FOURIER SERIES OF CONTINUOUS FUNCTIONS AT GIVEN POINTS

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Abstract. The purpose of this paper is to show that, for fixed x in the closed interval [0,1], the set of periodic continuous functions whose Fourier series diverges at x is dense in the set of all periodic continuous functions.

Contents

1. Introduction

The purpose of this paper is to prove a result from Fourier Analysis which is not itself standard, but which does contrast a standard result in a very interesting way. The theorem to be proved in this paper states that, given a point x, the set of continuous functions $\mathcal{A}$ defined over a compact subset T of $\mathbb{R}$ whose Fourier series has unbounded partial sums at a point x are dense in the collection of all continuous functions over the compact set T. In comparison, the Carleson-Hunt theorem states that for finite p greater than 1, the Fourier series of any periodic function in an L^p space will diverge on at most a set of measure zero. Essentially, most nicely behaved functions have Fourier series which converge almost everywhere. It comes as quite a surprise then that every continuous function is arbitrarily close to a continuous function divergent at any given point.

The audience for this paper is anyone who has worked with vector spaces before. Those who have not had a real analysis course will need to read carefully the first section on Metric Spaces, and those who have not studied functional analysis before will need to study the first subsection of the section titled Functional Analysis carefully. I move at a fast pace through the definitions, and so it is important that the reader refer back to the definitions whenever the terms are used in later proofs in order to appreciate the subtleties needed. It is recommended that anyone familiar with real and functional analysis at least skim those first few sections to

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acquaint themselves to the notation used throughout, but the definitions are rather standard.

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2. Metric Spaces

In order to be as self-contained as possible, we begin with the necessary definitions and tools from metric space theory. We define a *metric space* to be a pair (X, ρ) where X is a space and ρ is a metric. Commonly, metrics are associated with our a priori notion of distance. A *metric* is any function $\rho : X \times X \to \mathbb{R}^+$ such that for any $x, y, z \in X$ and $\alpha \in \mathbb{R}$ we have

(1) Positive Definiteness: $\rho(x, y) \geq 0$. Moreover, $\rho(x, y) = 0$ if and only if $x = y$
(2) Symmetry: $\rho(x, y) = \rho(y, x)$
(3) Triangle Inequality: $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$

One of the most important aspects of metric spaces is the ability to create very nicely behaved subsets called *balls*. An open ball is denoted $B(x, r)$ where x is the center of the ball and r is the radius. It consists of the points y such that $\rho(x, y) < r$. A closed ball is denoted $\overline{B(x, r)}$. In this case, it consists of the points y such that $\rho(x, y) \leq r$. Note that this is a more general notion of open and closed intervals talked about in $\mathbb{R}$.

We next turn our attention to *Cauchy sequences*. Given a sequence $\{x_n\}$, we say that it is Cauchy if for arbitrary $\epsilon > 0$ there exists an N such that for any $n, m > N$ we have that $\rho(x_n, x_m) < \epsilon$. If a point is the limit of a Cauchy sequence, then there exist balls of arbitrary radius around that point which contain all but finitely many points of the Cauchy sequence. A space that contains all of the limit points to its Cauchy sequences is said to be *complete*. Thus, in a natural way, the *completion* of a space is the union of the space with the limit points of the Cauchy sequences. The fundamental example of a complete space is $\mathbb{R}$.

This leads into the notion of a set being dense in a space. A set X is *dense* if every point of the space is the limit of a Cauchy sequence in X. Two of the most insightful and useful examples are the rational numbers as well as the irrational numbers in $\mathbb{R}$ (and of course $\mathbb{R}$ is dense in $\mathbb{R}$) using the metric $\rho(x, y) = |x - y|$.

