ON MINIMAL FINITE MODELS

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Abstract. A minimal finite model of a topological space is the smallest finite topological space that is weakly homotopic to that particular space. So far, we have only found literatures on the minimal finite models for spheres and finite graphs, and they are concluded in Barmak and Minian's paper [1]. In this paper, we will take a look at these minimal finite models and push our scope a bit further to more complicated spaces, giving a computational method to find such minimal finite models for these spaces.

Contents

1. Preliminaries

Before we plunge into the exploration of minimal finite models, we want to first reveal the connection between finite partially ordered sets and finite topological spaces, which will provide us some tools to investigate our problem.

The reason that we restricted to finite topological spaces here is because we are going to make use of the Alexandroff property, which states that open sets are closed under arbitrary intersection. (In fact, this connection between partially ordered sets and topological spaces remains valid for infinite Alexandroff spaces.) Since any intersection in a finite space is finite, it follows that all finite topological spaces have the Alexandroff property, which allows us to define the following:

Definition 1.1. In a finite topological space $(X, \mathscr{U})$, the *open hull* U_x of a point x is the intersection of all open subsets containing x.

In other words, U_x is the "smallest" open set in X that contains x. Now we want to define a relation $\leq$ by letting $x \leq y$ if $x \in U_y$. We claim that this gives rise to a preorder structure on the same underlying set:

Proposition 1.2. *The relation $\leq$ is a preorder defined on X. In particular, if X is T_0, then $\leq$ is in fact a partial order.*

Date: August 30th, 2010.

Proof. The relation $\leq$ is obviously reflexive. For transitivity, suppose $x \leq y$ and $y \leq z$, then by definition, $x \in U_y$ and $y \in U_z$. But then since U_z is an open subset that contains y, $U_y \subset U_z$, which implies that $x \in U_z$. Thus $x \leq z$.

Now suppose X is T_0. If there is $x \leq y$ and $y \leq x$, then $x \in U_y$ and $y \in U_x$, i.e. any open set containing x also contains y and vice versa. Therefore, x and y cannot be distinguished by an open neighborhood and hence they are the same point in a T_0 space. □

This proposition asserts that any finite topological space corresponds to a finite preordered set with the same underlying subset. In the reverse direction, if we start with a finite preordered set $(X, \leq)$, we only need to define U_x to be the set of all y such that $y \leq x$, and it is not hard to verify that this collection of $\{U_x\}_{x \in X}$ is a basis for a topology on X. Particularly, the resulting topology will be T_0 if we start with a partially ordered set.

In fact, these two correspondences are inverses of each other. As we shall conclude after the following proposition, the category of finite topological spaces and the category finite preordered sets are actually isomorphic via this bijective correspondence.

Proposition 1.3. *Let X and Y be finite topological spaces with the corresponding preorder defined on the same underlying sets. A map $f : X \to Y$ is continuous if and only if it is order preserving: $x \leq y$ in X implies $f(x) \leq f(y)$ in Y.*

Proof. First suppose f is continuous. If $x_1 \leq x_2$ in X, then $x_1 \in U_{x_2} \subset f^{-1}(U_{f(x_2)})$, which implies $f(x_1) \in U_{f(x_2)}$, i.e. $f(x_1) \leq f(x_2)$.

Now suppose f is order preserving. Let V be an open set in Y containing $f(x_1)$, then by construction we know that $U_{f(x_1)} \subset V$. For any $x \in U_{x_1}$, we have $x \leq x_1$, and hence $f(x) \leq f(x_1)$. But this implies that $f(x) \in U_{f(x_1)} \subset V$, and thus $U_{x_1} \in f^{-1}(V)$, i.e. $f^{-1}(V)$ is open. Therefore, f is a continuous map. □

As May [2] points out, when we are studying problems in finite topological spaces up to homotopy type, there is no loss of generality to just consider the T_0 spaces and their corresponding partially ordered sets, since we can always identify a point with its open hull together, and the quotient map is a homotopy equivalence. Furthermore, if X and Y are finite topological space and we let X_0 and Y_0 be the quotient T_0 spaces with quotient maps q_X and q_Y, respectively, then for any continuous map $f : X \to Y$, it naturally induces a unique continuous map $f_0 : X_0 \to Y_0$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q_Y} \\ X_0 & \xrightarrow{f_0} & Y_0 \end{array}$$

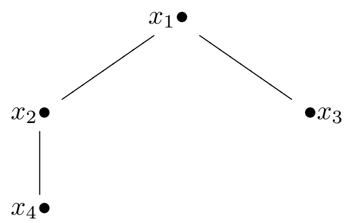
Unless otherwise specified, for the rest of this paper, finite spaces are assumed to be T_0.

The reason that we want this correspondence between finite *posets* (partially ordered sets) and finite T_0 spaces is because the poset structure can often give a more direct picture and provide some useful tools such as the following:

Definition 1.4. The *Hasse diagram* of a finite poset $(X, \leq)$ is the directed graph $H(X) = (V(H(X)), E(H(X))$ whose vertices are points in X. In the diagream, (x, y) is an edge if $x < y$ and there is no z in X such that $x < z < y$.

Normally when we draw the Hasse diagram, we want to put y above x to indicate $y > x$. The following is one example of a Hasse diagram:

Example 1.5.

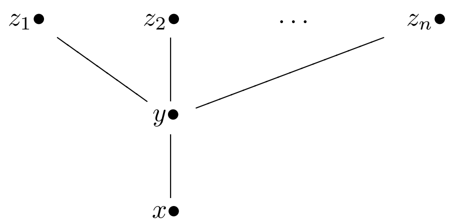


If we define the *opposite ordering* $\leq^{op}$ on a poset $(X, \leq)$ to be the new relation that $x \leq^{op} y$ if and only if $y \leq x$, then this new ordering is a valid partial order. The Hasse diagram for this opposite ordering is exactly the upside down of the original Hasse diagram.

