FRIEZE GROUPS IN $\mathbb{R}^2$

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Abstract. Focusing on the Euclidean plane under the Pythagorean Metric, our goal is to classify the frieze groups, discrete subgroups of the set of isometries of the Euclidean plane under the Pythagorean metric whose translation subgroups are infinite cyclic. We begin by developing a normal form for representing all isometries. To simplify the problem of composing isometries written in normal form, we shall discuss the properties of compositions of reflections across different axes. Such a discussion will naturally lead to a classification of all isometries of the plane. Using such knowledge, we can then show that there are only seven geometrically different types of frieze groups.

Contents

1. Metric Spaces, Groups, and Isometries

Before we can begin to talk about frieze groups, we must discuss certain necessary preliminary concepts. We begin by formalizing the concept of distance between points in a set.

Definition 1.1. A **metric** on a set X is a map $d : X \times X \to \mathbb{R}$ such that
i) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
iii) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

A set X with a metric d is called a **metric space**, written (X, d). We are specifically interested in distance preserving maps from a metric space to itself and so we present Definition 1.2.

Definition 1.2. An **isometry** of a metric space (X, d) is a bijection $u : X \to X$ such that, for all $x, y \in X$, $d(x, y) = d(xu, yu)$. (Note that for a map $u : X \to Y$ and an element $x \in X$ we denote the image of x by xu instead of $u(x)$.)

We let $\mathrm{Isom}(X, d)$ denote the set of isometries of a metric space (X, d). In other words, $\mathrm{Isom}(X, d)$ is the set of functions from a set to itself that preserve distance. While there are many such sets, we shall deal almost exclusively with the set of isometries of the Euclidean plane under the Pythagorean metric. We shall refer to this set as $\mathbb{E}$, formally defined as $\mathbb{E} := \mathrm{Isom}(\mathbb{R}^2, d)$ where $d(x,y) = \sqrt{\sum_{i=1}^{2}(x_i - y_i)^2}$, $x = (x_1, x_2)$, and $y = (y_1, y_2)$.

Diverging from the topic of metric spaces, we present the following definitions, formalizations of fundamental algebraic concepts:

Definition 1.3. A **group** is a set X with a binary operation such that

i) there is an **identity** element $1 \in X$ such that, for all $x \in X$, $1x = x1 = x$;

ii) the binary operation is **associative**, i.e. for all $x, y, z \in X$, $(xy)z = x(yz)$;

iii) for every $x \in X$, there exists some $y \in X$ (called the **inverse** of x and written x^{-1}) such that $xy = yx = 1$.

Definition 1.4. A **subgroup** is a subset of a group G that forms a group under the same binary operation as G. We denote a subgroup H of G by writing $H \leq G$.

Definition 1.5. A subgroup H of a group G is said to be **normal** if $Hx = xH$ for all $x \in G$. We denote a normal subgroup H of G by writing $H \lhd G$.

There is, however, an equivalent definition of a normal subgroup that will be useful for later proofs.

Theorem 1.6. *$H \lhd G$ if and only if $x^{-1}Hx = H$ for all $x \in G$.*

Proof. Let $x \in G$ and $h \in H$. Assume $H \lhd G$. By definition, $Hx = xH$, so $hx = xh'$ for some $h' \in H$. Right multiplying both sides by x^{-1} yields $x^{-1}hx = h' \in H$, from which it follows that $x^{-1}Hx \subset H$. Because $x \in G$ is arbitrary, xHx^{-1} is also a subset of H. Thus, $H = x^{-1}xHx^{-1}x \subset x^{-1}Hx$ and we have proved the theorem in one direction.

For the other direction, assume $x^{-1}Hx = H$ for all $x \in G$. Let $x \in G$ and $h \in H$. By the assumption, $x^{-1}hx = h' \in H$ and right multiplication by x yields $hx = xh' \in xH$. Thus, $Hx \subset xH$. Because $x^{-1}Hx = H$ for all $x \in G$, it follows that $xhx^{-1} = h'' \in H$. By similar logic, $xH \subset Hx$. Thus, $xH = Hx$ and $H \lhd G$. $\square$

The concepts of isometries and groups now combine in the following theorem. Note that, building off the previous notation, we write composition of maps $f \circ g$ as gf and $(f \circ g)(x) = f(g(x)) = xgf$.

Theorem 1.7. *The set of isometries of a set X, i.e.* $\mathrm{Isom}(X)$*, forms a group under composition of maps.*

Proof. First, we need to show the existence of an identity element. Consider the identity map $1 : X \to X, x \mapsto x$. The identity map 1 is obviously a bijection. Furthermore, for all $x, y \in X$, $d(x1, y1) = d(x, y)$ as $x1 = x$ and $y1 = y$. Thus, $1 \in \mathrm{Isom}(X)$. Let $u \in \mathrm{Isom}(X)$ and $x \in X$. Because $x1u = xu = xu1$, the identity map $1 \in \mathrm{Isom}(X)$ is the identity element such that, for all $u \in \mathrm{Isom}(X)$, $u1 = 1u = u$.

Composition of maps is obviously associative for $x(tu)v = ((xt)u)v = xt(uv)$.

Next, we need to show the existence of inverses, so let $u \in \mathrm{Isom}(X)$. Because $u \in \mathrm{Isom}(X)$, u is a bijection. Thus, there exists a map u^{-1} such that $uu^{-1} = u^{-1}u = 1$. Let $x, y \in X$. Because $d(xu^{-1}, yu^{-1}) = d(xu^{-1}u, yu^{-1}u) = d(x1, y1) = d(x, y)$, $u^{-1} \in \mathrm{Isom}(X)$. Thus, for every $u \in \mathrm{Isom}(X)$, there exists $u^{-1} \in \mathrm{Isom}(X)$ such that $uu^{-1} = u^{-1}u = 1$.