2.1. Topology.

It will be important to understand some extensions of notions from metric spaces into topological spaces. A *topological space* is a space X endowed with a topology $\mathcal{T}$ where a *topology* is defined to be a family $\{U_\alpha\} = \mathcal{T}$ such that

(1) $X \in \mathcal{T}$
(2) $\emptyset \in \mathcal{T}$, where $\emptyset$ is the empty set
(3) $\cup_{a \in A} U_a \in \mathcal{T}$
(4) $\cap_{n=1}^{N} U_n \in \mathcal{T}$

We refer to the sets U_α as *open sets*. The complement of an open set is said to be a *closed* set (it is clear from basic set theory that the complement of a closed set is open). It follows along with De Morgan's laws that closed sets then have similar properties. In words, the entire space is closed, the empty set is closed, arbitrary intersections of closed sets are closed, and finite unions of closed sets are closed. Any metric space is a topological space, where the open sets are composed of the necessary unions and intersections of open balls.

Given a set $\mathcal{A}$, we say that x is an *interior* point of $\mathcal{A}$ if there exists some open set U_α such that $x \in U_\alpha$ and $U_\alpha \subseteq \mathcal{A}$. Note that every point of an open set $\mathcal{O}$ is then an interior point of $\mathcal{O}$.

It will later prove necessary to talk about the smallest closed set containing a given set $\mathcal{A}$. We refer to this closed set as the *closure* of $\mathcal{A}$ and denote it by $\bar{\mathcal{A}}$. A formal definition for $\bar{\mathcal{A}}$ would be the intersection of all closed sets containing $\mathcal{A}$ as a subset. It is then clear that the closure of a closed set is itself. In metric spaces, a closed set is one containing all of its limit points. We take note that another characterization for a set being *dense* is that the closure of the set must be the whole space.

3. Functional Analysis

3.1. **Background.** The *supremum* (abbreviated *sup*) of a set of real numbers $\{r_\alpha\}$ is a number r such that

(1) $r \geq r_\alpha$ for all r_α
(2) there is no s satisfying 1) such that $s < r$

A supremum can then be approximated from below by a sequence of real numbers in the set. The supremum is only defined for the real numbers $\mathbb{R}$. That every set of real numbers has a supremum is, along with its ordering and field structure, a defining characteristic, being equivalent to the completeness of $\mathbb{R}$.

A *norm* is any function $||\cdot||$ from a space X to the non-negative real numbers obeying the following properties:

(1) $||x|| = 0$ if and only if $x = 0$
(2) $||\alpha x|| = |\alpha|||x||$ for any complex number α and for any x in X
(3) $||x + y|| \leq ||x|| + ||y||$ for any x, y in X

A familiar example of a norm is the standard norm on the plane $\mathbb{R}^2$ where the norm of a vector $\mathbf{x} = (x, y)$ is $||\mathbf{x}|| = \sqrt{x^2 + y^2}$. An important observation to make is that given a norm, it is possible to induce a metric by defining the distance between two points x, y as $||x - y||$.

Given two spaces X and Y, we define the set of all linear, bounded functions between them to be the space $L(X, Y)$. Elements of this space get added point-wise – that is to say that $(T + S)(x) = T(x) + S(x)$. A linear function T is one with the property that $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$. A bounded linear function then is a linear function such that for any bounded subset $E \in X$, $||f(E)||_Y$ is bounded. This notion allows us to define an important norm to be used later in the paper - the *operator norm.* Given a bounded linear function T between X and Y, we define the operator norm $||\cdot||_{X\to Y}$ by $||T||_{X\to Y} = \sup_{x\in X} ||T(\frac{x}{||x||})||_Y$.

One other notion needed from linear functions is the notion of dual spaces. Given a vector space X over a field $\mathcal{F}(= \mathbb{R}$ or $\mathbb{C})$ we can define bounded linear functionals $f_n : X \to \mathcal{F}$. The collection of all bounded linear functionals will be called the *dual space* of X and be denoted X^*. As an interesting aside, it is easy to verify that X^* is also a vector space over $\mathcal{F}$, and so its dual is also well defined and can be denoted as X^{**}.

3.2. **Tools from Functional Analysis.** Finally we are ready to begin deriving the theory needed. We begin with the Baire Category Theorem.