The next thing that we are going to investigate is the notion of a "beat point" proposed by Stong [3].

Definition 1.6. Let X be a finite space. A point $x \in X$ is *upbeat* if there is a $y > x$ such that $z > x$ implies $z \geq y$. A point $x \in X$ if *downbeat* if there is a $y < x$ such that $z < x$ implies $z \leq y$. A finite poset with no upbeat or downbeat points is called a *minimal finite space*. For any finite poset X, a *core* of X is a subspace Y that is a minimal finite space and a deformation retract of X. That is, there exists a map $r : X \to Y$ together with the inclusion map $i : Y \to X$ such that $r \circ i = \text{id}$ and $i \circ r$ is homotopic to the identity map.

The following is a part of a Hasse diagram that contains an upbeat point x, and a diagram for a downbeat point will be similar to this one but upside down.



On one hand, May [2] shows that the map that removes an upbeat point (or equivalently, a downbeat point) is a deformation retract. Therefore, one can always reach the core of a finite poset by removing upbeat and downbeat points one by one. This shows that any finite poset always has a core. In particular, if a poset has a maximum or a minimum, then we can always remove the rest of the points one by one until there is only the maximum or the minimum left, and this shows the following:

Corollary 1.7. *A poset with a maximum or a minimum is contractible.*

On the other hand, he also shows that for any minimal finite space X, the only map that is homotopic to the identity map is the identity map itself. Therefore, we have the following:

Theorem 1.8. *Two finite spaces are homotopic to each other if and only if their cores are homeomorphic to each other.*

Proof. It is easy to see that if two finite spaces have homeomorphic cores then they are homotopic. Now suppose that the two finite spaces are homotopic, then by the definition of core, their cores are also homotopic. Let X and Y be the cores and $f : X \to Y$ be a homotopy equivalence with inverse $g : Y \to X$, then $g \circ f$ is homotopic to the identity map on X and $f \circ g$ is homotopic to the identity map on Y. But this implies $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Thus their cores are homeomorphic. □

However, when we are studying minimal finite models, what we actually care about is weak homotopy equivalence rather than homotopy equivalence. The definition of weak homotopy equivalence is the following:

Definition 1.9. Let (X, x_0) and (Y, y_0) be two based topological spaces. A map $f : X \to Y$ is said to be a *weak homotopy equivalence* if the induced map

$$f_i : \pi_i(X, x_0) \to \pi_i(Y, y_0)$$

is an isomorphism for all $i \in \mathbb{N}$. Two topological spaces X and Y are said to be *weak homotopy equivalent* if there is a finite sequence of spaces

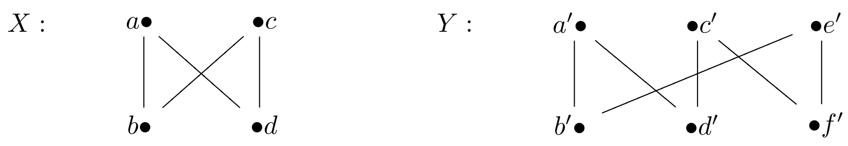
$$X = X_0, X_1, X_2, \ldots, X_n = Y$$

such that there exist weak homotopy equivalences $X_{j-1} \to X_j$ or $X_j \to X_{j-1}$ for each $1 \le j \le n$.

This gives rise to our notion of minimal finite model:

Definition 1.10. A finite topological sapce Y is a *finite model* of a topological space X if Y is weak homotopy equivalent to X. Y is a *minimal finite model* if The space Y is a finite model with the minimum cardinality.

Example 1.11. Consider the following example of finite models of circle. Let X and Y be finite spaces whose Hasse diagrams are given as below:



Take the unit circle in the complex plane. Let $f : S^1 \to X$ and $g : S^1 \to Y$ be given by

$$f(x) = \begin{cases} a & \text{if } x = 1 \\ b & \text{if } x = e^{i\theta}, 0 < \theta < \pi \\ c & \text{if } x = -1 \\ d & \text{if } x = e^{i\theta}, \pi < \theta < 2\pi \end{cases} \qquad g(x) = \begin{cases} a' & \text{if } x = 1 \\ b' & \text{if } x = e^{i\theta}, 0 < \theta < 2\pi/3 \\ c' & \text{if } x = e^{2\pi i/3} \\ d' & \text{if } x = e^{i\theta}, 2\pi/3 < \theta < 4\pi/3 \\ e' & \text{if } x = e^{4\pi i/3} \\ f' & \text{if } x = e^{i\theta}, 4\pi/3 < \theta < 2\pi. \end{cases}$$

One can verify that both of these are weak homotopy equivalences. Therefore, both X and Y are finite models of the circle. In fact, as we shall show later, X is the unique minimal finite model of the circle. Notice that even though X and Y are weak homotopy equivalent to each other, they have different homotopy type since they are both minimal spaces and not homeomorphic.

Next we want to introduce two very useful functors first defined by McCord [4].

Definition 1.12. Let $\mathscr{PO}$ be the category of finite partially ordered sets and $\mathscr{SC}$ be the category of finite simplicial complices. McCord defined the following:

The $\mathcal{K}$ functor: The $\mathcal{K}$ functor goes from $\mathscr{PO}$ to $\mathscr{SC}$. For a poset $(X, \leq)$, $\mathcal{K}(X)$ is a simplicial complex with an n-subsimplex $\{x_0, x_1, \ldots, x_n\}$ for each totally ordered subset $x_0 \leq x_1 \leq \cdots \leq x_n$ of X, i.e. the vertices of $\mathcal{K}(X)$ are in fact the elements in the poset X. It is not hard to see that $\mathcal{K}$ also takes an order preserving map f between posets X and Y to a map $\mathcal{K}(f)$ of simplicial complices, simply by taking the vertices of $\mathcal{K}(X)$ to the vertices of $\mathcal{K}(Y)$ under f, and since f is order preserving, $\mathcal{K}(f)$ takes subsimplex to subsimplex and is well defined.

