THE INFINITE SYMMETRIC PRODUCT AND HOMOLOGY THEORY

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Abstract. Following the work of Aguilar, Gitler, and Prieto, I define the infinite symmetric product of a pointed topological space. The infinite symmetric product allows the construction of a reduced homology theory on CW complexes in terms of the homotopy groups. This relationship between homology and homotopy gives a means to convert Moore spaces that are CW complexes into Eilenberg-Mac Lane spaces, and I give an explicit construction for any finitely generated Abelian groups of CW complexes which are Moore spaces.

Contents

1. Introduction

The basic elements of the field of algebraic topology are divided generally into two categories: homotopy theory, which considers the homotopy groups of a topological space; and homology theory, which considers instead the homology groups. Each of these approaches has its particular benefits and drawbacks. The homotopy groups are relatively easy to define, and are in some sense a more powerful tool than homology, but are in general not easily computed. The homology groups, which up to natural isomorphism can be constructed in a number of seemingly disparate ways, while perhaps a less powerful tool are much more easily computable, giving them considerable practical value.

In this paper I follow the program of Aguilar, Gitler, and Prieto given in [1], supplemented by the original work of Dold and Thom in [2], which relate homology theory to homotopy theory by means of the infinite symmetric product construction, a functor from the category of pointed spaces to itself. Section 2 of this paper defines the infinite symmetric product and develops some of the basic results about it, which find use in the later sections. The infinite symmetric product acts in some sense to "simplify" the homotopical structure of a space, such that we find that the homotopy groups of the infinite symmetric product form a reduced homology

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theory on CW complexes, with some care being taken with spaces that are not connected. The proof that this is in fact a reduced homology theory on CW complexes is the content of Section 3. Constructing the homology groups in this manner at first seems counterproductive, as it does not capitalize on the advantage of the homology groups, that they can be more easily computed, but if anything nullifies this advantage by giving them as homotopy groups of certain spaces. However, by relating homotopy and homology, this construction transforms the problem of constructing Eilenberg-Mac Lane spaces, spaces with one specified nontrivial homotopy group, into the problem of constructing Moore spaces, the homological equivalent, an easier problem. Section 4 explores this relation, constructing Moore spaces and the corresponding Eilenberg-Mac Lane spaces.
As a convention, the basepoint of any pointed topological space will be written $*$ unless otherwise specified. For maps f and g, $f \simeq g$ will denote that f is homotopic to g. Any function referred to as a map between topological spaces is understood to be continuous, and in the case of pointed spaces to send basepoint to basepoint. For pointed spaces, constructions such as the cone, mapping cylinder, and mapping cone will be taken to be their reduced forms, so that the resulting space is also pointed.

2. The infinite symmetric product

The infinite symmetric product of a pointed topological space can be thought of as a natural topology on the set of finite unordered tuples drawn from that space. We will develop the infinite symmetric product in the same manner as [1], constructing it as the colimit of topologies on the sets of unordered n-tuples. Key to this construction is that the symmetric group S_n acts on X^n by permuting elements, namely, $\sigma(x_1, x_2, \ldots, x_n) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for all σ in S_n and $(x_1, x_2, \ldots, x_n)$ in X^n.

Definition 2.1. Let $(X, *)$ be a pointed topological space. For all natural numbers n, the *nth symmetric product of X* is the orbit space

$$\mathrm{SP}^n X = X^n / S_n$$

of the permutation action described above, where X^n has the usual product topology and the orbit space is given the quotient topology. $\mathrm{SP}^n X$ is given the basepoint $(*, *, \ldots, *)$.

There is an embedding of $\mathrm{SP}^n X$ into $\mathrm{SP}^{n+1} X$ given by $(x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2, \ldots, x_n, *)$. So $\mathrm{SP}^n X$ can naturally be considered as a subset of $\mathrm{SP}^{n+1} X$, giving a sequence $\mathrm{SP}^1 X \subset \mathrm{SP}^2 X \subset \cdots \subset \mathrm{SP}^n X \subset \mathrm{SP}^{n+1} \subset \cdots$ that permits the definition of an infinite symmetric product.

Definition 2.2. Let $(X, *)$ be a pointed topological space. The *infinite symmetric product of X* is the colimit

$$\mathrm{SP}\, X = \operatorname{colim} \mathrm{SP}^n X$$

according to the sequence above.

The processes of taking the orbit space and the colimit involved in constructing the infinite symmetric product can be reversed without affecting the end result.

Theorem 2.3. *Let $X^{<\omega}$ be the set of sequences in X such that only finitely many terms are not the basepoint, with the topology given by $X^{<\omega} = \operatorname{colim} X^n$, where X^n is consider as a subset of $X^{<\omega}$ by inclusion as the first n terms. Then $\mathrm{SP}\, X \cong X^{<\omega}/S_\omega$ where S_ω is the countable symmetric group with the action given by permuting the indices of the sequence.*

Proof. Let $\rho : X^{<\omega} \to X^{<\omega}/S_\omega$ and $\varrho_n : X^n \to \mathrm{SP}^n X$ be the quotient maps. Using the canonical inclusions $X^n \hookrightarrow X^{n+1}$, clearly $\varrho_{n+1}|_{X^n} = \varrho_n$. Hence we can define a map $\varrho : X^{<\omega} \to \mathrm{SP}\, X$ as $\varrho = \operatorname{colim} \varrho_n$, which we would like to be a quotient map. If ϱ is a quotient, both ρ and ϱ induce the same equivalence classes in $X^{<\omega}$, namely, $x \sim y$ if and only if x and y have the same set (counting multiplicities) of non-basepoint terms. That is, we clearly have that for any $x \in \mathrm{SP}\, X$, $\varrho^{-1}(x)$ is an equivalence class under ρ. Chasing across the colimits and quotients, we have that $F \subseteq \mathrm{SP}\, X$ is closed if and only if $F \cap \mathrm{SP}^n X$ is closed for all n, which are closed if and only if $\varrho_n^{-1}(F \cap \mathrm{SP}^n X) = \varrho_n^{-1}(F) \cap \varrho^{-1}(\mathrm{SP}^n X) = \varrho^{-1}(F) \cap \varrho^{-1}(\mathrm{SP}^n X)$ are closed. Evidently $\varrho^{-1}(\mathrm{SP}^n X) = X^n$, hence $F \subseteq \mathrm{SP}\, X$ is closed if and only if $\varrho^{-1}(F) \cap X^n$ is closed for all n, which since $X^{<\omega}$ has the colimit topology is closed if and only if $\varrho^{-1}(F)$ is closed. So we have that ϱ is a quotient inducing the same equivalence classes as ρ, and so $\rho \circ \varrho^{-1}$ is a homeomorphism $\mathrm{SP}\, X \to X^{<\omega}/S_\omega$. □