Definition 3.1. A space X such that any countable intersection of open dense sets is dense is said to be a *Baire Space.*

Theorem 3.2 (Baire Category Theorem)**.** *Every complete metric space is a Baire Space.*

Baire's theorem attempts to capture the sense that complete metric spaces are full to the brim with points which are all very cozy and close together. The way the theorem will be proved is a standard method where we manage to nest open balls inside one another to obtain a Cauchy sequence from their centers. By completeness, we then have a point known to be both in our arbitrary open set and in the countable intersection of our dense open subsets. This finding holds for all open sets, and hence the intersection of open dense sets is arbitrarily close to any point in the space. But this is what it means to be dense since we can approximate any point in the space by a Cauchy sequence with points from the intersection.

Proof. Let U_n be a countable collection of open dense subsets of our complete metric space X. It suffices to show that for any given open subset $\mathcal{O}$ in X, there is a point x of $\bigcap U_n$ which is contained in $\mathcal{O}$. By assumption, U_1 is dense in X and hence intersects $\mathcal{O}$. So, combined with the axioms of topology, we know that the intersection of U_1 and $\mathcal{O}$ is itself a non-trivial open set and so we can place an open ball $B(x_1, r_1)$ inside their intersection, and inside of that we can fit a closed ball $\overline{B(x_1, r_1/2)}$ containing the open ball $B(x_1, r_1/2)$. This last step guarantees that our soon-to-be-defined sequence of points will not end up at the boundary of $\mathcal{O}$ which could result in convergence issues. This open ball of radius $r_1/2$ now replaces the role of $\mathcal{O}$ and we can repeat the process for U_n for $n \in \mathbb{N}$ where $r_1 > r_2 > ...$ and $r_n \to 0$. To make clear this process, here is the second iteration. Using $B(x_1, r_1/2)$ as our open set, we intersect it with U_2 to get a non-trivial open set in which we place an open ball $B(x_2, r_2)$. Inside of this ball we place a closed ball $\overline{B(x_2, r_2/2)}$ which contains the open ball $B(x_2, r_2/2)$. This open ball becomes the open set used in the next iteration. Given ϵ, n can be chosen such that $r_n/2 < \epsilon$ since $r_n \to 0$. Hence the centers of the balls, the x_n, form a Cauchy sequence where for $n, m > N$, $r_N < \epsilon$ we have that $x_n, x_m \in B(x_N, r_N/2)$, and so $\rho(x_n, x_m) < \epsilon$. This Cauchy sequence converges to some point $x \in \bigcap U_n$ since x was placed inside of a closed ball which contains all of its limit points and X is complete by assumption. Clearly x is in $\mathcal{O}$ since x is in $\overline{B(x_1, r_1/2)}$, and it was just shown to be in $\bigcap U_n$. □

Definition 3.3. A set U is said to be *nowhere dense* if $\bar{U}$ has dense complement.

Corollary 3.4 (Corollary to the Baire Category Theorem)**.** *A non-empty complete metric space is never a countable union of nowhere dense sets.*

Since we just finished showing that complete metric spaces are full of points, we wish to show that such spaces cannot be made up of countably many very small sets. The proof is a very straight forward contradiction employing only our previous result and basic set theory.

Proof. Let X be a non-empty complete metric space. Then X is a Baire space by Theorem 2.2, and hence any countable intersection of open dense sets in X form themselves a dense set. We assume for a contradiction that X is the countable union of nowhere dense sets, $\{U_n\}$. We know that for each U_n, the complement of the closure of U_n, $\complement\bar{U}_n$, is dense in X by assumption and is also open by virtue of being the complement of a closed set. Thus $\bigcap \complement\bar{U}_n$ is dense in X by the property of Baire spaces. But $\bigcap \complement\bar{U}_n = \complement \bigcup \bar{U}_n$ by De Morgan's law. We assumed that $X = \bigcup U_n$ so that $\complement \bigcup \bar{U}_n$ is empty. A contradiction follows. $\Rightarrow\Leftarrow$ $\square$

This result is quite powerful. It allows for complete metric spaces to be broken down into countably many parts (the union of which is the whole space) from which we are given the existence of at least one part which is not no-where dense. If this sounds like a minor accomplishment, consider the implication of not being a no-where dense set $\mathcal{A}$. It means that the complement of the closure of the set $\complement\bar{\mathcal{A}}$ is not dense in the space X. Hence, there is at least one point in X which cannot be approximated by sequences of points in $\complement\bar{\mathcal{A}}$. This means that there is an entire ball not contained in $\complement\bar{\mathcal{A}}$, and hence that ball is contained in $\bar{\mathcal{A}}$.