The $\mathcal{X}$ functor: The $\mathcal{X}$ functor goes from $\mathscr{SC}$ to $\mathscr{PO}$. For a simplicial complex K, what $\mathcal{X}$ does is to assign an element for each subsimplex of K, and to define a partial ordering on the set of these elements by setting $\sigma \leq \tau$ if $\sigma \subset \tau$ as a subsimplex. One can easily verify that this partial ordering is well defined, and $\mathcal{X}$ takes a map g of simplical complices to a order preserving map $\mathcal{X}(g)$ of posets.

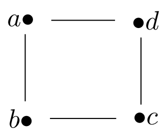
McCord also showed the following theorem, which we are going to take for granted and omit the proof here:

Theorem 1.13. *For any finite poset X, there exists a weak homotopy equivalence $\psi : |\mathcal{K}(X)| \to X$, and for any finite simplicial complex K, there exists a weak homotopy equivalence $\phi : |K| \to \mathcal{X}(K)$.*

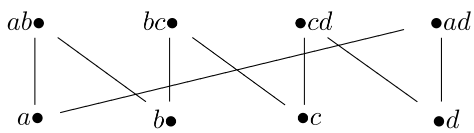
Also, Whitehead's theorem tells us that if X is a minimal finite model of a compact polyhedron Y, then Y must be homotopy equivalent to $|\mathcal{K}(X)|$. Therefore, to find a minimal finite model is essentially to find the poset X with the least elements such that $|\mathcal{K}(X)|$ is homotopic to the object that we are interested in.

The following example will demonstrate how these two functors work:

Example 1.14. Take the space X from Example 1.11. $\mathcal{K}(X)$ will be a simplicial complex with four vertices a, b, c and d with four edges $\{a, b\}$, $\{b, c\}$, $\{c, d\}$ and $\{a, d\}$ representing the four totally ordered subsets. As we can see, the resulting simplicial complex $\mathcal{K}(X)$ has a geometric realization that is homotopic to a circle.



Now if we take this "4-point circle" as a simplicial complex and apply $\mathcal{X}$ to it, then we need four new elements for the four edges, say they are ab, bc, cd, ad, then the resulting poset, represented by its Hasse diagram, would be the following:



Remark 1.15. Notice that if a topological space can be realized as a finite simplicial complex, then a finite model of this space always exists by an application of the $\mathcal{X}$ functor, and this finite model will give an upper bound on the size of a minimal finite model.

2. Minimal Finite Models of Spheres

In this section, we are going to show that the unique minimal finite model of an n-sphere is a poset with $2n+2$ points. First we want to define the non-Hausdorff suspension:

Definition 2.1. The *non-Hausdorff suspension* of a finite space X, denoted by $\mathbb{S}X$, is the finite space $X \cup \{+,-\}$, whose open sets are those of X originally together with $X \cup \{+\}$, $X \cup \{-\}$ and $X \cup \{+,-\}$. The *nth non-Hausdorff suspension* is $\mathbb{S}^n X = \mathbb{S}\left(\mathbb{S}^{n-1}X\right)$.

In the language of posets, the non-Hausdorff suspension simply adds two elements on top of the Hasse diagram and draws edges between the new elements and the original maximal points.

When looking for a minimal finite model, we always care about the size of the posets. Since a minimal finite model cannot have any upbeat or downbeat points, any point in the Hasse diagram of a minimal finite model should have at least two points that are greater than it (unless it is a maximal point) and two points that are less than it (unless it is a minimal point). Therefore, it seems that the maximum size of a totally ordered subset would give rise to a good approximation for the lower bound of the size of a minimal finite model. Let us call the maximum size of a totally ordered subset the *height* of the poset X, denoted by $h(X)$, then we have the following theorem:

Theorem 2.2. *If X is a minimal finite space with more than one point, then X has at least $2h(X)$ points. In particular, if X has exactly $2h(X)$ points, then it is homeomorphic to $\mathbb{S}^{h(X)-1}S^0$, where $S^0 = \{+,-\}$, the zero dimensional sphere.*

Proof. Let $h(X) = n$ and let $x_1 \leq x_2 \leq \cdots \leq x_n$ be a totally ordered subset with n elements. Since none of these elements is an upbeat point, there must be a y_{i+1} for each x_i $(1 \leq i \leq n-1)$ such that $y_{i+1} > x_i$ but $y_{i+1} \ngeq x_{i+1}$.

We claim that all the y_i's are distinct from the x_i's and distinct from each other. Note that $y_i \neq x_j$ for $j \leq i$ by construction. If $y_i = x_j$ for $j > i$, then $y_i = x_j > x_i$, a contradiction. Now suppose $y_i = y_j$ $(i < j)$, then $y_i = y_j > x_{j-1} \geq x_i$, a contradiction. Therefore, the y_i and x_i are distinct.

Also, since X is a minimal finite space, X cannot have a minimum. Thus we need one more point y_1 and $y_1 \ngeq x_1$. Note that all the x_i's and y_i's that we put in before are greater or equal to x_1, hence y_1 should be distinct from all of them. Therefore, now we have the lower bound of $2n$ points.

Now suppose we have a minimal finite space X with $2n$ points, we want to show that it is homeomorphic to the $(n-1)$th non-Hausdorff suspension.

Since $x_1 \leq x_2 \leq \cdots \leq x_n$ is already the biggest totally ordered subset of X, y_i and x_i should be incomparable for all i, otherwise we could extend the totally ordered subset. We are going to show that $y_i < x_j$ and $y_i < y_j$ for all $i < j$ by induction on j.

