Representations of Matrix Lie Algebras

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Abstract

Building upon the concepts of the matrix Lie group and the matrix Lie algebra, we explore the natural connections between the Lie groups and Lie algebras via the exponential map. We later introduce the matrix commutator as a Lie bracket operation to aid our investigation of Lie algebra representations, which we illustrate with the example of the adjoint representation on the special unitary group.

1 Introduction

1.1 Significance

The theory of representation of Lie algebras allows for the classification of Lie groups, which has broad applications to analyses of continuous symmetries in mathematics and physics. In mathematics Lie group classification reveals symmetries in differential equations and provides a method of considering geometries in terms of their invariant properties. In physics representation theory yields natural connections between representations of Lie algebras and the properties of elementary particles.

Limiting the current study to matrix groups and algebras allows us to work from a Lie group to its adjoint representation with little more theoretical background than basic analysis, group theory and linear algebra. According to Ado's theorem, every finite-dimensional Lie algebra can be viewed as a Lie algebra of $n \times n$ matrices; thus, scaling the problem down to one of matrix groups does not limit the current study beyond any mathematical or physical application.

1.2 Preliminary Definitions

Definition 1. A **Lie group** is a group that is also a smooth manifold such that the group action is compatible with differential structure.

Example 1. *The set of complex numbers of absolute value 1 together with complex multiplication satisfies the group axioms and has the structure of a topological group, as multiplication and inversion are continuous functions over the multiplicative group of nonzero complex numbers. This set, called the* ***circle group*** *and denoted $\mathbb{T}$, is a Lie group.*

Definition 2. A **matrix group** is a subset of the $n \times n$ matrices that forms a group with matrix multiplication. Naturally, a **matrix Lie group** is a Lie group that is also a matrix group.

Example 2. *The set of $n \times n$ invertable matrices together with matrix multiplication is a matrix Lie group called the* ***general linear group*** *and denoted* $\mathbf{GL}(n)$.

Definition 3. A real vector space $\mathfrak{g}$ over a field F is called a **Lie algebra** if it has a bracket operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the **Lie bracket**, with the following properties:

- $[\cdot,\cdot]$ is anti-symmetric, i.e. for all $v, w \in \mathfrak{g}$,

$$[v, w] = -[w, v]$$

- $[\cdot,\cdot]$ is bilinear, i.e. for all $a, b \in F$ and $v_1, v_2, v_3 \in \mathfrak{g}$,

$$[av_1 + bv_2, v_3] = a[v_1, v_3] + b[v_2, v_3]$$

  and

$$[v_1, av_2 + bv_3] = a[v_1, v_2] + b[v_1, v_3]$$

- $[\cdot,\cdot]$ satisfies the Jacobi identity, i.e. for all $v_1, v_2, v_3 \in \mathfrak{g}$,

$$[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0$$

Example 3. *The standard Lie bracket for matrix Lie algebras, called the* ***commutator bracket****, is defined for $A, B \in \mathfrak{g}$ as $[A, B] = AB - BA$.*

2 The Lie Algebra of a Matrix Lie Group

2.1 Theory: the Tangent Space

Definition 4. The **tangent space** of a matrix Lie group G at a point $\mathbf{p}$ in G is the n-dimensional plane in $\mathbb{R}^{n^2}$ tangent to G at $\mathbf{p}$. The tangent space at the identity is denoted L(G).

Let G be a matrix Lie group, and define a smooth matrix-valued function $\gamma : \mathbb{R} \to G$ of a real variable t that satisfies $\gamma(0) = \mathbf{p}$ for some $\mathbf{p}$ in G. The $n \times n$ matrix $\gamma(t)$ has entries $\gamma_{ij}(t)$ that are smooth functions of t. The image of γ is a smooth curve in G that passes through $\mathbf{p}$. The differential γ' at $t = 0$ is tangent to γ at $t = 0$. Since the image of γ lies on the manifold G, we know that $\gamma'(0)$ is tangent to G at $\mathbf{p}$. The differential $\gamma'(t)$ is an $n \times n$ matrix: the Jacobian of $\gamma(t)$. The tangent space of G at $\mathbf{p}$ is the set of all matrices of the form $\gamma'(0)$ for some smooth curve $\gamma : \mathbb{R} \to G$ that satisfies $\gamma(0) = \mathbf{p}$. Letting $\mathbf{p} = I_n$ defines the tangent space of G at the identity, denoted L(G).