Remark 3.5. The existence of a ball contained in $\bar{\mathcal{A}}$ as claimed above is essential to a step used later. We make a quick detour to verify its guaranteed existence. It is clear that there is at least one point x in X which cannot be approximated by a sequence of points in $\complement\bar{\mathcal{A}}$, for if that were not the case then $\complement\bar{\mathcal{A}}$ would be dense which is false by assumption. Since we are working with metric spaces, we note that if there were not a ball contained in $\bar{\mathcal{A}}$, then that would mean that for all ϵ we would have $B(x, \epsilon)$ containing points of $\complement\bar{\mathcal{A}}$. But then we would could approximate x with points from $\complement\bar{\mathcal{A}}$ by considering the sequence of balls $\{B(x, 1/n)\}$ where n is taken over the natural numbers. Thus, there must exist a ball around x contained in $\bar{\mathcal{A}}$.

Equipped with the fact that the closure of a no-where dense set has at least one open ball contained in it - hence an interior point - we proceed with the Uniform Boundedness Principle.

Definition 3.6. A function (in particular, a linear function) T from X to Y is said to be *continuous* if for every open set $\mathcal{O}_Y$ in Y we have that $T(\mathcal{O}_X) = \mathcal{O}_Y$ for some open set $\mathcal{O}_X$ in X. Equivalently, T is continuous if for every closed set $\mathcal{C}_Y$ in Y we have that $T(\mathcal{C}_X) = \mathcal{C}_Y$ for some closed set $\mathcal{C}_X$ in X.

Theorem 3.7 (Uniform Boundedness Principle)**.** *Let X be a complete normed vector space (a Banach space) and Y a normed vector space. If $F = \{F_\alpha\}$ is a family of continuous linear functions from X to Y, then if for all $x \in X$ we have that $\sup_{F_\alpha \in F} ||F_\alpha(x)||_Y < \infty$, it follows that $\sup_{F_\alpha \in F} ||F_\alpha||_{X \to Y} < \infty$.*

The intent of this theorem is to relate a point-wise boundedness result into a global boundedness result. This enables us to pass from one norm to the other. The supremums are very important in portraying the inner mechanisms of the theorem. We said nothing about the functions other than in what space they lived, and so being able to take these limits and still maintain boundedness is a subtle but important feature. As for the proof of the theorem, there are two distinct parts. The first part defines a countable number of sets which cover the entire space. It then takes a little bit of work in order to find an interior point in one of these sets using the Baire Category Corollary. The second half of the proof is to explicitly find an upper bound for the operator norms. This is done by using the ball obtained from the interior point. We translate this ball to the origin and are able to make a remark on the effect of the operator on points of this relocated ball. As a concluding step, we are able to use the definition of the operator norm to find our explicit upper bound.