To complete the proof, we need to show that $\mathrm{Isom}(X)$ is closed under composition of maps. In other words, we need to show that, for all $u, v \in \mathrm{Isom}(X)$, $uv \in \mathrm{Isom}(X)$. Let $u, v \in \mathrm{Isom}(X)$ and let $x, y \in X$. By definition, $d(x, y) = d(xu, yu)$ and $xu, yu \in X$. It follows that $d(xu, yu) = d(xuv, yuv)$. By the transitivity of equality, $d(x, y) = d(xuv, yuv)$. Therefore, $uv \in \mathrm{Isom}(X)$. □

2. $\mathbb{E}$ and the Normal Form Theorem

This section deals exclusively with $\mathbb{E}$, the set of isometries of the Euclidean plane under the Pythagorean metric. The most well-known isometries of the plane are perhaps translations, rotations, and reflections. We begin by formally defining these isometries. Later, the Normal Form Theorem will show that every isometry of $\mathbb{R}^2$ can be written as a composition of a reflection, rotation, and translation.

Definition 2.1. A **translation** t is a map that moves every point a fixed distance in a fixed direction. In symbols, for some $x \in \mathbb{R}^2$ with Cartesian coordinates (x_1, x_2)

$$t : (x_1, x_2) \mapsto (x_1 + a_1, x_2 + a_2)$$

where a_1 and a_2 are constants that define a constant vector $\mathbf{a} = (a_1, a_2)$. We denote such a translation by writing $t = t(\mathbf{a})$ and call the direction of $\mathbf{a}$ the **axis** of t.

Translations are orientation preserving (OP) and have no fixed points, unless, of course, the translation is the trivial translation through the zero vector.

Translations compose according to the following rule: If $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ are constant vectors in $\mathbb{R}^2$, then $t(\mathbf{a})t(\mathbf{b}) = t(\mathbf{a} + \mathbf{b})$ where t maps a point (x_1, x_2) in the plane to $(x_1 + a_1 + b_1, x_2 + a_2 + b_2)$. It follows from this fact that the set of all translations in $\mathbb{E}$ form a subgroup of $\mathbb{E}$.

Definition 2.2. A **rotation** s of the plane is a map that moves every point through a fixed angle about a fixed point, called the **center**.

Taking the center O to be the origin of a polar coordinate system,

$$s : (\rho, \theta) \mapsto (\rho, \theta + \alpha)$$

where (ρ, θ) are the polar coordinates of an arbitrary point in $\mathbb{R}^2$ and α is a fixed angle. We denote such a rotation by writing $s = s(O, \alpha)$.

Rotations are order preserving and non-trivial rotations have only one fixed point, namely the center.

Rotations with the same center compose according to the following rule:

$$s(O, \alpha)s(O, \beta) = s(O, \alpha + \beta)$$

where $s(O, \alpha + \beta) : (\rho, \theta) \mapsto (\rho, \theta + \alpha + \beta)$ and α and β are constant angles. It follows that rotations about the same center O form a subgroup of $\mathbb{E}$.

Definition 2.3. A **reflection** r is a map that moves every point in the plane to its mirror image across a line l. This line l is called the **axis** of r and we denote such a reflection by writing $r = r(l)$. In symbols, given a point $P \in \mathbb{R}^2$, if $P \in l$ then $Pr = P$. If $P \notin l$, then Pr is the unique point in $\mathbb{R}^2$ such that l is the perpendicular bisector of P and Pr.

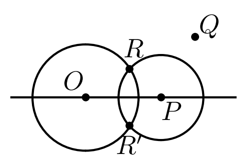
Reflections are orientation reversing and fix only the points on the axis of the reflection. The inverse of a reflection is itself, i.e. $r(l)^2 = 1$. The composition of reflections across different axes, as well as the composition of rotations about different centers, is the topic of discussion in Section 4.

The proofs that translations, rotations, and reflections are isometries of $\mathbb{R}^2$ and satisfy their described properties regarding orientation, fixed points, and composition are somewhat trivial and will not be discussed here.

Having formalized the notions of translation, rotation, and reflection, we seek to show that every isometry of the plane can be written as a composition of a reflection, rotation, and translation. With that goal in mind, the following lemma, that every isometry of $\mathbb{R}^2$ is determined by its effect on 3 non-collinear points, will be a key first step towards proving the Normal Form Theorem.

Lemma 2.4. *Let O, P, Q be 3 non-collinear points of $\mathbb{R}^2$ and let $u_1, u_2 \in \mathbb{E}$ such that $Ou_1 = Ou_2, Pu_1 = Pu_2$ and $Qu_1 = Qu_2$. Then $u_1 = u_2$.*

Proof. Define $u = u_1 u_2^{-1}$. Thus, $O = Ou, P = Pu$, and $Q = Qu$. $\mathbb{E}$ is a group by Theorem 1.7, so $u \in \mathbb{E}$. Let $R \in \mathbb{R}^2$. Because $u \in \mathbb{E}$, $d(O, R) = d(Ou, Ru) = d(O, Ru)$. Thus, Ru lies on circle C_1 center O radius $d(O, R)$. By similar logic, Ru lies on circle C_2 center P radius $d(P, R)$. Thus, $R, Ru \in C_1 \cap C_2$. Because O, P, Q are non-collinear, $O \neq P$. It follows that $|C_1 \cap C_2| \in \{1, 2\}$. If $|C_1 \cap C_2| = 1$, then $R = Ru$ and we are done. Otherwise, $|C_1 \cap C_2| = 2$, $C_1 \cap C_2 = \{R, R'\}$, and $Ru \in \{R, R'\}$. It follows that line $\overline{OP}$ is the perpendicular bisector of $\overline{RR'}$. Equivalently, for any point $S \in \mathbb{R}^2$, $d(S, R) = d(S, R')$ if and only if $S \in \overline{OP}$. Because O, P, Q are non-collinear, $Q \notin \overline{OP}$, so $d(Q, R) \neq d(Q, R')$. Because $d(Q, R) = d(Qu, Ru) = d(Q, Ru)$, $R' \neq Ru$. Therefore, $R = Ru$ for all $R \in \mathbb{R}^2$ and we have proved the lemma. $\square$