Both the finite symmetric products and the infinite symmetric products are functorial from the category of pointed topological spaces to itself. A map $f : X \to Y$ between pointed topological spaces induces maps $f^n : X^n \to Y^n$ given by $f^n(x_1, x_2, \dots, x_n) = (f(x_1), f(x_2), \dots, f(x_n))$. These maps evidently respect the permutation action, and hence induce maps $\mathrm{SP}^n f : \mathrm{SP}^n X \to \mathrm{SP}^n Y$ between the orbit spaces. Passing to the colimit, these maps in turn induce a map $\mathrm{SP}\, f : \mathrm{SP}\, X \to \mathrm{SP}\, Y$. Immediately from the construction of these maps, we find $\mathrm{SP}^n \mathrm{id}_X = \mathrm{id}_{\mathrm{SP}^n X}$ and $\mathrm{SP}\, \mathrm{id}_X = \mathrm{id}_{\mathrm{SP}\, X}$, and for maps $f : X \to Y$ and $g : Y \to Z$ we find $\mathrm{SP}^n(f \circ g) = \mathrm{SP}^n f \circ \mathrm{SP}^n g$ and $\mathrm{SP}(f \circ g) = \mathrm{SP}\, f \circ \mathrm{SP}\, g$.

Theorem 2.4. *If $f : X \to Y$ and $g : X \to Y$ are such that $f \simeq g$, then $\mathrm{SP}\, f \simeq \mathrm{SP}\, g$.*

Proof. The proof is analogous to the construction of $\mathrm{SP}\, f$ above, but for a homotopy instead. Let $H : X \times I \to Y$ be a homotopy from f to g. For all natural numbers n, define $H^n : X^n \times I \to Y^n$ by $H^n(x_1, x_2, \dots, x_n, t) = (H(x_1, t), H(x_2, t), \dots, H(x_n, t))$. Then H^n is continuous, as its projection onto each coordinate is continuous, being H acting on that coordinate. We have S_n act on $X^n \times I$ by permuting the coordinates of X^n and fixing I, and H^n respects this action. So H^n induces a map $\mathrm{SP}^n H : \mathrm{SP}^n X \times I \to \mathrm{SP}^n Y$, which passing to the colimit induces a map $\mathrm{SP}\, H : \mathrm{SP}\, X \times I \to \mathrm{SP}\, Y$. Letting $h_t : X \to Y$ take $x \mapsto H(x, t)$, by construction we have $(\mathrm{SP}\, H)(x, t) = (\mathrm{SP}\, h_t)(x)$. Hence $(\mathrm{SP}\, H)(x, 0) = (\mathrm{SP}\, f)(x)$ and $(\mathrm{SP}\, H)(x, 1) = (\mathrm{SP}\, g)(x)$, and $\mathrm{SP}\, f \simeq \mathrm{SP}\, g$. □

Corollary 2.5. *If X and Y are homotopy equivalent, then $\mathrm{SP}\, X$ and $\mathrm{SP}\, Y$ are homotopy equivalent. If X is contractible, then $\mathrm{SP}\, X$ is contractible.*

Proof. The first statement is immediate; given a map $f : X \to Y$ with homotopy inverse $g : Y \to X$, then $\mathrm{SP}\, g$ must be a homotopy inverse of $\mathrm{SP}\, f$. The second follows from the first after noting that $\mathrm{SP}\{*\} = \{*\}$. □

Theorem 2.6. *An inclusion of a closed (or open) subspace $i : A \hookrightarrow X$ induces closed (or open) inclusions $\mathrm{SP}^n i : \mathrm{SP}\, A \hookrightarrow \mathrm{SP}\, X$ and $\mathrm{SP}\, i : \mathrm{SP}\, A \hookrightarrow \mathrm{SP}\, X$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} A^n & \xhookrightarrow{i^n} & X^n \\ \downarrow \rho_A & & \downarrow \rho_X \\ \mathrm{SP}^n A & \xrightarrow{\mathrm{SP}^n i} & \mathrm{SP}^n X \end{array}$$

where ρ_A and ρ_X are the quotient maps. If $F \subseteq \mathrm{SP}^n A$ is closed, then $\rho_A^{-1}(F)$ is closed and invariant under the permutation action, hence $i^n \circ \rho_A^{-1}(F)$ is closed and respects the permutation action, and finally $(\mathrm{SP}^n i)(F) = \rho_X \circ i^n \circ \rho_A^{-1}(F)$ is closed. Similarly, if $E \subseteq \mathrm{SP}^n A$ is open, then $(\mathrm{SP}^n i)(E)$ is open. So $\mathrm{SP}^n i$ is a closed (open) inclusion, and passing to the colimit $\mathrm{SP}\, i$ is a closed (open) inclusion. □

The Dold-Thom theorem regarding quasifibrations and the infinite symmetric product will prove useful in demonstrating several important results. Before stating the theorem, we will establish some necessary definitions and show the long exact sequence generated by a quasifibration, which is what will make quasifibrations useful. For a proof of the Dold-Thom theorem, see [1] or [2].

Definition 2.7. A *quasifibration* is a map $\rho : E \to B$ such that for all $b \in B$ and $e \in \rho^{-1}(b)$, $\rho_* : \pi_n(E, \rho^{-1}(b)) \to \pi_n(B)$ is an isomorphism, where the homotopy functors are based on e and b, respectively.

The following theorem establishing the exact sequence associated to a pair of pointed spaces provides the basis for associating a sequence to a quasifibration, which here will be the purpose of considering quasifibrations. For the proof of this theorem, see [1].

Theorem 2.8. *A pair of pointed spaces (X, A) induces an exact sequence*

$$\cdots \longrightarrow \pi_n(A) \longrightarrow \pi_n(X) \longrightarrow \pi_n(X, A) \longrightarrow \pi_{n-1}(A) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_1(X, A) \longrightarrow \pi_0(A) \longrightarrow \pi_0(X)$$

Theorem 2.9. *Let $\rho : E \to B$ be a quasifibration. Given $b \in B$ and $e \in \rho^{-1}(b)$, there exists a long exact sequence*

$$\cdots \longrightarrow \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{\rho_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \longrightarrow \cdots$$

where $F = \rho^{-1}(b)$ is the fiber and $i : F \hookrightarrow E$ is the inclusion.