Proof. Let $X_n = \{x \in X : \sup_{F_\alpha \in F} ||F_\alpha(x)||_Y \leq n\}$. We observe that $\bigcup X_n = X$ since we assumed that for all $x \in X$ we have that $\sup_{F_\alpha \in F} ||F_\alpha(x)||_Y < \infty$ which means that every x must fall into one of the X_n. Since X is a Baire Space with the inherited metric from the norm, we know that not every X_n can be nowhere dense by the corollary to the Baire Category Theorem. The X_n are closed since each X_n is the intersection over all α of $\mathcal{X}_{n,\alpha} = \{x \in X : ||F_\alpha(x)||_Y \leq n\}$. The equality of these two sets can be seen by noting that $\cap_\alpha \mathcal{X}_{n,\alpha} \subseteq X_n$ since $\mathcal{X}_{n,\alpha} \subseteq X_n$ for every α. To see that $X_n \subseteq \cap_\alpha \mathcal{X}_{n,\alpha}$, observe that any point x in X_n must be in $\mathcal{X}_{n,\alpha}$ for every α by definition of x being in X_n. Since the $\mathcal{X}_{n,\alpha}$ are closed for each α by continuity - and so the arbitrary intersection of the $\mathcal{X}_{n,\alpha}$ will also be closed by the axioms of topology - x is in $\cap_\alpha \mathcal{X}_{n,\alpha}$. Thus, $X_n = \cap_\alpha \mathcal{X}_{n,\alpha}$ and so X_n is closed for every n. Since all of the X_n are closed, we know that at least one X_m has an interior point by virtue of not being nowhere dense and Remark 3.5. We associate with this interior point $x \in X$ a radius δ representing the open ball $\mathcal{O}$ in X_m. Pick a point $p \in X$ such that $||p||_X \leq \delta$. Then both x and $x-p$ are in $\mathcal{O}$ and so in X_m. Then for every $F_\alpha \in F$ we have that $||F_\alpha(p)||_Y \leq ||F_\alpha(x-p)||_Y + ||F_\alpha(x)||_Y \leq m+m = 2m$ whenever $||p||_X \leq \delta$. The first inequality is just the triangle inequality, and the last inequality follows from the definition of X_m. We can divide the left and right hand sides by $|\delta|$ to get $\frac{||F_\alpha(p)||_Y}{|\delta|} \leq \frac{2m}{|\delta|}$. Since F_α is linear, this is the same as $||F_\alpha(\frac{p}{|\delta|})||_Y \leq \frac{2m}{|\delta|}$ where $\frac{p}{|\delta|} \leq 1$. Taking the supremum over p of both sides, where we recall that $||p||_X \leq \delta$, we conclude that $||F_\alpha||_{X \to Y} \leq \frac{2m}{|\delta|}$ by the definition given for the operator norm. As a quick technical point, note that we could write $p = rd$, where $||r||_X$ is at most 1 and d is chosen such that $||d||_X = \delta$, and so it is clear that for finding the operator norm it is optimal to have p with norm δ by linearity. Since m and δ were fixed, we have a fixed bound for the operator norm as desired. □

This theorem is a very useful tool, although it will not be used directly in this paper. Rather, we will require a corollary to this theorem. Essentially, we need to use the contrapositive. But even that is not a strong enough statement. It is clear that the contrapositive guarantees us at least one point in X such that $\sup_{F_\alpha \in F} ||F_\alpha(x)|| = \infty$. However, given the structure of the functions used, we can make a far stronger statement. Since linear functions have the characteristic property that $T(\alpha x+\beta y) = \alpha T(x)+\beta T(y)$, if we know that x is in the set of points such that $\sup_{F_\alpha \in F} ||F_\alpha(x)|| = \infty$, then $(\alpha x + \beta y)$ is also going to be in that set (for $\alpha \neq 0$). This condition forces a large number of points in the space to share this common property, and in fact we will show that this collection of points will be dense in the space.

Corollary 3.8 (Corollary to the Uniform Boundedness Principle)**.** *If the family $F = \{F_\alpha\}$ of linear operators in $L(X, Y)$ does not have a uniform bound for the operator norms of its members, then the set $S = \{x \in X : \sup_{F_\alpha \in F} ||F_\alpha(x)|| = \infty\}$ is dense in X.*

This proof will seem very familiar since it follows the same path as the previous proof. As mentioned previously, a set which is not dense contains an interior point. In the previous proof, once we found an interior point we were able to show that all of the linear functions had to be bounded. But this contradicts our assumption, and so there is no interior point. But this is the same as proving denseness.