For the base case $j = 1$, and since there is no i smaller than 1, the claim is true. Suppose now the claim is true for $j \leq k < n$. Then when $j = k+1$, since x_{k+1} is not a downbeat point, there must be a point $z < x_{k+1}$ but $z \nleq x_k$. Now we need to choose an element from X to be such a z. We can choose this element by eliminating the impossible ones:

By our inductive hypothesis, $y_i < x_k$ for all $i < k$.

(1) $x_i \leq x_k$ for $i \leq k$, and $x_i \geq x_{k+1}$ for $i \geq k+1$.
(2) $y_i > x_{i-1} \geq x_{k+1}$ for $i \geq k+2$.
(3) y_{k+1} is incomparable with x_{k+1}.

Therefore, the only possible candidate is y_k, and this forces $y_k < x_{k+1}$. By the inductive hypothesis, $y_i \leq y_k < x_{k+1}$ for all $i \leq k$.

Also, since y_{k+1} is not a downbeat point, there must be a point $w < y_{k+1}$ but $w \nleq x_k$. We again choose w by elimination.

(1) By our inductive hypothesis, $y_i < x_k$ for all $i < k$.
(2) $x_i \leq x_k$ for $i \leq k$.
(3) y_{k+1} is not greater than x_i for $i \geq k+1$, otherwise there will be $y_{k+1} > x_i \geq x_{k+1}$, contradicting the fact that y_{k+1} and x_{k+1} are incomparable.
(4) y_{k+1} is not greater than y_i for $i \geq k+2$, otherwise there will be $y_{k+1} > y_i > x_{i-1} \geq x_{k+1}$, the same contradiction.

Therefore, the only possible candidate is y_k, and this forces $y_k < y_{k+1}$. By the inductive hypothesis, $y_i \leq y_k < y_{k+1}$ for all $i \leq k$.

Now we have all the information about the ordering on X, namely: for $1 \leq i < j \leq n$, there are $x_i < x_j$, $x_i < y_j$, $y_i < x_j$, $y_i < y_j$, and x_i is incomparable with y_i. This ordering is exactly the ordering on $\mathbb{S}^{n-1}S^0$, and hence X is homeomorphic to $\mathbb{S}^{n-1}S^0$. □

Now we are ready to show our goal of this section, i.e. the unique minimal finite model of S^n.

Theorem 2.3. *The n-sphere S^n has a unique minimal finite model with $2n+2$ points.*

Proof. Since S^0 is itself a finite space and is not contractible, it is its own unique minimal finite model. For $n \geq 1$, suppose X is a minimal finite model of S^n, then by the definition of minimal finite model, we know that the homotopy group $\pi_i(X, x_0) \cong \pi_i(S^n, s_0)$ for all $i \in \mathbb{N}$. In particular, $\pi_n(X, x_0) \cong \pi_n(S^n, x_0) \ncong 0$.

Now by the Hurewicz Theorem, $H_n(|\mathcal{K}(X)|) \cong \pi_n(|\mathcal{K}(X)|, x_0) \cong \pi_n(X, x_0) \ncong 0$. Therefore, the dimension of $\mathcal{K}(X)$ must be at least n and the height of X must be at least $n+1$. Also notice that S^n is not contractible and hence X must have more than one point. By the previous theorem, X has at least $2n+2$ points. But notice that $|\mathcal{K}(\mathbb{S}^n S^0)|$ is in fact homotopy equivalent to S^n, thus we have a minimal finite model with $2n+2$ points. Furthermore, since any minimal finite space with $2n+2$ points is homeomorphic to $\mathbb{S}^n S^0$, this proves the uniqueness of this minimal finite model. □

3. Hasse Diagram and Fundamental Group

One good use of the Hasse diagram and the associated finite poset is, as we shall show, to compute the fundamental group of a finite space. The so-called H-loop group $\mathscr{H}(X, x_0)$ in the Hasse diagram of a given based poset (X, x_0), as we are about to define, turns out to be isomorphic to the fundamental group $\pi_1(X, x_0)$. In this section, we are going to fully present all these materials, originated from Barmak and Minian [1].

First let us start with the definition of the H-loop group.

Definitions 3.1. For any poset X with a base point x_0, let $H(X)$ be the associated Hasse diagram. We call an ordered pair $e = (x, y)$ an *H-edge* if $(x, y) \in E(H(X))$

or $(y, x) \in E(H(X))$. The point x is called the *origin* of e, denoted by $o(e)$ and the point y is called the *end* of e, denoted by $e(e)$. The *inverse* of an H-edge $e = (x, y)$ is the H-edge $e^{-1} = (y, x)$.

If we have a sequence of H-edges $e_1, e_2, \dots, e_n$ with $e(e_i) = o(e_{i+1})$ for all $1 \leq i \leq n-1$, we can connect them together to get an *H-path* $\xi = e_1 e_2 \dots e_n$. Typically we say the *origin* of this H-path is $o(\xi) = o(e_1)$ and the *end* of this H-path is $e(\xi) = e(e_n)$. The *inverse* of an H-path $\xi = e_1 e_2 \dots e_n$ is the H-path $\xi^{-1} = e_n^{-1} e_{n-1}^{-1} \dots e_1^{-1}$.

An H-path $\xi = e_1 e_2 \dots e_n$ is said to be *monotonic* if either $e_i \in E(H(X))$ for all $1 \leq i \leq n$ or $e_i^{-1} \in E(H(X))$ for all $1 \leq i \leq n$.

For two H-paths $\xi_1 = e_1 e_2 \dots e_n$ and $\xi_2 = f_1 f_2 \dots f_m$ with $e(\xi_1) = o(\xi_2)$, it makes sense to define a *composition* of ξ_1 and ξ_2:

$$\xi_1 \xi_2 = e_1 e_2 \dots e_n f_1 f_2 \dots f_m.$$

An *H-loop at x_0* is an H-path ξ such that $o(\xi) = e(\xi) = x_0$.