Theorem 2.5. *(Normal Form Theorem) Fix a point O and a line $l \in \mathbb{R}^2$ such that $O \in l$. Any $u \in \mathbb{E}$ can be written uniquely as*

$$u = r^{\epsilon} s t$$

where r is a reflection over axis l, $\epsilon \in \{0, 1\}$, s is a rotation about center O, and t is a translation.

Proof. Let t be a translation such that $Ou = Ot$. Thus, $Out^{-1} = O$. Let $P \in l$ such that $P \neq O$. It follows that $0 < d(O, P) = d(Out^{-1}, Put^{-1}) = d(O, Put^{-1})$. Thus, both P and Put^{-1} lie on the circle center O with radius $d(O, P) = d(O, Put^{-1})$. It follows that there exists a rotation s about O such that $Ps = Put^{-1}$. Thus, $P = Put^{-1}s^{-1}$. Because s is a rotation about O, $Os = O$, from which it follows that $O = Out^{-1}s^{-1}$. Let $Q \notin l$. Furthermore, $d(O, Q) = d(Out^{-1}s^{-1}, Qut^{-1}s^{-1}) = d(O, Qut^{-1}s^{-1})$ and $d(P, Q) = d(Put^{-1}s^{-1}, Qut^{-1}s^{-1}) = d(P, Qut^{-1}s^{-1})$. By similar logic as in Lemma 2.4, $Qut^{-1}s^{-1}$ equals either Q or the reflection of Q over line l. In the case of the former, $\epsilon = 0$. In the case of the latter, $\epsilon = 1$. In both cases, O and P remain fixed. It follows that, since O, P, Q are non-collinear, $u = r^{\epsilon}st$ by Lemma 2.4.

To prove uniqueness, suppose $r^{\epsilon}st = r^{\delta}s't'$ where $\epsilon, \delta \in \{0, 1\}$, s and s' are rotations about O, and t and t' are translations. If this isometry is orientation preserving, then $\epsilon = \delta = 0$. Otherwise, this isometry is orientation reversing and $\epsilon = \delta = 1$. Right multiplying both sides by r, if necessary, yields $st = s't'$. Further multiplication yields $s'^{-1}s = t't^{-1}$ which is both a translation and a rotation about O. Because the rotation fixes O and only the trivial translation has fixed points, $s'^{-1}s = t't^{-1}$ must equal the identity map 1. It follows that $s = s'$ and $t = t'$. □

3. Generators and Relations

We now take a short break from isometries of the Euclidean plane to discuss the notation that will be used to describe and classify the frieze groups.

We write $G = \langle X|R\rangle$ where the symbols X, R, G have the following meaning:

The set X of **generators** consists of symbols, usually finite in number, say $x_1, ..., x_n, n \in \mathbb{N} \cup \{0\}$. We think of the symbols $x_i^{\pm 1}, 1 \le i \le n$, as letters in an alphabet $X^{\pm}$ from which **words** can be formed. The **length** of a word is the number of its letters, assumed to be finite, and we allow the empty word e of length zero. A word is **reduced** if it does not involve the letters $x_i^{\pm 1}$ in adjacent places for any $i \in \{1, ..., n\}$, and the set of all reduced words is denoted by $F(X)$.

The set R of **defining relations** consists of relations, that is, equations between words, usually finite in number, say $u_i = v_i$, where $u_i, v_i \in F(X)$, $i \in \{1, ..., m\}$, $m \in \mathbb{N} \cup \{0\}$.

We say that $\langle X|R\rangle$ is a **presentation** of a group G or, equivalently, $G = \langle X|R\rangle$, if the following three conditions are satisfied:

i) every element of G can be written as a word in $X^{\pm}$;

ii) the equations in R all hold in G;

iii) any equation between words in $X^{\pm}$ that holds in G is a consequence of the relations in R.

Before we forget the definition of generators, we shall present the following lemma that will later be useful in the classification of frieze groups.

Lemma 3.1. *Let $T \le G$, t be a generator of T, and $r \in G$. Then, $r^{-1}tr$ is a generator of the group $r^{-1}Tr = \{r^{-1}xr | x \in T\}$.*

Proof. Let $x \in r^{-1}Tr$. Because t is a generator of T, it follows that $x = r^{-1}t^l r$ for some $l \in \mathbb{Z}$. Note that $(r^{-1}tr)^l = r^{-1}trr^{-1}tr...r^{-1}tr = r^{-1}t^l r$. Thus, $x = (r^{-1}tr)^l$. Because $x \in r^{-1}Tr$ is arbitrary, $r^{-1}tr$ is a generator of $r^{-1}Tr$. □

4. Compositions of Reflections Across Different Axes

Returning once again to isometries of the plane, we now take a look at compositions of reflections across different axes. Doing so provides a powerful tool with which to compose and classify isometries in $\mathbb{E}$ and will furthermore simplify the classification of frieze groups. As different axes of reflection can easily become confused, many diagrams have been included in this section to illustrate the various axes and lines. We now begin with the composition of two reflections across different axes.