Proof. Suppose that $\sup_\alpha ||F_\alpha||_{X\to Y} = \infty$. We borrow the same sets as from above: $X_n = \{x \in X : \sup_{F_\alpha \in F} ||F_\alpha(x)||_Y \leq n\}$. In order to show that the set $S = \{x \in X : \sup_{F_\alpha \in F} ||F_\alpha(x)||_Y = \infty\}$ is dense in X, we assume for contradiction that some X_n has a non-empty interior. We now follow rather closely to the proof of the Uniform Boundedness Principle. We pick some X_m to have an interior point. We associate with it a ball $B(\delta, x)$ contained in X_m. Pick p such that $||p||_X \leq \delta$. Then for every $F_\alpha \in F$, we have that $||F_\alpha(p)||_Y \leq ||F_\alpha(x-p)||_Y + ||F_\alpha(x)||_Y \leq m+m = 2m$. Dividing the left and right by $|\delta|$ and taking supremums over x, we conclude that $||F_\alpha||_{X\to Y} \leq \frac{2m}{|\delta|}$. But we have assumed that not all F_α were bounded. Hence, it must not be the case that any X_n has an interior point. We conclude that S is dense in X by complementation. □

We now have all of the tools needed from functional analysis. In a moment of reflection, it is important to examine what we have achieved up to this point. We proved the Baire Category Theorem which enabled us to easily prove the corollary needed in this paper, allowing us to know the existence of an interior point if we divide up our space in the right way. From this corollary we were able to elegantly prove the Uniform Boundedness Principle. Then, in much the same way, we proved the Uniform Boundedness Corollary which is a stronger statement than merely being the contrapositive of the principle. This corollary allows us to say something about a particular type of dense subset of a space based on the behavior of the norm on the space. This is a useful result which will nicely tie together the proof for the big-time theorem.

4. Big-Time Theorem

The purpose of this paper is to prove a result about Fourier series. Hence, some basic facts about Fourier series need to be discussed before the theorem can be shown. We begin by defining the Fourier series. It is only defined for the set of periodic functions (the periodic functions can be defined on anything as abstract as locally compact abelian groups, but it will suffice to think of $\mathbb{R}$ or $\mathbb{C}$), where a periodic function is one such that there exists a p, known as the period, such that $f(p+x) = f(x)$. Thinking of a periodic function defined on the real line is the same as thinking of a function defined on a circle. For the sake of clarity, identify the circle T with the unit interval $[0, 1]$. Given a continuous function f, we define the k^{th} *Fourier coefficient* of f to be

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_0^1 f(t) e^{-2\pi i k t} dt.$$

We define the Fourier series of a function f to be

$$S(f)(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=\mathbb{Z}} \hat{f}(k) e^{2\pi i k x}.$$

To make sense of this definition, we define the partial sums of the Fourier series

$$S_N(f)(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-N}^{k=N} \left(\frac{1}{\sqrt{2\pi}} \int_0^1 f(t) e^{-2\pi i k t} dt \right) e^{2\pi k x} = \frac{1}{2\pi} \int_0^1 f(t) \sum_{k=-N}^{k=N} e^{2\pi i k (x-t)} dt.$$

To simplify notation, and since it possesses many useful properties, we define the Dirichlet Kernel D_N to be

$$D_N(x) = \sum_{k=-N}^{k=N} e^{2\pi i k x}$$

Hence, the calculation above shows

$$S_N(f)(x) = \frac{1}{2\pi} \int_0^1 f(t) D_N(x-t) dt.$$

This will be the form used from now on. The Fourier series is the limit of S_N as $N \to \infty$.

We state a standard result about the Dirichlet Kernel used in the upcoming proof. The Dirichlet kernel can be rewritten as a ratio of two sine functions by the equation

$$D_N(t) = \frac{\sin((2N+1)t\pi)}{\sin(t\pi)}.$$

To show this, we merely need to massage the definition of the Dirichlet Kernel. But first we make a quick substitution of $e^{2\pi i t} = \omega$ to simplify notation. We begin with

$$D_N(t) = \sum_{k=-N}^{k=N} \omega^k = \omega^{-N} \left(\sum_{k=0}^{k=2N} \omega^k \right).$$

This is just a shifting of the terms. Next we see that

$$\omega^{-N} \left(\sum_{k=0}^{k=2N} \omega^k \right) = \omega^{-N} \left(\frac{1-\omega^{2N+1}}{1-\omega} \right) = \frac{\omega^{-N} - \omega^{N+1}}{1-\omega} = \frac{\omega^{-(N+1/2)} - \omega^{N+1/2}}{\omega^{-1/2} - \omega^{1/2}}.$$

Substituting back in $e^{2\pi i t}$ for ω and applying the exponential definition for the sine function: $\sin(t) = \frac{1}{2i}(e^{it} - e^{-it})$, we conclude that

$$D_N(t) = \frac{\sin((2N+1)t\pi)}{\sin(t\pi)}$$

as desired.

