, however, we know $V_e = E_e$, and thus, inductively, $3F = 2E_i + E_e$.This implies,

$$\sum_{j,k=1} \theta_{jk} = 2\pi E_i + \pi E_e - \sum_{j,k=1} \varphi_{jk}.$$

We note that the vertices must belong to either some T_j or a C_i, so $V_e = V_{et} + V_{ec}$, and then, since the sum of the angles around each internal vertex is 2π,

$$\sum_{j,k=1} \theta_{jk} = 2\pi E_i + \pi E_e - 2\pi V_i - \pi V_{et} - \sum_i (\pi - \theta_i).$$

By adding πE_e and subtracting πE_e to the right hand side of the above equation, we have,

$$\sum_{j,k=1} \theta_{jk} = 2\pi E - 2\pi V + \sum_i \theta_i.$$

Now, we collect the terms to find:

$$\sum_{i=0}^{k} \int_{C_i} k_{g(s)} ds + \iint_R \kappa d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi(F - E + V).$$

But, by the definition of a triangulation $F - E + V = \chi(R)$, thus,

$$= 2\pi\chi(R).$$

□

5. Applications

Remark 5.1. If R is a simple region, then

$$\chi(R) = 1.$$

Proof. We note that R is homeomorphic to the disk, and likewise, homeomorphic to a single triangle. Thus, by the definition of the Euler characteristic,

$$\chi(R) = V - E + F = 3 - 3 + 1 = 1.$$

□

Corollary 5.2. *If R is a simple region of a surface S, then,*

$$\sum_{i=0}^{k} \int_{C_i} k_{g(s)} ds + \iint_S \kappa d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi.$$

This is trivially proved by the above remark.

The next corollary is often the presented form for the Gauss-Bonnet theorem. We simply take into account that the boundary of a compact surface is empty, and therefore it can be thought of as a region with empty boundary. Thusly, the terms that depend on the boundary drop out of the equation and we are left with the following powerful statement.

Corollary 5.3. *If S is an orientable compact surface, then:*

$$\iint_S \kappa d\sigma = 2\pi\chi(S).$$

Remark 5.4. The Euler Characteristic $\chi(S)$ can be written

$$\chi(S) = 2 - 2g,$$

where g is the genus of a surface, i.e., the number of holes in the surface.

Remark 5.5. This elegant formulation of the theorem introduces the notion that the total curvature depends exclusively on the topological characteristic, i.e., the genus. This implies that all possible embeddings of a surface of genus g have the same total curvature, which is a highly non-intuitive result.

Definition 5.6. A **vector field** is a map from $\mathbb{R}^n \to \mathbb{R}^n$ such that for every $\mathbf{x} \in \mathbb{R}^n$ the map assigns a vector $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^n$ to that point.

Definition 5.7. The **index** I of v at the singular point p is defined as follows. Let $\mathbf{x}$ be an orthogonal parametrization such that $\mathbf{x}(0,0) = p$ and the orientation is compatible with that of the surface S. Let $\alpha : [0,l], \to S$ be a closed simple regular parametrized curve such that it is the boundary of a simple region $R \subset S$, where the only singular point in R is p. Now, we have a function $\varphi(t)$ with $t \in [0,l]$, such that it measures the angle from $\mathbf{x_u}$ to the restriction of v to α, then,

$$2\pi I = \varphi(l) - \varphi(0) = \int_0^l \frac{d\varphi}{dt} dt.$$

Proposition 5.8. *The index is independent of the choice of parametrization* $\mathbf{x}$.

Proposition 5.9. *The index is independent of the choice of α.*

Theorem 5.10 (Poincare-Hopf Index Theorem)**.** *The sum of the indices of a differentiable vector field v with isolated singular points on a compact surface S is equal to the Euler Characteristic of S.*

Proof. Let $S \subset \mathbb{R}^3$ be a compact surface and v a differentiable vector field with exclusively isolated singular points. We notice that, due to compactness, these singular points must be finite in number otherwise there would exist a non-isolated singular point as a limit point for the others. We let $\{\mathbf{x_a}\}$ be a family of parametrizations such that each is compatible with the orientation of S. Now, we let J be a triangulation of S with the conditions that each T in J is contained in a coordinate neighborhood of $\{\mathbf{x_a}\}$, each triangle T contains at most one singular point, and each triangle is positively oriented with no singular points on its boundary. Now, we apply the local Gauss-Bonnet theorem to each triangle and sum up the result. However, we recall that each triangle appears twice in this formulation, in opposite orientation, therefore, we have

$$\iint_S \kappa d\sigma - 2\pi \sum_{i=1}^{k} I_i = 0,$$

to which we apply the most general form of Gauss-Bonnet to obtain,

$$\sum_{i=1}^{k} I_i = \frac{1}{2\pi} \iint_S \kappa d\sigma = \chi(S).$$

□

Remark 5.11. This result guarantees that a vector field on any surface homeomorphic to a sphere must have at least two isolated singular points, because the Euler characteristic of the 2-sphere is two. The solution is particularly remarkable because it shows that the sum of the indices of a vector field does not depend on v, but rather the topology of S, which is a non-intuitive idea. More tangibly, the Poincare-Hopf Index Theorem implies the previous theorem.

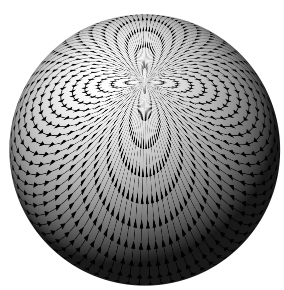

FIGURE 2. An isolated singular point guaranteed by the Poincare-Hopf Theorem.

Theorem 5.12 (The Hairy Ball Theorem)**.** *You can't comb a hairy ball without a bald spot. Or, let f be a continuous function that assigns a vector in $\mathbb{R}^3$ to each point p on the 2-sphere S such that $f(p)$ is tangent to S at p. Then, there is at least one p such that $f(p) = 0$.*

Proof. Assume, in order to find a contradiction, that $f(p) \neq 0$ for all p on the 2-sphere. Then, there exists a vector field v on S^2 such that v has no singular points. However, given the above theorem, this is a contradiction, as we know that there must exist two distinct isolated singular points on this vector field, thusly there can be no such vector field on the sphere or any topological homeomorphism. □

6. Acknowledgements

I would like to thank Daniel Studenmund for his patience and assistance, without which this would have not been possible.

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