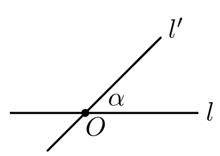
Theorem 4.1. *Let $r, r' \in \mathbb{E}$ be reflections across distinct lines l, l', respectively. If $l \nparallel l'$, then rr' is a rotation about the point of intersection of l and l' through* *an angle twice that from l to l'. If $l \parallel l'$, then rr' is a translation in the direction normal to l and l' through distance twice that from l to l'.*

Proof. Let r and r' be reflections across lines l and l', respectively. Assume $l \neq l'$. Either $l \nparallel l'$ or $l \parallel l'$.

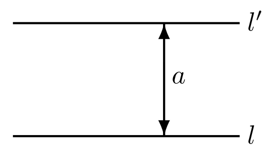
Case 1: Suppose $l \nparallel l'$. Thus, $l \cap l' = \{O\}$ where O is the point of intersection of lines l and l' (see figure). Let $P \in \mathbb{R}^2$. Taking O as the origin and l as the axis, we can write P in terms of polar coordinates $P = (\rho, \theta)$. Thus, $r : (\rho, \theta) \mapsto (\rho, -\theta)$. Let $Pr' = P' = (\rho, \phi)$. Because l' bisects $\angle P'OP$, $\frac{\theta+\phi}{2} = \alpha$ and it follows that $\phi = 2\alpha - \theta$. Thus, $rr' : (\rho, \theta) \mapsto (\rho, 2\alpha + \theta)$ and, therefore, $rr' = s(O, 2\alpha)$. In other words, rr' is a rotation about O through twice the angle from l to l'.

Case 2: Suppose $l \parallel l'$. Let $P \in \mathbb{R}^2$. Taking l as the x-axis, we can write P in terms of Cartesian coordinates $P = (x, y)$. Thus, $r : (x, y) \mapsto (x, -y)$ and $r' : (x, y) \mapsto (x, z)$. Note that $\frac{y+z}{2} = a$, so $z = 2a - y$. It follows that $rr' : (x, y) \mapsto (x, 2a + y)$. Therefore, $rr' = t(0, 2a)$, i.e. a translation through twice the distance between lines l and l'. □

It is worth noting that Theorem 4.1 also indicates that any translation or rotation can be written as the composition of two reflections. In the case of a rotation s about some center O, simply draw two lines l, l', both containing O, such that $l \nparallel l'$ and the angle from l to l' is half the angle of rotation. It follows from Theorem 4.1 that $r(l)r(l') = s$. Similarly, in the case of a translation t through a vector $\mathbf{a}$, draw two parallel lines l, l' such that l and l' are perpendicular to $\mathbf{a}$ and the distance between l and l' equals half the magnitude of $\mathbf{a}$. By Theorem 4.1, $r(l)r(l') = t$.



Case 1: $l \nparallel l'$



Case 2: $l \parallel l'$

Before continuing, we introduce a new isometry that will appear in the next theorem.

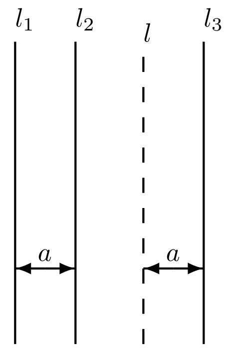
Definition 4.2. Given a pair of distinct points P, P' on a line l, the isometry $q(P, P') = r(l)t(\overrightarrow{PP'})$ is called a **glide reflection**. (Note that $t(\overrightarrow{PP'})$ is the unique translation such that $Pt = P'$.)

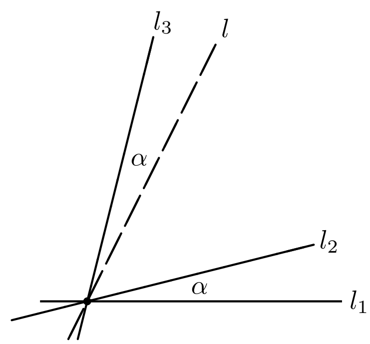
Unlike the isometries defined in Section 2, glide reflections are both orientation reversing and have no fixed points. As the following theorem shows, glide reflections also arise from the composition of three reflections across distinct axes.

Theorem 4.3. *The product of 3 reflections in $\mathbb{E}$ is either a reflection or glide reflection according as the number of points of intersection of distinct axes is less than or greater than $\frac{3}{2}$.*

Proof. Let $r_1, r_2, r_3 \in \mathbb{E}$ be reflections across distinct lines l_1, l_2, l_3 respectively. Because two distinct lines can have at most 1 point in common, it follows that the total number n points of intersection of 3 distinct lines is either 0,1,2, or 3.

Suppose $n = 0$. It follows by Theorem 4.1 that r_1r_2 is a translation in the direction normal to l_1 through distance $2a$ where a is the distance between lines l_1 and l_2. By Theorem 4.1 there is a line l such that reflection r across line l composed with r_3 is a translation in the direction normal to l through distance $2a$. Because $l \parallel l_3 \parallel l_1$, $r_1r_2 = rr_3$. Thus, $r_1r_2r_3 = rr_3r_3 = r$. Therefore, $r_1r_2r_3$ is reflection r across line l. Similarly, for the case $n = 1$, we can draw the line l such that $r_1r_2 = rr_3$ where r is a reflection across line l. By similar logic, $r_1r_2r_3$ is reflection r across line l.