Proof. This sequence is obtained from the exact sequence of the pair (E, F) by substituting $\pi_n(B)$ for $\pi_n(E, F)$ according to the isomorphism ρ_*. □

Definition 2.10. Given a subspace $A \subset X$, we say that a neighborhood $B \supset A$ is *deformable to* A if there exists a homotopy $H : X \times I \to X$ such that

$$H(x, 0) = x \text{ for all } x$$
$$H(A \times I) \subseteq A$$
$$H(B \times I) \subseteq B$$
$$H(B \times \{1\}) \subseteq A$$

Theorem 2.11. (Dold-Thom) *Let X be a Hausdorff pointed space, and A a closed path connected subspace with a neighborhood deformable to it. If $\rho : X \to X/A$ is the quotient map, then $\operatorname{SP}\rho$ is a quasifibration with fiber $(\operatorname{SP}\rho)^{-1}(x)$ homotopy equivalent to $\operatorname{SP} A$ for all x in $\operatorname{SP} X/A$*

This has an immediate corollary by application to mapping cones which will prove useful more than once.

Corollary 2.12. *Let X and Y be Hausdorff pointed spaces such that Y is path connected, with a map $f : X \to Y$. Let $\rho : C_f \to \Sigma X$ be the quotient map from the mapping cone of f to the suspension of X which collapses $Y \times \{0\}$ to a point. Then $\operatorname{SP}\rho : \operatorname{SP} C_f \to \operatorname{SP}\Sigma X$ is a quasifibration with fiber $\operatorname{SP} X$.*

Proof. As a subset of C_f, Y is closed and path connected, so all that remains is to find a deformable neighborhood of Y. The subspace $Y \cup (X \times [0, \frac{1}{2}])$ is deformable to Y by the homotopy which fixes Y and takes

$$([(x,t)], s) \mapsto \begin{cases} [(x, t(1-s))] & t \in [0, \frac{1}{2}] \\ [(x, t(1+s) - s] & t \in (\frac{1}{2}, 1] \end{cases}$$

where $[\cdot]$ denotes the equivalence class of that point in the cone. Note that this deformation is similar to that of the mapping cylinder. □

Theorem 2.13. *Let X be a path connected Hausdorff pointed space. Then for all $n \geq 0$, $\pi_n(\operatorname{SP} X) \cong \pi_{n+1}(\operatorname{SP}\Sigma X)$. Further, this isomorphism is natural.*

Proof. Applying the previous corollary to the identity map and noting that $C_{\mathrm{id}} \cong CX$, the long exact sequence of the quasifibration $\operatorname{SP}\rho : \operatorname{SP} CX \to \operatorname{SP}\Sigma X$ gives

$$\cdots \longrightarrow \pi_{n+1}(\operatorname{SP} CX) \longrightarrow \pi_{n+1}(\operatorname{SP}\Sigma X) \longrightarrow \pi_n(\operatorname{SP} X) \longrightarrow \pi_n(\operatorname{SP} CX) \longrightarrow \cdots$$

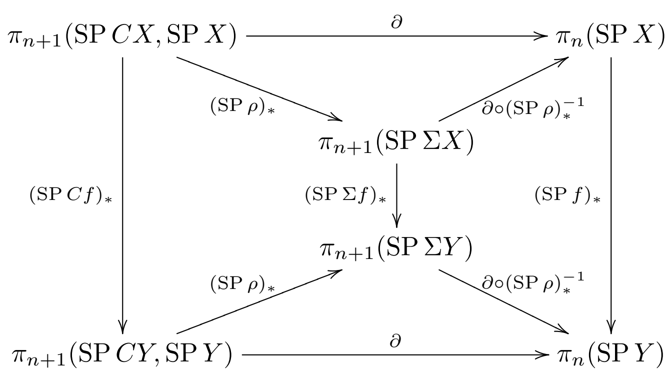
Since CX is contractible, $\operatorname{SP} CX$ is contractible and has trivial homotopy, so this segment of the sequence reduces to

$$0 \longrightarrow \pi_{n+1}(\operatorname{SP}\Sigma X) \longrightarrow \pi_n(\operatorname{SP} X) \longrightarrow 0$$

which implies that $\pi_n(\operatorname{SP} X) \cong \pi_{n+1}(\operatorname{SP}\Sigma X)$. In particular, we have

$$0 \longrightarrow \pi_1(\operatorname{SP}\Sigma X) \longrightarrow \pi_0(\operatorname{SP} X) \cong 0$$

at the end of the sequence, so $\pi_1(\operatorname{SP}\Sigma X) \cong 0$.
Now it remains to show that this isomorphism is natural. Given a map $f : X \to Y$, this is equivalent to that the right square of



commutes for all n, where ∂ is the connecting homomorphism for the homotopy sequence of a pair, and ρ is the quotient from the cone to the suspension, as above. Having proven that $\partial \circ \rho_*^{-1}$ is an isomorphism, the upper and lower triangles, which commute definitionally, are seen to be comprised of isomorphisms. We see that the outer square commutes easily from the definition of the connecting homomorphism as the restriction to the second term of the pair. To see that the left square commutes, note that

$$\begin{array}{ccc} (CX, X) & \xrightarrow{\rho} & (\Sigma X, *) \\ {\scriptstyle Cf}\downarrow & & \downarrow{\scriptstyle \Sigma f} \\ (CY, Y) & \xrightarrow{\rho} & (\Sigma Y, *) \end{array}$$

commutes, and induces the left square under the functor $\pi_{n+1} \circ \mathrm{SP}$. These together force the right square to commute, demonstrating that the isomorphism is natural. □

We will now analyze the nature of the infinite symmetric product on spheres, with the goal of characterizing the homotopy groups of $\mathrm{SP}\,\mathbb{S}^n$. This result will also later be useful in constructing Eilenberg-Mac Lane spaces.

Theorem 2.14. *For all n, $\mathrm{SP}^n\,\mathbb{S}^2 \cong \mathbb{CP}^n$, hence $\mathrm{SP}\,\mathbb{S}^2 \cong \mathbb{CP}^\infty$.*

Proof. The premise of this proof will be to take the coordinates of $\mathrm{SP}^n\,\mathbb{S}^2$ as the roots of a polynomial which will provide the homogeneous coordinates for $\mathbb{CP}^n$. The Riemann sphere projection gives a homeomorphism $\mathbb{S}^2 \to \mathbb{C} \cup \{\infty\}$ taking ∞ as the basepoint of $\mathbb{C} \cup \{\infty\}$, hence there is a homeomorphism $\mathrm{SP}\,\mathbb{S}^2 \to \mathrm{SP}(\mathbb{C} \cup \{\infty\})$, so we can consider $(x_1, \ldots, x_n) \in \mathrm{SP}\,\mathbb{S}^n$ as an unordered n-tuple of elements from $\mathbb{C} \cup \{\infty\}$. Then $(x_1, \ldots, x_n)$ can be mapped to $\mathbb{CP}^n$ by taking the coefficients of a polynomial over $\mathbb{C}$ whose roots[1] are $x_1, \ldots, x_n$, a map which is well-defined by the uniqueness of the polynomial up to scalar multiples and evidently homeomorphic. So $\mathrm{SP}^n\,\mathbb{S}^2 \cong \mathbb{CP}^n$ and by passing to the colimit $\mathrm{SP}\,\mathbb{S}^2 \cong \mathbb{CP}^\infty$. □