2.2 An Example: SU(n)

Let $\mathbf{SU}(n)$ denote the special unitary group, the set of all $n \times n$ unitary matrices with determinant 1 together with matrix multiplication. Clearly, $\mathbf{SU}(n)$ is a matrix Lie group, as it is a subgroup of $\mathbf{GL}(n, \mathbb{C})$.

Let $\gamma : \mathbb{R} \to \mathbf{SU}(n)$ be a smooth matrix-valued function of a real variable t that satisfies $\gamma(0) = I_n$. Then $\gamma(t)$ is an $n \times n$ unitary matrix, and we have the identity

$$\gamma(t) \cdot \gamma(t)^* = I_n \tag{1}$$

where A^* represents the conjugate transpose of the matrix A.

Lemma: product rule for matrix functions. Express matrix-valued functions $A, B : \mathbb{R} \to M_n(\mathbb{C})$ of a real variable t in terms of their elements: $A(t) = (a_{ij}(t))_{n\times n}$ and $B(t) = (b_{ij}(t))_{n\times n}$. Then $(A(t) \cdot B(t))_{ij} = \sum_{k=1}^{n} a_{ik}(t)b_{kj}(t)$. The derivatives of A and B are taken element-wise; thus,

$$\begin{aligned}
(A(t) \cdot B(t))'_{ij} &= \sum_{k=1}^{n} (a_{ik}(t)b_{kj}(t))' \\
&= \sum_{k=1}^{n} a_{ik}(t)b'_{kj}(t) + \sum_{k=1}^{n} a'_{ik}(t)b_{kj}(t)) \\
&= (A(t)B'(t) + A'(t)B(t))_{ij}
\end{aligned}$$

Note that differentiation commutes with the complex conjugacy operation because differentiation acts element-wise. Then we can unambiguously express an immediate implication of (1):

$$\gamma(t) \cdot \gamma'(t)^* + \gamma'(t) \cdot \gamma(t)^* = 0_n \tag{2}$$

Now let $t = 0$. Then by the assumption $\gamma(0) = I_n$, we have

$$\gamma'(0) + \gamma'(0)^* = 0_n \tag{3}$$

Let W denote the set of all matrices of the form $\gamma'(0)$ that satisfy (3) for smooth γ, i.e. the set of $n \times n$ traceless skew-Hermitian matrices. Then the tangent space of $\mathbf{SU}(n)$ at the identity is a subspace of W. Later we will show that W and the tangent space of $\mathbf{SU}(n)$ are the same space.

2.3 Theory: the Exponential Map

If G is a matrix Lie group, then clearly there exists a group homomorphism $\gamma : (\mathbb{R}, +) \to \mathbf{GL}(n, \mathbb{R})$. If for all $1 \le i, j \le n$, γ_{ij} is continuous, then γ is continuous. The existence of a continuous group homomorphism implies that such a function is smooth.

Theorem 1. *If $\gamma : (\mathbb{R}, +) \to \mathbf{GL}(n, \mathbb{R})$ is a continuous group homomorphism, then $\gamma(t) = exp(tA)$ for $A = \gamma'(0)$.*

Proof. It is beyond the scope this discussion to prove Theorem 1 in general. Below is a proof of the 1-dimensional case. Since $\mathbf{GL}(1, \mathbb{R})$ is isomorphic to $\mathbb{R} \setminus \{0\}$, it is sufficient to prove that if $\gamma : (\mathbb{R}, +) \to (\mathbb{R}, \cdot)$ is a continuous group homomorphism, then $\gamma(t) = exp(ta)$ for $a = \gamma'(0)$.