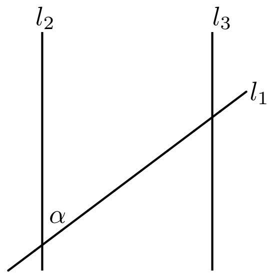


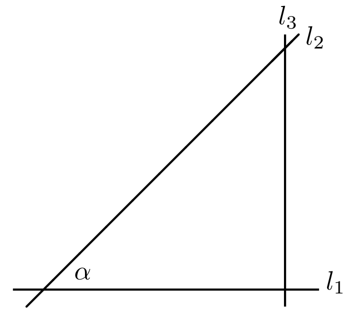


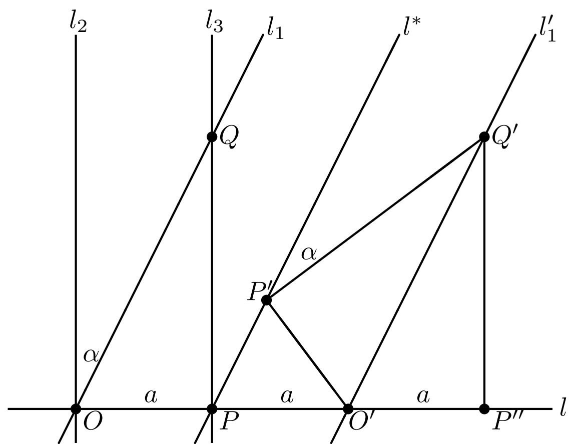
Consider the cases $n \geq 2$. Note that rotating axes l_1 and l_2 about their point of intersection while keeping l_3 fixed does not change r_1r_2. Thus, we can transform any case in which $n \geq 2$ to the case $n = 2$ and $l_2 \parallel l_3$ without affecting the composition $r_1r_2r_3$. So assume $n = 2$ and $l_2 \parallel l_3$. Let O be the point of intersection of lines l_1 and l_2 and let Q be the point of intersection of lines l_1 and l_3 (see figure below). Draw line l such that $O \in l$ and $l \perp l_2$. Note that $l \perp l_3$. Finally, let P be the point of intersection of lines l and l_3. It follows that either $P = Q$ or $P \neq Q$.

Suppose $P \neq Q$. Let $O' = Or_1r_2r_3$ and similarly define Q'. Because r_1 fixes O, $O' = Or_1r_2r_3 = Or_2r_3$. By similar logic, $Q' = Qr_2r_3$. By Theorem 4.1, O' and Q' are simply the images of O and Q, respectively, under translation in the direction of l through distance $2a$ where a is the distance between lines l_2 and l_3. Because $r_1r_2r_3 \in \mathbb{E}$, $d(O,P) = d(Or_1r_2r_3, Pr_1r_2r_3)$, $d(O,Q) = d(Or_1r_2r_3, Qr_1r_2r_3)$, and $d(P,Q) = d(Pr_1r_2r_3, Qr_1r_2r_3)$. It follows by "Side-Side-Side" that $\triangle OPQ$ and $\triangle O'P'Q'$ are congruent. Thus, $Pr_1r_2r_3$ can be one of two points as indicated by P' and P'' in the figure below. However, because $r_1r_2r_3$ is orientation reversing, it follows that $Pr_1r_2r_3 = P'$. Now, draw line l^* through P and P'. Let q be the composition of translation t along l^* through distance $d(P,P')$ followed by a reflection r across axis l^*. Note that $q = tr = rt$. Inspection of the figure below reveals that $O' = Oq$, $P' = Pq$, $Q' = Qq$. By Lemma 2.4, $q = r_1r_2r_3 = tr = rt$. Therefore, $r_1r_2r_3$ is a glide reflection.

Suppose $P = Q$. Thus, $l = l_1$, $l_1 \perp l_2$, and $l_1 \perp l_3$. It follows that r_1 is the reflection r across line l and r_1r_2 is a translation t through distance $2a$ along line l. As in the previous case, we have $r_1r_2r_3 = rt = tr$ and $r_1r_2r_3$ is a glide reflection. □







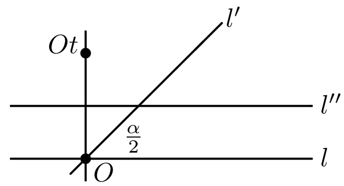
After having discussed the compositions of two and three reflections, the obvious next step is to cover the composition of four reflections, which, in turn, shall describe the composition of any number of reflections. In particular, the composition of five reflections allows for an alternative normal form as presented in the statement of the following theorem.

Theorem 4.4. *Every non-trivial isometry of $\mathbb{R}^2$ is the product of at most 3 reflections, and is either a rotation, translation, reflection, or glide reflection according to the following table:*

Fixed points?	*Yes*	*No*
OP	*Rotation*	*Translation*
OR	*Reflection*	*Glide Reflection*

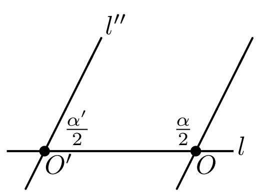
Proof. Consider the product u of four reflections. By Theorem 4.1, u is either the composition of two translations, two rotations, or one of each. If u is the composition of two translations, then u is a translation.

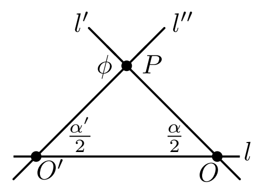
Suppose $u = st$ where $s = s(O, \alpha)$ is a rotation and $t = t(a)$ is a translation. Let l be the line through O perpendicular to the direction a. Let l' be the line through O such that the angle between l' and l is $\frac{\alpha}{2}$. Let l'' be the perpendicular bisector of O and Ot. It follows by Theorem 4.1 that $s = r(l')r(l)$ and $t = r(l)r(l'')$. Thus, $u = st = r(l')r(l)r(l)r(l'') = r(l')r(l'')$. Because $l' \nparallel l''$, u is a rotation.