Theorem 2.15. *For all $n \geq 1$, $\pi_n(\mathrm{SP}\,\mathbb{S}^n) \cong \mathbb{Z}$ and $\pi_i(\mathrm{SP}\,\mathbb{S}^n) \cong 0$ if $i \neq n$.*

Proof. Consider $\mathbb{S}^1$, from which the general theorem follows by taking suspensions. We have evidently that $\mathbb{S}^1$ is homotopy equivalent to $\mathbb{S}^2 - \{0, \infty\}$ (using the Riemann sphere coordinates). This allows us to consider $\mathrm{SP}^n\,\mathbb{S}^1$ as the set of polynomials over $\mathbb{C}$ of degree at most n that have neither 0 nor ∞ as a root. Taking the homeomorphism $\mathrm{SP}^n\,\mathbb{S}^2 \to \mathbb{CP}^n$ given above to map $(x_1, \ldots, x_n) \mapsto (a_1, \ldots, a_n)$, $\mathrm{SP}^n\,\mathbb{S}^1$ is homeomorphic to $\mathbb{CP}^n$ without the planes $a_0 = 0$ and $a_n = 0$, that is, without the subspaces $\{0\} \times \mathbb{CP}^{n-1}$ and $\mathbb{CP}^{n-1} \times \{0\}$. Using the construction of $\mathbb{CP}^n$ by gluing a disk $\mathbb{D}^{2n}$ along its boundary to $\mathbb{CP}^{n-1}$, we have that $\mathbb{CP}^n - (\{0\} \times \mathbb{CP}^{n-1}) \cong \mathring{\mathbb{D}}^{2n}$. Accounting for this first removal, the second subspace removed then looks like $(\mathbb{CP}^{n-1} \times \{0\}) - (\{0\} \times \mathbb{CP}^{n-2} \times \{0\}) = (\mathbb{CP}^{n-1} - \{0\} \times \mathbb{CP}^{n-2}) \times 0 \cong \mathring{\mathbb{D}}^{2n-2} \times \{0\}$. Hence $\mathrm{SP}^n\,\mathbb{S}^1$ is homotopy equivalent to $\mathbb{CP}^n - (\{0\} \times \mathbb{CP}^{n-1} \cup \mathbb{CP}^{n-1} \times \{0\}) \cong \mathring{\mathbb{D}}^{2n} - \mathring{\mathbb{D}}^{2n-2} \times \{0\} = \mathring{\mathbb{D}}^{2n-2} \times (\mathring{\mathbb{D}}^2 - \{0\})$. Clearly $\mathring{\mathbb{D}}^{2n-2} \times (\mathring{\mathbb{D}}^2 - \{0\})$ is

[1] "Roots" at ∞ may effectively be ignored. It is natural to consider a polynomial of degree $m < n$ as having $n - m$, or indeed any additional number, of roots at ∞. With ∞ taken as the basepoint of $\mathbb{C} \cup \{\infty\}$, doing so preserves the intuition of the inclusion $\mathrm{SP}^m(\mathbb{C} \cup \{\infty\}) \hookrightarrow \mathrm{SP}^n(\mathbb{C} \cup \{\infty\})$ as the inclusion of polynomials of degree at most m into those of degree at most n, and more importantly is precisely how they must be considered to make the map homeomorphic.

homotopy equivalent to $\mathbb{S}^1$, collapsing the contractible space $\mathring{\mathbb{D}}^{2n-2}$ and projecting $\mathring{\mathbb{D}}^2 - \{0\}$ to $\mathbb{S}^1$ as its outer boundary. Passing to the colimit, we have $\mathrm{SP}\,\mathbb{S}^1 = \mathring{\mathbb{D}}^\infty \times (\mathring{\mathbb{D}}^2 - \{0\})$, which by similar reasoning is homotopy equivalent to $\mathbb{S}^1$.
Using that $\Sigma\mathbb{S}^n \cong \mathbb{S}^{n+1}$, we thus have by induction with theorem 2.13 that for all n, $\pi_n(\mathrm{SP}\,\mathbb{S}^n) \cong \mathbb{Z}$ and $\pi_i(\mathrm{SP}\,\mathbb{S}^n) \cong 0$ if $i \neq n$, noting for $i < n$ that the homotopy groups "added" to the bottom by taking suspensions are trivial, as was noted in the proof of 2.13. □

3. The homology groups

The infinite symmetric product may be used to define a reduced homology theory of pointed CW complexes, which as shown in [4] in turn determine a complete homology theory of topological spaces. The proof below that the infinite symmetric product can be used to define a reduced homology theory follows a variant of the proofs given in [1] and [2].

Definition 3.1. A *reduced homology theory on pointed CW complexes* $\tilde{E}_*$ is a set of functors $\tilde{E}_n$ for all $n \geq 0$ from the category of pointed CW complexes to the category of Abelian groups satisfying the following axioms:

(i) Exactness: If A is a subcomplex of X, then the sequence
$$\tilde{E}_n(A) \to \tilde{E}_n(X) \to \tilde{E}_n(X/A)$$
induced by the inclusion and the quotient map is exact.

(ii) Suspension: There is a natural isomorphism between $\tilde{E}_n(X)$ and $\tilde{E}_{n+1}(\Sigma X)$, where ΣX is the suspension of X.

(iii) Additivity: If $X = \bigvee_{\lambda\in\Lambda} X_\lambda$ is a wedge of CW complexes, then the inclusions $X_\alpha \hookrightarrow X$ induce an isomorphism
$$\tilde{E}_n(X) \to \bigoplus_{\lambda\in\Lambda} \tilde{E}(X_\lambda)$$

The purpose of this section will be to demonstrate that the homotopy groups of the infinite symmetric product form such a theory.

Definition 3.2. For $n \geq 0$, the *nth reduced homology group* of a path connected pointed CW complex X is
$$\tilde{H}_n(X) = \pi_n(\mathrm{SP}\,X)$$
If X is a pointed CW complex that is *not* path connected, instead take
$$\tilde{H}_n(X) = \pi_{n+1}(\mathrm{SP}\,\Sigma X)$$
to define the nth reduce homology of X. These functors together form $\tilde{H}_*$, the *reduced homology of X.*

For $n \geq 2$, $\tilde{H}_n$ is by composition evidently a functor from the category of pointed CW complexes to the category of Abelian groups. However, composition only guarantees that $\tilde{H}_1$ is to the category of groups, and, even worse, only that $\tilde{H}_0$ is to the category of sets. This difficulty will be addressed by first considering the suspension axiom.