Assume $\gamma(1) < 0$. Since $\gamma(0) = 1$, by the Intermediate Value Theorem there exists a real t_0 such that $\gamma(t_0) = 0$, but zero does not belong to the codomain of γ ※. Thus, $b = \gamma(1) > 0$.

Subpoint: $\gamma(mc) = (\gamma(c))^m$ for $c \in \mathbb{R}$, $m \in \mathbb{Z}^+$. The case where $m = 1$ is trivially true. Assume the case where $m = k$. Then $\gamma((k+1)c) = \gamma(kc+c) = \gamma(kc)\gamma(c)$ since γ is a group homomorphism, and $\gamma(kc)\gamma(c) = (\gamma(c))^k\gamma(c) = (\gamma(c))^{k+1}$ by the induction hypothesis. ✓

Subpoint: $\gamma(q) = b^q$ for $q \in \mathbb{Q}$. Let $c = 1$. Then clearly $\gamma(m) = b^m$. Now let $s \in \mathbb{Z}^+$. $\gamma(1/s) = b^{1/s}$ follows from $(b^{1/s})^s = b = \gamma(1) = \gamma(s(1/s)) = (\gamma(1/s))^s$. Then for $r \in \mathbb{Z}^+$, we have $b^{r/s} = (b^{1/s})^r = (\gamma(1/s))^r = \gamma(r/s)$. $\gamma(-r/s) = b^{-r/s}$ follows from $1 = \gamma(0) = \gamma(r/s+(-r/s)) = \gamma(r/s)\gamma(-r/s) = b^{r/s}\gamma(-r/s)$. ✓

Subpoint: $\gamma(t) = b^t$ for $t \in \mathbb{R}$. Define a sequence of rationals $\{t_i\}_{i\in\mathbb{Z}^+}$ for each t such that $t = lim_{i\to\infty}t_i$. Since γ is continuous, we can write

$\gamma(t) = \gamma(lim_{i\to\infty}t_i) = lim_{i\to\infty}\gamma(t_i) = lim_{i\to\infty}b^{t_i}$. Since b^t is continuous, $\gamma(t) = lim_{i\to\infty}b^{t_i} = b^{lim_{i\to\infty}t_i} = b^t$. ✓

Let $a = ln(b)$. Then $\gamma(t) = b^t = (exp(a))^t = exp(at)$ and $\gamma'(0) = a$. □

As a corollary, $\gamma(t)$ is smooth. This follows from the fact that $exp(ta)$ is smooth. The extension of this theorem to complex numbers involves a similar proof.

2.4 An Example: $\mathbf{SU(n)}$

Lemma 1. *For $A \in W$, $exp(tA)exp(tA)^* = I_n$.*

Proof. We wish to show

$$\begin{aligned} exp(tA)exp(tA)^* &= exp(tA)exp(tA^*) & (4)\\ &= exp(tA)exp(-tA) & (5)\\ &= exp(tA - tA) & (6)\\ &= I_n & (7) \end{aligned}$$

(4) follows from the fact that for matrices A, B we have $(AB)^* = B^*A^*$ since complex conjugation commutes with matrix transposititition and distibrutes over complex multiplication. (5) follows from the fact that (tA) is skew-Hermitian. (6) follows from the fact that (tA) and $(-tA)$ commute. (7) is trivial. □

Lemma 2. *For $A \in W$, $det(exp(tA)) = 1$.*

Proof. We wish to show

$$\begin{aligned} det(exp(tA)) &= exp(tr(tA)) & (8)\\ &= exp(0) & (9)\\ &= 1 & (10) \end{aligned}$$

For a field F, let an $n \times n$ matrix M be such that the eigenvalues of M belong to F. Then by the Jordan normal form theorem, M is upper triangularizable, so there exist an invertable matrix L and an upper triangular matrix C such that $M = LCL^{-1}$. Then

$$\begin{aligned} exp(A) &= exp(LCL^{-1})\\ &= \sum_{k=0}^{\infty} \frac{1}{k!}(LCL^{-1})^k\\ &= \sum_{k=0}^{\infty} \frac{1}{k!}LC^kL^{-1}\\ &= Lexp(C)L^{-1} \end{aligned}$$