Remark 4.1. We define

$$\mathcal{S}_N : \mathcal{C}([0,1]) \to \mathbb{C}$$

by

$$\mathcal{S}_N(f) = S_N(f)(x)$$

where $S_N(f)(x)$ denotes the N^{th} partial sum of the Fourier series of f at x. Note that each $\mathcal{S}_N$ has associated with it an implicit, fixed x. Since $S_N(f)(x) = \frac{1}{2\pi} \int_0^1 f(x) D_N(x-t) dt$, we see that

$$\begin{aligned}\mathcal{S}_N(\alpha f+\beta g) &= S_N(\alpha f+\beta g)(x)\\ &= \frac{1}{2\pi}\int_0^1 (\alpha f+\beta g)(x)D_N(x-t)dt\\ &= \frac{\alpha}{2\pi}\int_0^1 f(x)D_N(x-t)dt+\frac{\beta}{2\pi}\int_0^1 g(x)D_N(x-t)dt\\ &= \alpha\mathcal{S}_N(f)+\beta\mathcal{S}_N(g).\end{aligned}$$

So then the $\mathcal{S}_N$ are linear functionals.

Theorem 4.2. *The set of continuous functions on the circle $T = [0,1]$ whose Fourier series are infinite at x is dense in $\mathcal{C}([0,1])$.*

Proof. We equip the continuous functions with the supremum norm. We claim the operator norm of $\mathcal{S}_N$ is $||\mathcal{S}_N|| = \frac{1}{2\pi}\int_0^1 |D_N(x-t)|dt$. Indeed,

$$\frac{|\mathcal{S}_N(f)|}{||f||} = \frac{1}{2\pi}\left|\int_0^1 \frac{f(t)}{|f(t)|}D_N(x-t)dt\right| \le \frac{1}{2\pi}\int_0^1 |D_N(x-t)|\,dt.$$

Since we do need our inequality to be an equality, we note that the Dirichlet Kernel is continuous and equality occurs when $f(t) = D_N(x-t)$. That D_N is continuous is easily verified. Being the sum of exponential functions, all of which are continuous, continuity of the Dirichlet Kernel follows.

Now we aim to show that $\int_0^1 |D_N(x-t)|dt$ approaches infinity as N approaches infinity. Since the circle is translation invariant and since the Dirichlet Kernel is an even function, the operator norm of $\mathcal{S}_N$ is equal to $\frac{1}{2\pi}\int_0^1 |D_N(t)|dt$. We want to show that $||\mathcal{S}_N||$ is unbounded, so it suffices to show that $\frac{1}{2\pi}\int_0^1 |D_N(t)|dt$ is unbounded. Because the function $f(x) = \sin x - x$ is not positive for $x > 0$, we get that

$$|D_N(t)| = \left|\frac{\sin{(2N+1)t\pi}}{\sin t\pi}\right| \ge \left|\frac{\sin{(2N+1)t\pi}}{t\pi}\right|.$$

Hence,

$$\frac{1}{2\pi}\int_0^1 |D_N(t)|\,dt \ge \frac{1}{2\pi}\int_0^1 \left|\frac{\sin{(2N+1)t\pi}}{t\pi}\right| dt.$$