Suppose $u = ts$. Thus, $u^{-1} = s^{-1}t^{-1}$. By similar logic, u^{-1} is a rotation from which it follows that u is a rotation.

Finally, suppose u is the composition of two rotations $s(O, \alpha)$ and $s'(O', \alpha')$. If $O = O'$, then u is a rotation through $\alpha + \alpha'$. Suppose $O \neq O'$. Let l be the line through O and O'. Let l' be the line through O such that the angle from l' to l is $\frac{\alpha}{2}$. Let l'' be the line through O' such that the angle from l'' to l is $\frac{\alpha'}{2}$. It follows by Theorem 4.1 that $s = r(l')r(l)$ and $s' = r(l)r(l'')$. Thus, $u = r(l')r(l'')$ as $r(l)r(l) = 1$. There are now two cases: $l' \parallel l''$ and $l' \nparallel l''$. If $l' \parallel l''$, then u is a translation by Theorem 4.1 and this happens if and only if the angles $\frac{\alpha}{2}$ and $\frac{\alpha'}{2}$ are supplementary (i.e. sum up to angle π). Otherwise, $l' \nparallel l''$. In this case, u is a rotation $s(P, 2\phi)$ where $P = l' \cap l''$ and $2\phi = \alpha + \alpha'$.





In summary, the product of four reflections is either a translation or rotation which, in turn, is the product of two reflections by Theorem 4.1. It follows that, when $n \geq 4$, a product of n reflections is a product of $n-2$ reflections. By an obvious induction, it follows that when $n \geq 4$, a product of n reflections is a product of at most three reflections.

Now, consider the Normal Form Theorem from Section 2 which stated that any isometry in $\mathbb{E}$ could be written as the composition of a reflection, rotation, and translation. It follows from Theorem 4.1 that any isometry in $\mathbb{E}$ can then be written as the product of five reflections, which, in turn, can be reduced to the product of at most three reflections. The cases that then result have already been discussed in Theorems 4.1 and 4.3 and as such complete the proof of Theorem 4.4. $\square$

5. Classification of Frieze Groups

The effect of composing reflections will be enormously useful and often cited in our discussion of subgroups of $\mathbb{E}$. Of the many such subgroups, we shall only concern ourselves with those that are discrete, which we now define.

Definition 5.1. A subgroup G of $\mathbb{E}$ is **discrete** if, for any point $O \in \mathbb{R}^2$, every circle center O contains only finitely many points in $\{Og | g \in G\}$.

Given a discrete subgroup $G \leq \mathbb{E}$, there is a translation subgroup $T \leq G$ defined as the set of all translations in G. Furthermore, there are, in fact, only three types of translation subgroups of a discrete subgroup: the trivial subgroup of no translations, the free abelian group on two generators (i.e. $\langle t_1, t_2 | t_1 t_2 = t_2 t_1 \rangle \cong \mathbb{Z}^2$), or the infinite cyclic group. We shall limit our discussion to groups with the last such type of translation subgroup.

Definition 5.2. A **frieze group** is a discrete subgroup of $\mathbb{E}$ whose translation subgroup is infinite cyclic, i.e. a subgroup of $\mathbb{E}$ whose translation subgroup is generated by only one translation.

There are, in fact, only seven frieze groups. Before classifying them, we shall need two lemmas that show some useful properties of translations.

Lemma 5.3. *Let r be a glide reflection and let t be a translation such that the axes of r and t are parallel. Then r and t commute.*

Proof. Taking the axis of r as the x-axis, we can write $t = t(a)$ where $a = (a_1, 0)$. Thus, rt and tr both map an arbitrary point with Cartesian coordinates $(x, y) \mapsto (x + a_1, -y)$. It follows that r and t commute. □

Lemma 5.4. *Let $T \leq G \leq \mathbb{E}$ where T is the translation subgroup of G. Then T is a normal subgroup of G.*

Proof. Let r be a reflection and $t = t(a)$ be a translation. Let $P \in \mathbb{R}^2$ be an arbitrary point. Taking the axis of r as the x-axis we can describe the point P and the vector a in terms of Cartesian coordinates $P = (x, y)$ and $a = (a_1, a_2)$, respectively. It follows that the image of P under rtr proceeds as follows: $(x, y) \mapsto (x, -y) \mapsto (x + a_1, -y + a_2) \mapsto (x + a_1, y - a_2)$. Therefore, $Prtr = Pt'$ where t' is translation by the vector $a^* = (a_1, -a_2)$. Thus, we have shown that if t is a translation and r is a reflection then rtr is also a translation.

Now, let $u \in G$ be an arbitrary isometry of $\mathbb{R}^2$ and let t be an arbitrary translation. We claim that $u^{-1}tu$ is a translation. If u is the identity, then $u^{-1}tu$ is obviously the translation t. Otherwise, u is the product of at most 3 reflections according to Theorem 4.4. We can then rewrite u as the product of reflections say $u = r_1r_2r_3$. It follows that $u^{-1}tu = r_3r_2r_1tr_1r_2r_3$. We have already shown that if r is a reflection then rtr is a translation. It thus follows from associativity that $r_3r_2r_1tr_1r_2r_3$ is also a translation. By similar logic, if u is the product of one or two reflections, then $u^{-1}tu$ is a translation. Because $u^{-1}tu$ is a translation, $u^{-1}tu \in T$. Therefore, by Theorem 1.6, T is a normal subgroup. □

At this point, we have all the tools we need to classify the seven types of frieze groups.