Then, where the λ_i are the diagonal entries of C,

$$\begin{aligned} det(Lexp(C)L^{-1}) &= det(L)det(exp(C))det(L^{-1}) = det(exp(C)) \\ &= \prod_{k=0}^{n} exp(\lambda_k) = exp\left(\sum_{k=0}^{n} \lambda_k\right) = exp(tr(C)) \end{aligned}$$

The second line follows from the fact that the diagonal entries of C are invariant over exponentiation. Then since $tr(C) = tr(LCL^{-1}) = tr(M)$, we have $det(exp(M)) = exp(tr(M))$. Then (8) follows from the fact that the eigenvalues of (tA) are complex. (9) follows from the fact that (tA) is traceless. (10) is trivial. □

Together, lemmas 1 and 2 demonstrate that W is a subspace of $\mathfrak{su}(n)$, the tangent space of $\mathbf{SU}(n)$ at the identity, which as we demonstrated above is a subspace of W; thus, $\mathfrak{su}(n) = W$. In other words, the tangent space of $\mathbf{SU}(n)$ at the identity is the space of $n \times n$ traceless skew-Hermitian matrices.

2.5 The Lie Bracket

If the tangent space L(G) at the identity of a Lie group G is equipped with a Lie bracket operation, it is called the Lie algebra $\mathfrak{g}$ of G. The standard Lie bracket for Lie algebras of matrix Lie groups is given by $[A, B] = AB - BA$.

From the previous examples and the definitions above, it is clear that the Lie algebra $\mathfrak{su}(n)$ of the special unitary group $\mathbf{SU}(n)$ is the space of $n \times n$ traceless skew-Hermitian matrices together with the commutator bracket.

3 Lie Algebra Representation

3.1 Theory: the Adjoint Representation

Definition 5. A **representation** of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\rho : \mathfrak{g} \to End(\mathfrak{g})$.

Definition 6. Given an element A of a Lie algebra $\mathfrak{g}$, we define the **adjoint action** of A on $\mathfrak{g}$ as the endomorphism $ad(A) : \mathfrak{g} \to \mathfrak{g}$ with $ad(A)(B) = [A, B]$ for all B in $\mathfrak{g}$. The map $ad : \mathfrak{g} \to End(\mathfrak{g})$ given by $A \mapsto ad(A)$ is called the **adjoint representation** of $\mathfrak{g}$.

It is not difficult to show that the adjoint representation is indeed a representation of $\mathfrak{g}$. Let $A, B, C \in \mathfrak{g}$. Then we have

$$\begin{aligned}([ad(A), ad(B)])(C) &= [A,[B,C]] - [B,[A,C]] \\ &= [A,[B,C]] + [B,[C,A]] \\ &= -[C,[A,B]] \\ &= [[A,B],C] = ad([A,B])(C)\end{aligned}$$

where the third line follows from the Jacobi identity. Since the adjoint representation commutes with the Lie bracket, it is a Lie algebra homomorphism and thus is indeed a representation of $\mathfrak{g}$.

Definition 7. In general the **Cartan subalgebra** is defined as the self-normalizing nilpotent subalgebra of a Lie algebra. Given a finite-dimensional Lie algebra over an algebraically-closed field of characteristic 0, the Cartan subalgebra is unique up to conjugation.

Consideration of this technical definition is not necessary to our analysis of $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$. For the purposes of this discussion, let the Cartan subalgebra, denoted $\mathfrak{h}$, of a Lie algebra $\mathfrak{g}$ be the subalgebra of $\mathfrak{g}$ that consists of elements, the adjoint representation matrices of which have diagonal entries.

Definition 8. Let $\mathfrak{g}$ be a Lie algebra over a field F. A **root** is a functional $\chi_v : \mathfrak{h} \to F$, with corresponding root **root vector** $v \in \mathfrak{g}$, that satisfies $[x, v] = \chi_v(x)v$ for all $x \in \mathfrak{h}$.