Two H-loops ξ and ξ' at x_0 are said to be *close* if there exist four H-paths ξ_1, ξ_2, ξ_3 and ξ_4 with ξ_2 and ξ_3 being monotonic, such that $\xi = \xi_1 \xi_4$ and $\xi' = \xi_1 \xi_2 \xi_3 \xi_4$. Denote this close relation by $\xi \simeq \xi'$.

Two H-loops ξ and ξ' at x_0 are said to be *H-equivalent* if there exists a sequence of loops at x_0, $\xi = \xi_0, \xi_1, \xi_2, \dots \xi_n = \xi'$ such that $\xi_{i-1} \simeq \xi_i$ for each $1 \leq i \leq n$.

It is not hard to verify that H-equivalence is actually an equivalence relation. Therefore, we obtain the equivalence classes for H-loops at x_0. Let us denote the equivalence class of the H-loop ξ by $\langle \xi \rangle$ and collect all the equivalence classes into the set $\mathscr{H}(X, x_0)$. Similar to the way we handle the idea of fundamental group, we can define a product on these equivalence classes by taking $\langle \xi_1 \rangle \langle \xi_2 \rangle = \langle \xi_1 \xi_2 \rangle$. It is not hard to show that this product is well defined. This gives a group structure on the set $\mathscr{H}(X, x_0)$, which is called the *H-loop group*.

When we apply the functor $\mathcal{K}$ to the finite poset X, we obtain a simplicial complex $\mathcal{K}(X)$, and there is another special kind of group called the edge-path group. Next we are going to define the edge-path group of $\mathcal{K}(X)$ and show that it is actually isomorphic to the H-loop group of the space (X, x_0).

Definition 3.2. For a simplicial complex K, an *edge-path* ξ is a finite sequence of vertices $v_0 v_1 v_2 \dots v_n$ such that either $\{v_{i-1}, v_i\}$ is an edge (1-dimensional subsimplex) of K or $v_{i-1} = v_i$. If we write the ordered pair $(v_{i-1}, v_i) = \epsilon_i$, then an edge-path can be written as $\xi = \epsilon_1 \epsilon_2 \dots \epsilon_n$. An *edge-loop* ξ at a vertex v is an edge-path such that $v_0 = v_n = v$. In particular, we set the *zero edge-loop* to be v.

The reason that we use ϵ instead of e to represent an edge here is because an edge in $\mathcal{K}(X)$ may not correspond to an H-edge in X. In fact, for an edge $\epsilon = (x, y)$ in $\mathcal{K}(X)$, x is comparable to y, and we can always find $x_1, x_2, \dots, x_n$ such that $(x, x_1)(x_1, x_2) \dots (x_n, y)$ is a monotonic H-path. Conversely, an H-edge in X always corresponds to an edge in $\mathcal{K}(X)$.

Definition 3.3. Two edge-loops at v are said to be *equivalent* if one can be obtained from the other by a series of the following move: for any $\{x, y, z\}$ that is a subset of a triangle (2-dimensional simplex), it is allowed to switch the edge (x, y) with two consecutive edges $(x, z)(z, y)$ (note that x, y and z need not be distinct). Denote this equivalence relation by $\approx$.

Notice that for any edge-loop at v, the start and end point v should never be changed under the move discribed above, and thus the move does not change the nature of being an edge-loop at v.

In fact, one can verify that the definition above gives an equivalence relation. If we denote the equivalence class of $\xi = \epsilon_1\epsilon_2 \dots \epsilon_n$ to be $[\epsilon_1\epsilon_2 \dots \epsilon_n]$ and put in the composition operation, this gives a group with the identity being the zero edge-loop. This group is called the *edge-path group*, and it is denoted by $E(K, v)$.

Remark 3.4. One basic fact from algebraic topology is that the edge-path group $E(K, v)$ of a simplicial complex K is isomorphic to the fundamental group $\pi_1(|K|, v)$ of the geometric realization of K. A proof of this can be found in Spanier [5].

Now we are ready to prove the following:

Theorem 3.5. *If (X, x_0) is a finite poset, then the edge-path group $E(\mathcal{K}(X), x_0)$ of $\mathcal{K}(X)$ is isomorphic to the H-loop group $\mathscr{H}(X, x_0)$.*

Proof. Define the map

$$\begin{aligned} \phi : \mathscr{H}(X, x_0) &\to E(\mathcal{K}(X), x_0) \\ \langle e_1e_2 \dots e_n\rangle &\mapsto [e_1e_2 \dots e_n]. \end{aligned}$$

We first want to show this map is well defined. Suppose $\xi_1\xi_2\xi_3\xi_4 \simeq \xi_1\xi_4$ as H-loops at x_0, where $\xi_2 = e_1e_2 \dots e_n$ and $\xi_3 = f_1f_2 \dots f_m$ are monotonic. Then without loss of generality, we can assume that ξ_2 is monotonically increasing, and thus

$$o(e_1) < e(e_1) = o(e_2) < e(e_2) = o(e_3) < \cdots < e(e_n),$$

which means that any three consecutive vertices are within one triangle. Therefore, by induction on the subscript i of e_i,

$$\begin{aligned} [\xi_1\xi_2\xi_3\xi_4] &= [\xi_1(o(e_1), e(e_1))(o(e_2), e(e_2)) \dots (o(e_n), e(e_n))\xi_3\xi_4] \\ &= [\xi_1(o(e_1), e(e_2)) \dots (o(e_n), e(e_n))\xi_3\xi_4] \\ &= [\xi_1(o(e_1), e(e_n))\xi_3\xi_4]. \end{aligned}$$

Similarly, we can replace ξ_3 with $(o(f_1), e(f_m))$ to get

$$[\xi_1(o(e_1), e(e_n))\xi_3\xi_4] = [\xi_1(o(e_1), e(e_n))(o(f_1), e(f_m))\xi_4].$$

But by the definition of close H-loops, we know that $o(e_1) = e(f_m)$ and $e(e_n) = o(f_1)$. Therefore, we can replace the middle part by the zero edge-path and get $[\xi_1\xi_2\xi_3\xi_4] = [\xi_1\xi_4]$.