Theorem 3.3. (Suspension) *For all pointed CW complexes and $n \geq 0$, there exists a natural isomorphism between $\tilde{H}_n$ and $\tilde{H}_{n+1}\Sigma$.*

Proof. The suspension axiom holds definitionally for spaces which are not path connected, so we will consider the path connected case. Using theorem 2.13, for a path connected pointed CW complex X we immediately have $\tilde{H}_n(X) = \pi_n(\mathrm{SP}\, X) \cong \pi_{n+1}(\mathrm{SP}\, \Sigma X) = \tilde{H}_{n+1}(\Sigma X)$, and this isomorphism is natural. $\square$

Now that we have the suspension axiom, we may assume that any pointed CW complex X is path connected, as if not we may use the suspension axiom to pass to the suspension of X, which is path connected, and prove the desired result there. So for the rest of the section all pointed CW complexes will be understood to be path connected.

Theorem 3.4. (Functoriality) *For all $n \geq 0$, $\tilde{H}_n$ is a functor from the category of pointed CW complexes to the category of Abelian groups.*

Proof. First, since π_n is a functor into the category of Abelian groups for all $n \geq 2$, we have that $\tilde{H}_{n+2}$ is a functor into the category of Abelian groups for all $n \geq 0$. We apply the natural isomorphism ϕ from the suspension axiom twice to construct the following commutative diagram.

$$\begin{array}{ccc}
\tilde{H}_n(X) & \xrightarrow{\tilde{H}_n f} & \tilde{H}_n(Y) \\
{\scriptstyle \phi_X}\big\downarrow {\scriptstyle \cong} & & {\scriptstyle \phi_Y}\big\downarrow {\scriptstyle \cong} \\
\tilde{H}_{n+1}(\Sigma X) & \xrightarrow{\tilde{H}_{n+1}\Sigma f} & \tilde{H}_{n+1}(\Sigma Y) \\
{\scriptstyle \phi_{\Sigma X}}\big\downarrow {\scriptstyle \cong} & & {\scriptstyle \phi_{\Sigma Y}}\big\downarrow {\scriptstyle \cong} \\
\tilde{H}_{n+2}(\Sigma^2 X) & \xrightarrow{\tilde{H}_{n+2}\Sigma^2 f} & \tilde{H}_{n+2}(\Sigma^2 Y)
\end{array}$$

So $\tilde{H}_n(X) \cong \tilde{H}_{n+2}(\Sigma^2 X)$, hence $\tilde{H}_n(X)$ is in fact an Abelian group and $\tilde{H}_n$ acts properly on objects. By the functoriality of $\tilde{H}_{n+2}$, $\tilde{H}_{n+2}\Sigma^2 f$ is a group homomorphism. Since the vertical arrows are isomorphisms, $\tilde{H}_n f$ is also a group homomorphism. Therefore $\tilde{H}_n$ handles morphisms appropriately, and is a functor from the category of pointed CW complexes to the category of Abelian groups. $\square$

Theorem 3.5. (Exactness) *If A is a subcomplex of X, then the sequence*

$$\tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A)$$

induced by the inclusion and the quotient map is exact.

Proof. Consider the diagram

$$\begin{array}{ccccc}
A & \overset{i}{\hookrightarrow} & X & \xrightarrow{\rho} & X/A \\
{\scriptstyle \cong}\big\downarrow {\scriptstyle h} & & {\scriptstyle \simeq}\big\downarrow {\scriptstyle j} & & {\scriptstyle \simeq}\big\uparrow {\scriptstyle k} \\
A \times \{1\} & \overset{i'}{\hookrightarrow} & M_i & \xrightarrow{\rho'} & C_i
\end{array}$$

which commutes in the right square and commutes in the left square up to homotopy, where $h : A \to A \times \{1\}$ is the canonical homeomorphism, $i : A \hookrightarrow X$ and $i' : A \times \{1\} \to M_i$ are the inclusion maps, $\rho : X \to X/A$ is the quotient map, M_i is the mapping cylinder of i and $j : X \hookrightarrow M_i$ is the canonical inclusion map of X into M_i, which is a homotopy equivalence, and $\rho' : M_i \to M_i/(A \times \{1\}) \cong C_i$ and $k : C_i \to C_i/(A \times I) \cong X/A$ are the quotient maps. Being a CW complex,

$A \cong A \times \{1\}$ is path connected and Hausdorff, and is closed in M_i. Further, $A \times (\frac{1}{2}, 1]$ is evidently deformable to A. Hence ρ' satisfies the conditions of the Dold-Thom theorem, and $\operatorname{SP} \rho'$ is a quasifibration with fiber $\operatorname{SP}(A \times \{1\}) \cong \operatorname{SP} A$.
Selecting from the long exact sequence generated by $\operatorname{SP} \rho'$, we have that

$$\begin{array}{ccccc}
\pi_n(\operatorname{SP} A) & \xrightarrow{i_*} & \pi_n(\operatorname{SP} X) & \xrightarrow{\rho_*} & \pi_n(\operatorname{SP} X/A) \\
\cong \downarrow (\operatorname{SP} h)_* & & \cong \downarrow (\operatorname{SP} j)_* & & \cong \uparrow (\operatorname{SP} k)_* \\
\pi_n(\operatorname{SP}(A \times \{1\})) & \xrightarrow{i'_*} & \pi_n(\operatorname{SP} M_i) & \xrightarrow{\rho'_*} & \pi_n(\operatorname{SP} C_i)
\end{array}$$

commutes and that the lower sequence is exact. Since $\operatorname{SP} h$, $\operatorname{SP} j$ and $\operatorname{SP} k$ are all homotopy equivalences, their induced homomorphisms are in fact isomorphisms, and so the upper sequence is also exact, which is precisely what we desired to prove. □

Definition 3.6. A *directed system* is a set of closed subspaces $\{X_\lambda\}_{\lambda \in \Lambda}$ of a space X such that $X = \operatorname{colim}_{\lambda \in \Lambda} X_\lambda$ and the partial order on Λ induced by the inclusions of the subsets satisfies that any two $\lambda, \gamma \in \Lambda$ have a common upper bound $\mu \geq \lambda, \gamma$.