This right hand of this inequality grows without bound as $N \to \infty$. To show this we first simplify notation by substituting $\frac{1}{2\pi}\int_0^1 \left|\frac{\sin{(2N+1)t\pi}}{t\pi}\right| dt$ by $\int_0^1 \left|\frac{\sin Nt}{t}\right| dt$ since unboundedness of one as $N \to \infty$ implies unboundedness of the other. For fixed N, $|\sin(Nt)|$ will equal zero $\lfloor\frac{N}{2\pi}\rfloor$ times, where $\lfloor\frac{N}{2\pi}\rfloor$ is the smallest integer less than $\frac{N}{2\pi}$. These zeros will be spaced out at even intervals. We recall that $|\sin(Nt)|$ takes on the value .5 when $t = \frac{\pi}{6N}, \frac{5\pi}{6N}, \frac{7\pi}{6N}, \frac{11\pi}{6N}, \frac{13\pi}{6N}, \frac{17\pi}{6N}, ..., \frac{12n\pi+\pi}{6N}, \frac{12n\pi+5\pi}{6N}, \frac{12n\pi+7\pi}{6N}, \frac{12n\pi+11\pi}{6N}$. Then we see that

$$\begin{aligned}
\int_0^1 \left|\frac{\sin Nt}{t}\right| dt &> \frac{1}{2}\sum_{n=1}^{n=\lfloor \frac{N}{2\pi}\rfloor} \int_{\frac{12n\pi+\pi}{6N}}^{\frac{12n\pi+5\pi}{6N}} \frac{1}{t}dt + \int_{\frac{12n\pi+7\pi}{6N}}^{\frac{12n\pi+11\pi}{6N}} \frac{1}{t}dt \\
&> \frac{1}{2}\sum_{n=1}^{n=\lfloor \frac{N}{2\pi}\rfloor} \int_{\frac{12n\pi+\pi}{6N}}^{\frac{12n\pi+5\pi}{6N}} \frac{6N}{12n\pi+5\pi}dt \\
&= \frac{1}{2}\sum_{n=1}^{n=\lfloor \frac{N}{2\pi}\rfloor} \left[\frac{1}{12n+5}\right][(12n+5)-(12n+1)] \\
&= \frac{1}{2}\sum_{n=1}^{n=\lfloor \frac{N}{2\pi}\rfloor} \left[\frac{4}{12n+5}\right]
\end{aligned}$$

In the first two lines, we restrict ourselves only to half of those intervals where the value of sine is greater than a half, and we then approximate the desired integral by a lower sum. The final term is just a version of the harmonic series, which is well known to diverge as N approaches infinity. Hence, as desired, $\frac{1}{2\pi}\int_0^1 \left|\frac{\sin(2N+1)t\pi}{t\pi}\right|$ grows without bound as $N \to \infty$.

We want to conclude that the family of operator norms over all $\mathcal{S}_N$, $\{||\mathcal{S}_N||_{[0,1]\to\mathbb{R}}\}$, is unbounded. The collection $\{\mathcal{S}_N\}$ is unbounded in $\mathcal{C}([0,1])^*$, the dual space of $\mathcal{C}([0,1])$. Therefore, applying the corollary of the uniform boundedness principle to $\mathcal{C}([0,1])^*$, we see that the set $\mathcal{A} = \{f \in \mathcal{C}([0,1]) : \sup_N ||\mathcal{S}_N(f)|| = \infty\}$ is dense in $\mathcal{C}([0,1])$. We recall that we made the evaluation point x be fixed. This demonstrates that the set of continuous functions whose Fourier series diverges at x is dense in $\mathcal{C}([0,1])$. So we conclude that, for any x in $[0,1]$, the set of continuous functions whose Fourier series diverges at x is dense in $\mathcal{C}([0,1])$. □

It is important to take note of what was used in this proof. The assumptions can be generalized in a very simple way. There was no property specific to Fourier series used other than the lack of a uniform bound for the partial sums. The corollary to the Uniform Boundedness Principle only requires a family of linear operators be unbounded. This implies that, given a space X, if a subset of the dual space of X is unbounded, we could make a similar argument. It would be interesting to see if the Fourier series of other types of functions might share this property. For example, instead of looking at $\mathcal{C}^0$ functions it might be possible to follow a similar proof for some other $\mathcal{C}^n$ space. At this point, the problem changes from one about Fourier Analysis and Fourier series to one about Functional Analysis and dual spaces.

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