Note that the map ϕ is a homomorphism by construction.

In the reverse direction, we can define another map

$$\begin{aligned} \psi : E(\mathcal{K}(X), x_0) &\to \mathscr{H}(X, x_0) \\ [\epsilon_1\epsilon_2 \dots \epsilon_n] &\mapsto \langle \xi_1\xi_2 \dots \xi_n\rangle, \end{aligned}$$

where each ξ_i is a monotonic H-path sharing the same origin point and end point with the edge ϵ_i. This in fact does not depend on the choice of the monotonic H-paths, because if ξ_i and ξ_i' are two possible choices, then

$$\begin{aligned} \xi_1\xi_2 \dots \xi_i \dots \xi_n &\simeq \xi_1\xi_2 \dots \xi_i\xi_i^{-1}\xi_i' \dots \xi_n \\ &\simeq \xi_1\xi_2 \dots \xi_i' \dots \xi_n. \end{aligned}$$

To show this map ψ is well defined, it is enough to show the move in Definition 3.3 for equivalent edge-loops does not change the image. Suppose

$$\epsilon_1\epsilon_2\ldots(x,z)(z,y)\ldots\epsilon_n \approx \epsilon_1\epsilon_2\ldots(x,y)\ldots\epsilon_n.$$

Let α, β, γ be the three monotonic paths corresponding to (x,z), (z,y) and (x,y), respectively and let ξ_i correspond to ϵ_i for the other i as usual. Since x, y and z are the three vertices of a triangle, without loss of generality, we can assume either $x < z < y$ or $x < y < z$.

If $x < z < y$, then $\alpha\beta$ is also monotonic, and

$$\begin{aligned}\xi_1\xi_2\ldots\alpha\beta\ldots\xi_n &\simeq \xi_1\xi_2\ldots\alpha\beta\beta^{-1}\alpha^{-1}\gamma\ldots\xi_n\\ &\simeq \xi_1\xi_2\ldots\gamma\ldots\xi_n.\end{aligned}$$

If $x < y < z$, then $\gamma\beta^{-1}$ is also monotonic, and

$$\begin{aligned}\xi_1\xi_2\ldots\alpha\beta\ldots\xi_n &\simeq \xi_1\xi_2\ldots\alpha\alpha^{-1}\gamma\beta^{-1}\beta\ldots\xi_n\\ &\simeq \xi_1\xi_2\ldots\gamma\ldots\xi_n.\end{aligned}$$

Thus ψ is well defined and it is a homomorphism by construction. Now we claim that ϕ and ψ are inverses of each other.

Pick any H-loop class $\langle\xi\rangle = \langle e_1e_2\ldots e_n\rangle$ in $\mathscr{H}(X,x_0)$ and apply $\psi\circ\phi$, we get

$$\psi\circ\phi(\langle\xi\rangle) = \psi([e_1e_2\ldots e_n]) = \langle e_1e_2\ldots e_n\rangle.$$

Now pick any edge-loop class $[\xi] = [\epsilon_1\epsilon_2\ldots\epsilon_n]$ in $E(\mathcal{K}(X),x_0)$ and apply $\phi\circ\psi$, we get

$$\phi\circ\psi([\xi]) = \phi(\langle\xi_1\xi_2\ldots\xi_n\rangle),$$

where each $\xi_i = e_{i,1}e_{i,2}\ldots e_{i,n_i}$ is a monotonic H-path that corresponds to the edge ϵ_i. But as we showed above,

$$\begin{aligned}\phi(\langle\xi_1\xi_2\ldots\xi_i\ldots\xi_n\rangle) &= [\xi_1\xi_2\ldots\xi_i\ldots\xi_n]\\ &= [\xi_1\xi_2\ldots(o(e_{i,1}),e(e_{i,n_i}))\ldots\xi_n]\\ &= [\xi_1\xi_2\ldots\epsilon_i\ldots\xi_n]\\ &= [\epsilon_1\epsilon_2\ldots\epsilon_n].\end{aligned}$$

Since $\psi\circ\phi$ is the identity on $\mathscr{H}(X,x_0)$ and $\phi\circ\psi$ is the identity on $E(\mathcal{K}(X),x_0)$, we conclude that $\mathscr{H}(X,x_0)$ is isomorphic to $E(\mathcal{K}(X),x_0)$. □

As we mentioned before, the edge-path group of a simplicial complex (K,v) is isomorphic to the fundamental group of its geometric realization. Therefore, we have the following:

Corollary 3.6. *For a finite poset (X,x_0), the following groups are isomorphic:*

(1) $\mathscr{H}(X,x_0)$;
(2) $E(\mathcal{K}(X),x_0)$;
(3) $\pi_1(|\mathcal{K}(X)|,x_0)$;
(4) $\pi_1(X,x_0)$.

Remark 3.7. The H-loop group in the Hasse diagram provides a way to compute the fundamental group of a topological space by just looking at its minimal finite model. As we know, a minimal finite model is weak homotopy equivalent to the original space, and hence all the information of every homotopy group is carried by its minimal finite model. However, it is not known yet whether there is an efficient way to extract the information of higher homotopy groups just from a minimal finite model.

4. Minimal Finite Models of Finite Graphs

Finite graphs are another class of geometric objects whose minimal finite models have been completely understood. One important fact about finite graphs that makes it easier to find their minimal models is that a finite graph is a 1-dimensional CW complex, i.e. a wedge sum of circles. Therefore, the weak homotopy type of a finite graph is determined by its Euler characteristic, and from this we can work out a way to compute minimal finite models of finite graphs.