Lemma 3.7. *Let X be the colimit of a closed sequence of spaces $X_1 \subset X_2 \subset X_3 \subset \cdots$, where each space is T_1. If $K \subset X$ is compact, then there exists i such that $K \subset X_i$.*

Proof. Take $K \subset X$. We can choose $x_i \in K$ for all natural numbers i such that $x_i \notin X_i$. Clearly $\{x_i\}_{i \in \mathbb{N}}$ has no limit points since no more than finitely many points are in any X_i, and so it and all its subsets are closed. Let $F_n = \{x_i\}_{i \geq n} \subset K$. Then any finite intersection $F_{n_1} \cap F_{n_2} \cap \cdots \cap F_{n_k} = F_{max\{n_i\}} \neq \emptyset$, so $\{F_n\}_{n \in \mathbb{N}}$ has the finite intersection property, and if K is compact should have nonempty intersection. However, $\cap_{n \in \mathbb{N}} F_n = \emptyset$, so K cannot be compact. □

Lemma 3.8. *Let X be a space which is the colimit of a directed system $\{X_\lambda\}_{\lambda \in \Lambda}$ such that for all compact $K \subset X$ there exists $\kappa \in \Lambda$ such that $K \subset X_\kappa$. Then $\pi_n(X) \cong \operatorname{colim}_{\lambda \in \Lambda} \pi_n(X_\lambda)$.*

Proof. The inclusions $i_\lambda : X_\lambda \to X$ induce a map $i_* : \operatorname{colim}_{\lambda \in \Lambda} \pi_n(X_\lambda) \to \pi_n(X)$. Given $f : \mathbb{S}^n \to X$, there exists $\kappa \in \Lambda$ such that $f(\mathbb{S}^n) \subset X_\kappa$, hence there exists a restriction of the range $f' : \mathbb{S}^n \to X_\kappa$ such that $f = i_\kappa \circ f'$. Hence i_* is surjective. Suppose $a, b \in \operatorname{colim}_{\lambda \in \Lambda} \pi_n(X_\lambda)$ are such that $i_*(a) = i_*(b)$. There there exists $\mu \in \Lambda$ such that there are $f : \mathbb{S}^n \to X_\mu$ and $g : \mathbb{S}^n \to X_\mu$ whose homotopy classes correspond respectively to a and b in the colimit. Extending the range to $f' : \mathbb{S}^n \to X$ and $g' : \mathbb{S}^n \to X$ we have that there exists $H : f' \simeq g'$. Then there exists $\nu \geq \mu$ such that $H(\mathbb{S}^n \times I) \subset X_\nu$. We may again define $f'' : \mathbb{S}^n \to X_\nu$ and $g'' : \mathbb{S}^n \to X_\nu$ by extending the range, and f'' and g'' still correspond to a and b in the colimit. But by restricting the range of H to X_ν we get a homotopy $H' : f'' \simeq g''$, so f'' and g'' are representatives of the same homotopy class, and $a = b$. Therefore i_* is injective and an isomorphism. □

Theorem 3.9. *Let X be a pointed CW complex that is the colimit of a directed system $\{X_\lambda\}_{\lambda \in \Lambda}$ of subcomplexes. Then for all n there is an isomorphism $\tilde{H}_n(X) \cong \operatorname{colim}_{\lambda \in \Lambda} \tilde{H}(X_\lambda)$.*

Proof. By theorem 2.6, the inclusions $i_\lambda : X_\lambda \hookrightarrow X$ induce inclusions $\operatorname{SP} i_\lambda : \operatorname{SP} X_\lambda \hookrightarrow \operatorname{SP} X$, which evidently agree on all intersections. So we can define $i : \operatorname{colim}_{\lambda\in\Lambda} \operatorname{SP} X_\lambda \to \operatorname{SP} X$ by $i = \operatorname{colim}_{\lambda\in\lambda} \operatorname{SP} i_\lambda$. From this construction we have that i is a continuous bijection, but not in general that i has a continuous inverse. Showing that $\tilde{H}_n i$ is nonetheless an isomorphism will rely on being able to place compact subsets of $\operatorname{SP} X$ nicely into pieces of the colimit. Let $K \subset \operatorname{SP} X$ be a compact set. First, considering $\operatorname{SP}^n X$ canonically as a directed system for $\operatorname{SP} X$, we have by lemma 3.7 above that $K \subset \operatorname{SP}^n K$ for some n. Let $\rho : X^n \to \operatorname{SP}^n X$ be the quotient map, and $K' = \rho^{-1}(K)$.
Claim. K' is compact.
Let $\{X_\alpha\}_{\alpha\in A}$ be an open cover of K'. Fix $y \in K$, and fix $x_y \in \rho^{-1}(y)$. We find $\rho^{-1}(y) = \{\sigma x_y\}_{\sigma\in S_n}$. For all y and σ, there exists $X_{y,\sigma} \in \{X_\alpha\}_{\alpha\in A}$ such that $\sigma x_y \in X_{y,\sigma}$. Let $G_y = \cup_{\sigma\in S_n} \sigma(\cap_{\sigma\in S_n} \sigma^{-1} X_{y,\sigma})$. So G_y is open, respects the permutation action, and $G_y \subseteq \cup_{\sigma\in S_n} X_{y,\sigma}$. Clearly $\{G_y\}_{y\in K}$ is an open cover of K', hence $\{\rho(G_y)\}_{y\in K}$ is an open cover of K, and permits a subcover $\{\rho(G_{y_i})\}_{i=1}^m$. This implies that $\{G_{y_i}\}_{i=1}^m$ covers K', and hence that $\{X_{y,\sigma}\}_{i=1,\sigma\in S_n}^m$ covers K'.
Claim. There exists $\lambda \in \Lambda$ such that $K \subset \operatorname{SP} X_\lambda$.
Let $K'' = \cup_{i=1}^n \pi_i(K')$, where π_i is the ith projection. Then K'' is compact. So K'' intersects only finitely many cells of X, hence there exists $\kappa \in \Lambda$ such that $K'' \subset X_\kappa$. Then $K' \subseteq (K'')^n \subset X_\kappa^n$, hence $K \subset \operatorname{SP} X_\lambda$.
Hence, given $f : \mathbb{S}^n \to \operatorname{SP} X$, there exists $\kappa \in \Lambda$ such that $f(\mathbb{S}^n) \subset \operatorname{SP} X_\kappa$, so we can restriction the range of f to get $f' : \mathbb{S}^n \to \operatorname{SP} X_\kappa$ where $f = i \circ f$. Hence i_* is surjective. If $f \simeq g : \mathbb{S}^n \to \operatorname{SP} X$, there exists a homotopy $H : f \simeq g$, and similarly there exists $\mu \in \Lambda$ such that $H(\mathbb{S}^n \times I) \subset X_\mu$. Thus the homotopy restricts to a homotopy $H' : \mathbb{S}^n \times I \to \operatorname{SP} X_\mu$, which passing to the colimit induces a homotopy $H'' : \mathbb{S}^n \times I \to \operatorname{colim} \operatorname{SP} X_\mu$. So i_* is injective and an isomorphism, giving $\tilde{H}_n(X) = \pi_n(\operatorname{SP} X) \cong \pi_n(\operatorname{colim}_{\lambda\in\Lambda} \operatorname{SP} X_\lambda)$.
To pass the colimit across the homotopy functor, we now see that $\{\operatorname{SP} X_\lambda\}_{\lambda\in\Lambda}$ satisfies the condition for lemma 3.8. If $K \subset \operatorname{colim}_{\lambda\in\Lambda} \operatorname{SP} X_\lambda$ is compact, then $i(K) \subset \operatorname{SP} X$ is compact. There exists $\kappa \in \Lambda$ such that $i(K) \subset \operatorname{SP} X_\kappa$, hence $K \subset \operatorname{SP} X_\kappa$. So by lemma 3.8, $\tilde{H}_n(X) = \pi_n(\operatorname{SP} X) \cong \pi_n(\operatorname{colim}_{\lambda\in\Lambda} \operatorname{SP} X_\lambda) \cong \operatorname{colim}_{\lambda\in\Lambda} \tilde{H}(X_\lambda)$. ☐