3.2 Roots of $\mathfrak{su}(2)$

Consider the Lie algebra $\mathfrak{su}(2)$. Since for all positive integers n the Lie algebra $\mathfrak{su}(n)$ consists of $n \times n$ traceless skew-Hermitian matrices, we can express an arbitrary element of $\mathfrak{su}(2)$ by

$$\begin{pmatrix} ix & -\overline{\beta} \\ \beta & -ix \end{pmatrix}$$

where $x \in \mathbb{R}$ and $\beta \in \mathbb{C}$. The obvious basis for $\mathfrak{su}(2)$ over $\mathbb{R}$ is $\{u_1, u_2, u_3\}$ where

$$u_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, u_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Complexify the basis vectors as follows:

$$
\begin{aligned}
u_+ &= -\frac{1}{2}(u_1 + iu_2) = \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix} \\
u_- &= -\frac{1}{2}(u_1 - iu_2) = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \\
u_z &= \frac{1}{2}u_3 = \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}
\end{aligned}
$$

Then we can express the adjoint actions $ad(u_+), ad(u_-), ad(u_z)$ by matrices corresponding to the basis $\{u_+, u_-, u_z\}$ over $\mathbb{C}$:

$$
ad(u_+) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 1 & 0 & 0 \end{pmatrix}, ad(u_-) = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, ad(u_z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

These matrices demonstrate that scalar multiples of u_z constitute the Cartan subalgebra of $\mathfrak{su}(2)$. Then u_+ is an eigenvector of $ad(u_z)$ with eigenvalue 1 and that u_- is an eigenvector of $ad(u_z)$ with eigenvalue -1. In other terms the root from u_z to 1 has root vector u_+, and the root from u_z to -1 has root vector u_-.

3.3 Roots of $\mathfrak{su}(3)$

As with $\mathfrak{su}(2)$ complexify the obvious basis over $\mathbb{R}$ to find the useful basis over $\mathbb{C}$. For $\mathfrak{su}(3)$ the useful basis vectors are as follows:

$$
t_+ = \begin{pmatrix} 0 & \frac{1-i}{2} & 0 \\ \frac{1+i}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, t_- = \begin{pmatrix} 0 & \frac{1+i}{2} & 0 \\ \frac{1-i}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, t_z = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
v_+ = \begin{pmatrix} 0 & 0 & \frac{1-i}{2} \\ 0 & 0 & 0 \\ \frac{1+i}{2} & 0 & 0 \end{pmatrix}, v_- = \begin{pmatrix} 0 & 0 & \frac{1+i}{2} \\ 0 & 0 & 0 \\ \frac{1-i}{2} & 0 & 0 \end{pmatrix}, u_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1-i}{2} \\ 0 & \frac{1+i}{2} & 0 \end{pmatrix}
$$

$$
u_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1+i}{2} \\ 0 & \frac{1-i}{2} & 0 \end{pmatrix}, y = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}
$$

Then we can express the adjoint actions over this basis. In the case of $\mathfrak{su}(3)$, we can simplify the computation by considering the adjoint action of an arbitrary linear combination x of t_z and y, the two basis vectors whose adjoint matrices are diagonal. For $a, b \in \mathbb{C}$, let $x = at_z + by$. Then

$$ad(x) = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}a + b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}a - b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}a + b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}a - b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Here we see that $t_+, t_-, t_z, v_+, v_-, u_+, u_-, y$ are eigenvectors of $ad(x)$ with eigenvalues $a, -a, 0, -\frac{1}{2}a + b, \frac{1}{2}a - b, \frac{1}{2}a + b, -\frac{1}{2}a - b$ and 0, respectively.

4 Conclusion

By working our way from the arbitrary matrix Lie group to its respective Lie algebra and its adjoint representation, we have demonstrated several fundamental properties of matrix Lie groups and matrix Lie algebras including the existence and form of the exponential map and the utility of the Lie bracket in the representation of Lie algebras. And by supplementing our theoretical results with the example of the special unitary group we inspire immediate physical applications. A more general investigation of Lie groups requires more advanced techniques in algebra and topology, and the current study explores basic concepts and methods that establish an introductory understanding to an extent that they may be expanded upon later.