Before we go into the actual argument, we would like to study the Hasse diagram a little further. As we know from the previous section, the edge-path group of $(\mathcal{K}(X), x_0)$ is isomorphic to the fundamental group of (X, x_0). But what is more about the Hasse diagram is that it can provide another way of looking at the fundamental group in terms of generators and relations. Here we first want to show how to get the generators.

Proposition 4.1. *Let (X, x_0) be a poset. If $x \in X$ is neither maximal nor minimal and $x \neq x_0$, then the inclusion $i : X - \{x\} \to X$ induces an epimorphism*

$$i_* : E(\mathcal{K}(X - \{x\}), x_0) \to E(\mathcal{K}(X), x_0).$$

Proof. Since every edge-loop at x_0 in $X - \{x\}$ has a natural image as an edge-loop in X under inclusion, i_* is natually a homomorphism. Therefore, to show i_* is an epimorphism, it is sufficient to check that every edge-loop in $\mathcal{K}(X)$ that goes through x is equivalent to an edge-loop that does not go through x.

Suppose $\epsilon_1\epsilon_2 \dots (y, x)(x, z) \dots \epsilon_n$ is an edge-loop. Then without loss of generality, we can assume either $y \leq x \leq z$ or $y \leq x$ and $z \leq x$.

If $y \leq x \leq z$, then $\{x, y, z\}$ is within a triangle and therefore we can apply the move from Definition 3.3 and deduce that

$$\epsilon_1\epsilon_2 \dots (y, x)(x, z) \dots \epsilon_n \approx \epsilon_1\epsilon_2 \dots (y, z) \dots \epsilon_n.$$

If $y \leq x$ and $z \leq x$, then since x is not maximal, we can find $w \in X - \{x\}$ such that $w > x$. Then

$$\begin{aligned} \epsilon_1\epsilon_2 \dots (y, x)(x, z) \dots \epsilon_n &\approx \epsilon_1\epsilon_2 \dots (y, w)(w, x)(x, w)(w, z) \dots \epsilon_n \\ &\approx \epsilon_1\epsilon_2 \dots (y, w)(w, z) \dots \epsilon_n. \end{aligned}$$

□

We know that for a path connected space, the fundamental group does not depend on the choice of the base point. Thus without loss of generality, we can always choose the base point x_0 to be one of the minimal points. Now imagine that if we eliminate all the points that are neither maximal nor minimal in X, then we will be left only with all the maximals and minimals. Call this subspace with only maximals and minimals Y. Then we have the following corollary:

Corollary 4.2. *For any finite poset (X, x_0), let (Y, x_0) be the subspace that consists of only maximals and minimals in (X, x_0). Then the inclusion induces an epimorphism $i_* : E(\mathcal{K}(Y), x_0) \to E(\mathcal{K}(X), x_0)$, or equivalently, $i_* : \pi_1(Y, x_0) \to \pi_1(X, x_0)$.*

Remark 4.3. Note that since there are only maximals and minimals in Y, $h(Y) \leq 2$. Typically, for a non-contratible space X, $h(Y) = 2$. Also, if X is connected, then removing middle points will not disconnect the space, i.e. Y remains connected.

Remark 4.4. When $h(Y) = 2$, we know that $\mathcal{K}(Y)$ is a finite 1-dimensional simplicial complex, i.e. a finite graph. Since a finite graph is always homotopy equivalent to a wedge sum of circles, we can assume that $\mathcal{K}(Y)$ is homotopy equivalent to $\bigvee_{i=1}^{m} S^1$. Therefore, we have

$$\pi_1(Y, x_0) \cong E(\mathcal{K}(Y), x_0) \cong \pi_1\left(\bigvee_{i=1}^{m} S^1, s_0\right) \cong \mathbb{Z}^{*m}.$$

Now we can go into the search for minimal finite models of finite graphs:

Theorem 4.5. *If X is a minimal finite model of $\bigvee_{i=1}^{n} S^1$, then $h(X) = 2$.*

Proof. Take the subspace of maximals and minimals Y. Since X is a minimal finite model of a noncontractible space, we know that $h(Y) = 2$. By Remark 4.4, $\pi_1(Y, x_0) = \mathbb{Z}^{*m}$.

By Proposition 4.1, there is an epimorphism $i_* : \pi_1(Y, x_0) \to \pi_1(X, x_0)$. Note that since $\pi_1(X, x_0) = \mathbb{Z}^{*n}$, thus we must have $m \geq n$.

Now consider $\mathcal{K}(Y)$. Since it is a finite graph, in other words, a wedge sum of m circles, there are m edges that are not contained in any maximal tree of the graph. If we remove $m - n$ of these edges by forgetting the relations between the vertices, we obtain a new finite space Z and $\mathcal{K}(Z)$ is homotopy equivalent to $\bigvee_{i=1}^{n} S^1$.

Note that $\#Z = \#Y \leq \#X$. But since X is a minimal finite model of $\bigvee_{i=1}^{n} S^1$, we also have $\#X \leq \#Z$. Therefore, $\#Z = \#Y = \#X$, which implies $X = Y$. □

The following theorem will conclude our search:

Theorem 4.6. *Let j be the number of maximal points in X and k be the number of minimal points in X. Then X is a minimal finite model of $\bigvee_{i=1}^{n} S^1$ if and only if $h(X) = 2$, $\#X = \min\{j + k \mid (j - 1)(k - 1) \geq n\}$ and the number of edges in $\mathcal{K}(X)$ is $\#X + n - 1$.*

Proof. We have shown that if X is a minimal finite model of $\bigvee_{i=1}^{n} S^1$, then $h(X) = 2$. Since j is the number of maximal points and k is th number of minimal points in X, we know that there can be at most jk many edges in $\mathcal{K}(X)$. Let E be the number of edges in $\mathcal{K}(X)$ and V be the number of vertices, then the Euler characteristic formula tells us that

$$1 - n = V - E \geq j + k - jk.$$

Therefore, we must have $(j - 1)(k - 1) = jk - j - k + 1 \geq n$, and hence $\#X = j + k \geq \min\{j + k \mid (j - 1)(k - 1) \geq n\}$.