Theorem 3.10. (Additivity) *If $X = \bigvee_{\lambda\in\Lambda} X_\lambda$ is a wedge of CW complexes, then the inclusions $X_\alpha \hookrightarrow X$ induce an isomorphism*

$$\tilde{H}_n(X) \to \bigoplus_{\lambda\in\Lambda} \tilde{H}(X_\lambda)$$

Proof. First we will consider the finite case, which follows by induction from considering a wedge of two spaces, and then we will consider the general case by passing to the colimit. Let X_1 and X_2 be pointed CW complexes. Consider the sequence

$$X_1 \underset{\rho_2}{\overset{i_1}{\rightleftarrows}} X_1 \bigvee X_2 \underset{i_2}{\overset{\rho_1}{\rightleftarrows}} X_2$$

where i is the inclusion map and ρ is the quotient map. Passing to homology, we consider

$$\tilde{H}_n(X_1) \underset{\rho_{2*}}{\overset{i_{1*}}{\rightleftarrows}} \tilde{H}_n(X_1 \bigvee X_2) \underset{i_{2*}}{\overset{\rho_{1*}}{\rightleftarrows}} \tilde{H}_n(X_2)$$

where we wish to prove that the sequence from left to right is a short exact sequence. We have exactness at $\tilde{H}_n(X_1 \bigvee X_2)$ by the exactness axiom, so all that remains is to prove that i_{1*} is a monomorphism and ρ_{1*} an epimorphism. Noting that $\rho_2 \circ i_1 = \mathrm{id}_{X_1}$, we have $\rho_{2*} \circ i_{1*} = \mathrm{id}_{\tilde{H}_n(X_1)}$, hence i_{1*} is injective and ρ_{1*} is surjective. Similarly, i_{2*} is injective and ρ_{1*} is surjective. So we have that the sequence is exact and splits. Applying the splitting lemma for the category of Abelian groups, we thus obtain $\tilde{H}_n(X_1 \bigvee X_2) \cong \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2)$. So by induction we have $\tilde{H}_n(X) \cong \bigoplus_{i=1}^r \tilde{H}(X_i)$ for all finite wedges. For the general case, consider a wedge $\bigvee_{\lambda \in \Lambda} X_\lambda$. Let $\mathcal{F}$ be the set of finite subsets of Λ. Then

$$\bigvee_{\lambda \in \Lambda} X_\lambda = \operatorname*{colim}_{\Gamma \in \mathcal{F}} \bigvee_{\gamma \in \Gamma} X_\gamma$$

and passing to the homology

$$\tilde{H}_n(\bigvee_{\lambda \in \Lambda} X_\lambda) = \tilde{H}_n(\operatorname*{colim}_{\Gamma \in \mathcal{F}} \bigvee_{\gamma \in \Gamma} X_\gamma) \cong \operatorname*{colim}_{\Gamma \in \mathcal{F}} \tilde{H}_n(\bigvee_{\gamma \in \Gamma} X_\gamma) \cong \operatorname*{colim}_{\Gamma \in \mathcal{F}} \bigoplus_{\gamma \in \Gamma} \tilde{H}_n(X_\gamma) = \bigoplus_{\lambda \in \Lambda} \tilde{H}(X_\lambda)$$

using theorem 3.9 above to pass the colimit into the category of Abelian groups. □

The given construction of $\tilde{H}_*$ as the homotopy groups of the infinite symmetric product thus satisfies the axioms for a reduced homology theory on CW complexes. These axioms in fact guarantee the uniqueness of this theory; that is, any two homology theories are equivalent up to natural isomorphism. From the basis of a reduced homology theory on CW complexes, an unreduced homology theory on pairs of CW complexes can be constructed, and both the reduced and unreduced homology theories can be extended from CW complexes to general topological spaces through the use of a CW approximation. For details on the uniqueness of homology theories and on CW approximation, see [4].

4. Moore spaces and Eilenberg-Mac Lane spaces

Eilenberg-Mac Lane spaces are an important type of space in algebraic topology characterized by having trivial homotopy groups for all but one homotopy group. Eilenberg-Mac Lane spaces can be used to define a cohomology theory, identifying the cohomology groups with coefficients in a give group with the homotopy classes of a space into Eilenberg-Mac Lane spaces for that group (see [1]). Further, by taking products of Eilenberg-Mac Lane spaces, one can construct a space with any given set of homotopy groups, assuming one can construct the needed Eilenberg-Mac Lane spaces. The infinite symmetric product, by converting homotopy groups into homology groups, can transform the construction of Eilenberg-Mac Lane spaces into the construction of Moore spaces, the homological analogy to Eilenberg-Mac Lane spaces.

Definition 4.1. Let G be an (Abelian) group. A connected pointed space X is a *Moore space of type* (G, n) if $\tilde{H}_n(X) \cong G$ and otherwise $\tilde{H}_i(X) \cong 0$ for $i \geq 1$.

Definition 4.2. Let G be an (Abelian) group. A connected pointed space X is an *Eilenberg-Mac Lane space of type* $K(G, n)$, or a $K(G, n)$, if $\pi_n(X) \cong G$ and otherwise $\pi_i(X) \cong 0$ for $i \geq 1$.