Theorem 5.5. *If F is a frieze group, then F is one of the seven possible groups:*

$F_1 = \langle t | \rangle$
$F_1^1 = \langle t, r | r^2 = 1, r^{-1}tr = t \rangle$
$F_1^2 = \langle t, r | r^2 = 1, r^{-1}tr = t^{-1} \rangle$
$F_1^3 = \langle t, r | r^2 = t, r^{-1}tr = t \rangle$
$F_2 = \langle t, s | t^s = t^{-1}, s^2 = 1 \rangle$
$F_2^1 = \langle t, s, , r | s^2 = 1, t^s = t^{-1}, r^2 = 1, t^r = t, (sr)^2 = 1 \rangle$
$F_2^2 = \langle t, s, r | s^2 = 1, t^s = t^{-1}, r^2 = t, t^r = t, (sr)^2 = 1 \rangle$

Proof. Let F be a frieze group and let T be the translation subgroup of F. Because F is a frieze group, $T = \langle t | \rangle$ where t is a translation that generates T.

Case 1: Suppose F contains no non-trivial rotations. If F contains no reflections, then $F = F_1 = \langle t | \rangle$. Otherwise, F contains a reflection or a glide reflection. Suppose

F contains a reflection r. By Lemma 3.1, $r^{-1}tr$ generates $r^{-1}Tr$. Note that T is a normal subgroup by Lemma 5.4. Thus, $r^{-1}Tr = T$ by Theorem 1.6. It follows that $r^{-1}tr$ generates T. Therefore, $r^{-1}tr = t^{\pm 1}$. Each case generates a different frieze group so either $F = F_1^1 = \langle t, r | r^2 = 1, r^{-1}tr = t\rangle$ or $F = F_1^2 = \langle t, r | r^2 = 1, r^{-1}tr = t^{-1}\rangle$. (Note: If F also contained a glide reflection q, then rq, being an orientation preserving isometry that has no fixed points, would be a translation in F. It would then follow that q would already be generated by r and t, and, therefore, not generate a new frieze group.)

There does, however, remain the case in which F contains a glide reflection but no reflection. So, instead of a reflection, suppose F contains a glide reflection r. Thus, $r^2 = t^h$ for some $h \in \mathbb{Z}$. Because r and t commute by Lemma 5.3, it follows that $(rt^k)^2 = r^2t^{2k} = t^{2k+h}$ for all $k \in \mathbb{Z}$. Choose $k \in \mathbb{Z}$ such that $2k + h = 0$ or 1 and define $r' = rt^k$. Because $r'^2 = t^{2k+h}$, $r'^2 = 1$ or t. Also, because $r' = rt^k$, it follows that the group generated by r and t is the same group as that generated by r' and t, i.e. $\langle r, t|\rangle = \langle r', t|\rangle$. If $r'^2 = 1$, then the the frieze group F_1^1 is generated. Otherwise, $r'^2 = t$ and we have a new frieze group $F_1^3 = \langle t, r | r^2 = t, r^{-1}tr = t\rangle$.

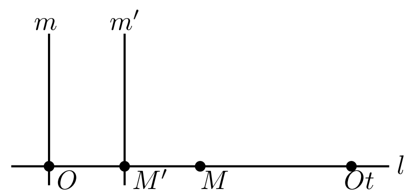
Case 2: Suppose F contains a non-trivial rotation s. By similar logic as in Case 1, $s^{-1}ts = t^{\pm 1}$. Because s is non-trivial, $s^{-1}ts = t^{-1}$. Thus, s is a rotation through π and, consequently, $s^2 = 1$. Suppose there exists some rotation $s' \in F$ such that $s' \neq s$. By similar logic, s' is a rotation through π. Because $s \neq s'$, it follows from our analysis of composition of rotations in the proof of Theorem 4.4 that ss' is a translation. Thus, $ss' \in T$. Multiplying both sides by s, we see that $s' \in sT$ and, thus, s' is generated by s and t. It follows that if F contains no reflections then $F = F_2 = \langle t, s | t^s = t^{-1}, s^2 = 1\rangle$.

Suppose F contains a reflection or glide reflection r with axis l'. By similar logic as before, $r^{-1}tr = t^{\pm 1}$. Thus, either $l' \parallel l$ or $l' \perp l$ where l is the axis of t. Suppose $l' \perp l$. It follows that $(sr)^{-1}t(sr) = t$ and sr is an orientation reversing isometry (i.e. a reflection or glide reflection) with axis l'' parallel to l. Because $(sr)^{-1}t(sr) = t$ and $rtr = t^{-1}$, it follows that $rstst = r$ and $tstr = (sr)$ Thus, we can rewrite the generators without changing the group, so let sr replace the generator r. This simplifies the cases $l' \parallel l$ or $l' \perp l$ to only the case $l' \parallel l$ as sr is an orientation reversing isometry with axis parallel to l.

Now we claim that $l' = l$ where l' denotes the axis of the orientation reversing isometry r and shall prove the claim by contradiction. Suppose $l' \neq l$ and consider the point Or and the isometry $r^{-1}sr$. It follows that $Orr^{-1}sr = Osr = Or$ as O is the center of s. Since $r^{-1}sr$ is orientation preserving and fixes Or, $r^{-1}sr$ must be a rotation by Theorem 4.4. To show that $r^{-1}sr$ is a non-trivial rotation, consider the some point $P \in \mathbb{R}^2$ such that $P \neq O$. The image of Pr under $r^{-1}sr$ is Psr. Because $P \neq O$, $Ps \neq P$ from which it follows that that $Psr \neq Pr$ as the isometry r preserves distance. Thus, Pr is not fixed under $r^{-1}sr$, so $r^{-1}sr$ is a non-trivial rotation of the plane. We have already established that Or is fixed and all non-trivial rotations in frieze groups are rotations through angle π so it follows that $r^{-1}sr = s(Or, \pi)$. Because $O \neq Or$ and $\frac{\pi}{2}$ and $\frac{\pi}{2}$ are supplementary, it follows from our analysis of composition of rotations in Theorem 4.4 that $s(O, \pi)s(Or, \pi)$ is a translation τ. By looking at the image of O under τ we see that $\tau = t(2\overrightarrow{OOr})$. Because $l \parallel l'$ and $l \neq l'$, τ is a translation that is not in the frieze group. This contradiction proves the claim that $l = l'$.