Theorem 4.3. $\mathrm{SP}\,\mathbb{S}^n$ *is a* $K(\mathbb{Z}, n)$*, and so* $\mathbb{S}^n$ *is a Moore space of type* $(\mathbb{Z}, n)$*.*

Proof. This is evident by combining the definitions of a Moore space and an Eilenberg-Mac Lane space with theorem 2.15. $\square$

Theorem 4.4. *$\mathbb{S}^n \cup_{\alpha_k^n} e^{n+1}$ is a Moore space of type $(\mathbb{Z}/k, n)$, where $\alpha_k^n : \mathbb{S}^n \to \mathbb{S}^n$ is the $(n-1)$-fold suspension of the monomial map of degree k.*

Proof. For all positive integers k and natural numbers n, define $\alpha_k^1 : \mathbb{S}^1 \to \mathbb{S}^1$ by $x \mapsto x^k$ and $\alpha_k^n : \mathbb{S}^n \to \mathbb{S}^n$ by $\alpha_k^n = \Sigma^{n-1}\alpha_k^1$. We can choose $\alpha_{k*}^1 = k \in \mathbb{Z}$, that is, α_k^1 induces multiplication by k (as opposed to $-k$). Using the homotopy equivalence from 2.15 we have that $\mathrm{SP}\,\alpha_k^1$ also induces multiplication by k, and applying the suspension axiom inductively, that $\mathrm{SP}\,\Sigma^{n-1}\alpha_k^1 = \mathrm{SP}\,\alpha_k^n$ induces multiplication by k.

Consider the diagram

$$\begin{array}{ccccccc}
\mathbb{S}^n & \xrightarrow{\alpha_k^n} & \mathbb{S}^n & \xhookrightarrow{i} & \mathbb{S}^n \cup_{\alpha_k^n} e^{n+1} & \xrightarrow{\rho} & \mathbb{S}^{n+1} \\
\cong\downarrow & & \simeq\downarrow & & \cong\downarrow & & \\
\mathbb{S}^n \times \{1\} & \xrightarrow{\alpha_k^n \times \{1\}} & M_{\alpha_k^n} & \xrightarrow{i'} & C_{\alpha_k^n} & &
\end{array}$$

which commutes up to homotopy, where i is the inclusion, and i' and ρ are the quotients. Passing part of this diagram to homology, for $q \neq n, n+1$ we have

$$0 \cong \tilde{H}_q(\mathbb{S}^n) \longrightarrow \tilde{H}_q(\mathbb{S}^n \cup_{\alpha_k^n} e^{n+1})) \longrightarrow \tilde{H}_q(\mathbb{S}^{n+1}) \cong 0$$

which is exact by the exactness axiom (3.5), hence $\tilde{H}_q(\mathbb{S}^n \cup_{\alpha_k^n} e^{n+1})) \cong 0$. For the remaining cases, we select from the homotopy sequence associated with the quasifibration $\mathrm{SP}\,i'$, expressing it as a homology sequence of i', to find that

$$\tilde{H}_{n+1}(M_{\alpha_k^n}) \longrightarrow \tilde{H}_{n+1}(C_{\alpha_k^n}) \longrightarrow \tilde{H}_n(\mathbb{S}^n \times \{1\}) \longrightarrow$$

$$\longrightarrow \tilde{H}_n(M_{\alpha_k^n}) \longrightarrow \tilde{H}_n(C_{\alpha_k^n}) \longrightarrow \tilde{H}_{n-1}(\mathbb{S}^n \times \{1\})$$

is exact, which using the homotopy equivalences from the original diagram shows that the sequence

$$0 \longrightarrow \tilde{H}_{n+1}(X) \longrightarrow \tilde{H}_n(\mathbb{S}^n) \xrightarrow{\tilde{H}\alpha_k^n} \tilde{H}_n(\mathbb{S}^n) \longrightarrow \tilde{H}_n(X) \longrightarrow 0$$

is exact, where $X = \mathbb{S}^n \cup_{\alpha_k^n} e^{n+1}$. Since $\tilde{H}\alpha_k^n = \mathrm{SP}\,\alpha_k^n$ is multiplication by k, we quickly see from the sequence that $\tilde{H}_n(\mathbb{S}^n \cup_{\alpha_k^n} e^{n+1}) \cong \mathbb{Z}/k$ and $\tilde{H}_{n+1}(\mathbb{S}^n \cup_{\alpha_k^n} e^{n+1}) \cong 0$. Hence $\mathbb{S}^n \cup_{\alpha_k^n} e^{n+1}$ is a Moore space of type $(\mathbb{Z}/k, n)$. $\square$

The following theorem from algebra will allow the extension of these results to any finitely generated Abelian group. For a proof, see theorems I.8.5 and III.7.7 in [3].

Theorem 4.5. *Any finitely generated Abelian group G can be written as a direct sum $G \cong \bigoplus_{i=1}^r \mathbb{Z} \oplus \bigoplus_{i=1}^l \mathbb{Z}/k_i$.*

Theorem 4.6. *If G is a finitely generated Abelian group factored as $G \cong \bigoplus_{i=1}^r \mathbb{Z} \oplus \bigoplus_{i=1}^l \mathbb{Z}/k_i$, then $(\bigvee_{i=1}^r \mathbb{S}^n) \bigvee (\bigvee_{i=1}^l (\mathbb{S}^n \cup_{\alpha_{k_i}^n} e^{n+1}))$ is a Moore space of type (G, n).*

Proof. Using the additivity axiom (3.10), we have that

$$\tilde{H}_n\left(\left(\bigvee_{i=1}^{r}\mathbb{S}^n\right)\bigvee\left(\bigvee_{i=1}^{l}\left(\mathbb{S}^n\cup_{\alpha_{k_i}^n}e^{n+1}\right)\right)\right)\cong\bigoplus_{i=1}^{r}\mathbb{Z}\oplus\bigoplus_{i=1}^{l}\mathbb{Z}/k_i\cong G$$

and for $q\neq n$,

$$\tilde{H}_q\left(\left(\bigvee_{i=1}^{r}\mathbb{S}^n\right)\bigvee\left(\bigvee_{i=1}^{l}\left(\mathbb{S}^n\cup_{\alpha_{k_i}^n}e^{n+1}\right)\right)\right)\cong\bigoplus_{i=1}^{r}0\oplus\bigoplus_{i=1}^{l}0\cong 0$$

which are the conditions to be a Moore space of type (G, n). □

Having constructed Moore spaces which are connected CW complexes, we immediately have Eilenberg-Mac Lane spaces by applying the infinite symmetric product.

Theorem 4.7. *If G is a finitely generated abelian group factored as $G\cong\bigoplus_{i=1}^{r}\mathbb{Z}\oplus\bigoplus_{i=1}^{l}\mathbb{Z}/k_i$, then $\mathrm{SP}((\bigvee_{i=1}^{r}\mathbb{S}^n)\bigvee(\bigvee_{i=1}^{l}(\mathbb{S}^n\cup_{\alpha_{k_i}^n}e^{n+1})))$ is a $K(G,n)$.*

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