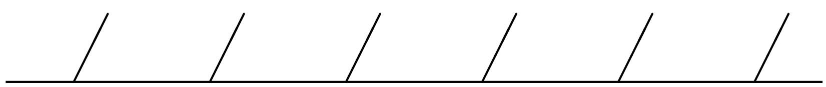
It now follows that if r is a reflection then sr is a reflection by Theorem 4.3 with axis $m \perp l$ at O. Thus, $F = F_2^1 = \langle t, s, r | s^2 = 1, t^s = t^{-1}, r^2 = 1, t^r = t, (sr)^2 = 1 \rangle$.

Otherwise, r is glide reflection. By similar logic as in Case 1, we can change generators to get $r^2 = t$. Because $r^2 = t$, l is the axis of both r and t. Let O be the center of the rotation s. Let M be the midpoint of line segment $\overline{OOt}$ and the M' be the midpoint of line segment $\overline{OM}$. Let m the line containing O such that $m \perp l$ and let m' be the line containing M' such that $m' \perp l$ as illustrated in the figure below.

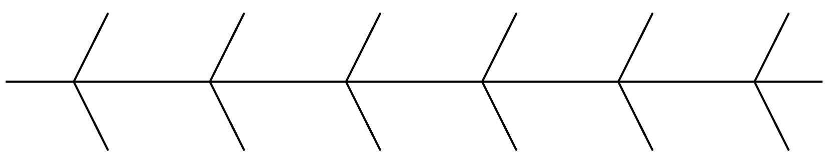


It follows by Theorem 4.1 that $s = r(m)r(l)$ and $r = r(l)r(m)r(m')$. Thus, $sr = r(m')$ and, consequently, $(sr)^2 = 1$. This gives the final frieze group $F_2^2 = \langle t, s, r | s^2 = 1, t^s = t^{-1}, r^2 = t, t^r = t, (sr)^2 = 1 \rangle$. $\square$

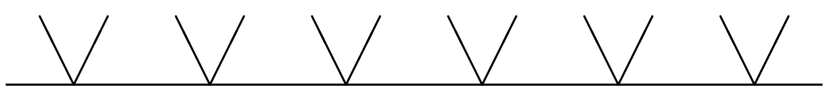
Having successfully completed our goal, we conclude with illustrations of the seven frieze groups, understanding the horizontal line to represent the axis of translation and the images to continue indefinitely.



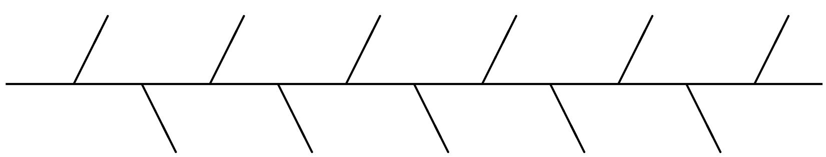
$F_1 = \langle t | \rangle$



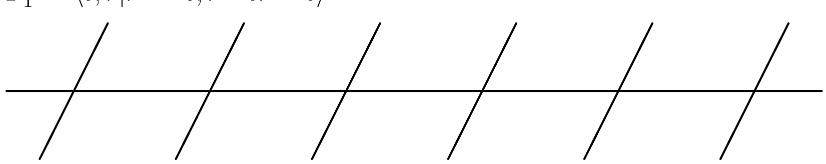
$F_1^1 = \langle t, r | r^2 = 1, r^{-1}tr = t \rangle$



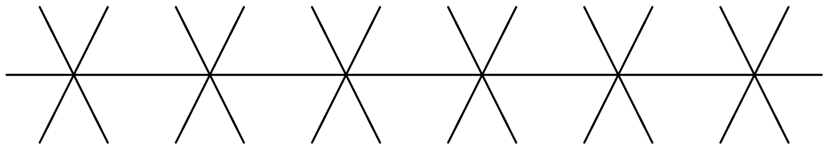
$F_1^2 = \langle t, r | r^2 = 1, r^{-1}tr = t^{-1} \rangle$



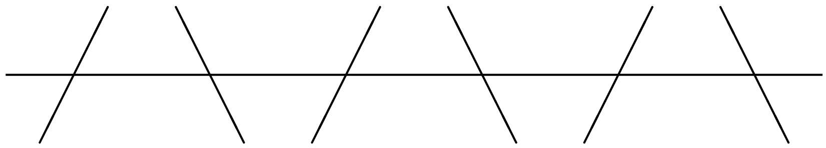
$F_1^3 = \langle t, r | r^2 = t, r^{-1}tr = t \rangle$



$F_2 = \langle t, s | t^s = t^{-1}, s^2 = 1 \rangle$



$F_2^1 = \langle t, s, , r | s^2 = 1, t^s = t^{-1}, r^2 = 1, t^r = t, (sr)^2 = 1\rangle$



$F_2^2 = \langle t, s, r | s^2 = 1, t^s = t^{-1}, r^2 = t, t^r = t, (sr)^2 = 1\rangle$

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[1] D. L. Johnson. Symmetries. Springer. 2001.