Now suppose we have j and k such that $(j - 1)(k - 1) \geq n$. Then consider the finite poset $W = \{x_1, x_2, \ldots, x_j, y_1, y_2, \ldots, y_k\}$ with $x_s > y_t$ for any $1 \leq s \leq j$ and $1 \leq t \leq k$. As we can see, W is a finite model of $\bigvee_{i=1}^{(j-1)(k-1)} S^1$. But then we can remove $(j - 1)(k - 1) - n$ edges from $\mathcal{K}(W)$ by forgetting the corresponding relations, and the resulting finite poset would be a finite model of $\bigvee_{i=1}^{n} S^1$.

Now since for any j and k with $(j - 1)(k - 1) \geq n$ we can find a finite model with $j + k$ points, we conclude that $\#X = \min\{j + k \mid (j - 1)(k - 1) \geq n\}$, and the number of edges just follows from the Euler characteristic formula.

Conversely, suppose we have a finite poset X with $h(X) = 2$, $\#X = \min\{j + k|(j-1)(k-1) \geq n\}$ and the number of edges in $\mathcal{K}(X)$ being $\#X + n - 1$. Note that if X is connected, then we are done, for the reason that $\mathcal{K}(X)$ will also be connected, and the three conditions will determine that $\mathcal{K}(X)$ is a finite graph with the Euler characteristic $1 - n$. Therefore, the only thing we need to show here is connectedness.

Suppose X is disconnected. Let $X_1, X_2, \ldots, X_l$ be distinct connected components in X. Let M_i be the set of maximal points in X_i and m_i be the set of minimal points in X_i. Then $j = \sum_{i=1}^{l} \#M_i$ and $k = \sum_{i=1}^{l} \#m_i$. Since $\#X = \min\{j + k \mid (j-1)(k-1) \geq n\}$, we must have $(j-2)(k-1) < n$. But at the same time, $n = E - j - k + 1$ by the Euler characteristic formula. Therefore,

$$(j-2)(k-1) < E - j - k + 1$$
$$jk < E + (k-1).$$

Note that jk is in fact the number of edges in the complete bipartite graph $\left(\bigcup_{i=1}^{l} m_i, \bigcup_{i=1}^{l} M_i\right)$. The inequality above shows that $\mathcal{K}(X)$ differs from the complete bipartite graph in less than $k-1$ edges.

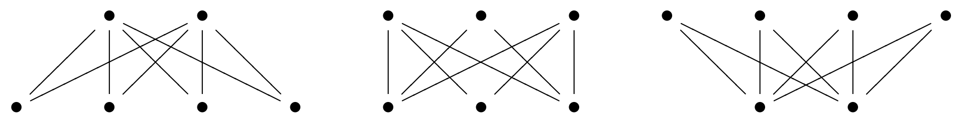
Since there are no edges between M_i and m_r for $i \neq r$, we have

$$k - 1 > \sum_{i=1}^{l} \#M_i(k - \#m_i) \geq \sum_{i=1}^{l} (k - \#m_i) = (l-1)k.$$

This forces $l = 1$ and hence X is connected. □

This theorem gives a method to compute minimal finite models of all finite graphs. Unlike the n-spheres, some finite graphs have more than one minimal finite model, with the same number of points but different arrangements. For example, the following three are minimal finite models of $\bigvee_{i=1}^{3} S^1$:

Example 4.7.



Up to this point are the minimal finite models that have been completely understood. But we want to push the frontier a little bit further to some slightly more complicated spaces and investigate the possible size of their minimal finite models.

5. Towards Realizing Groups with Finite Presentations

One fact from algebraic topology is that any group can be realized as the fundamental group of a geometric CW complex of dimension less than or equal to two. The way to do this is by taking a presentation of that group, and gluing a 1-cell to the base point for each generator of that group and a 2-cell along the 1-cells for each relation. (Note that if the starting group is free, then we only need the 1-cells, and the resulting CW complex would just be a graph.)

This makes us wonder whether we can do the same thing with finite spaces, i.e. realizing certatin groups just by finitely many points, and have a restriction on the height of the finite posets. Of course, we should point out that we can never realize a group that requires infinitely many generators, for the reason from Corollary 4.2

and Remark 4.4 that the fundamental group of any finite space is an epimorphic image from a finitely generated free group. Nevertheless, for a group with a finite presentation, we assert that we can always realize it with a finite poset, simply by subdividing the corresponding CW complex into a simplicial complex and applying the $\mathcal{X}$ functor. The resulting finite poset automactically has height no more than 3. However, we can even assure more with the following theorem:

Theorem 5.1. *For any finite poset (X, x_0), there exists a finite poset (X', x_0') with no more than $\#X$ many points, whose fundamental group is isomorphic to that of X and $h(X') \leq 3$. In other words, among all the realizations of a certain group, we can find such a poset with the least number of elements that has height of no more than 3.*

Proof. Without loss of generality, let us assume that x_0 is a minimal. We are going to construct X' explicitly as the follows.

First copy the subspace Y of all the maximals and minimals in X, call it X'. Then for any point x that is neither maximal nor minimal in X, put a point x' in X' with relations:

(1) For any maximal $\alpha \in Y$, let α' be the copy of α in X'. Then $x' < \alpha'$ if $x < \alpha$ in X.
(2) For any minimal $\beta \in Y$, let β' be the copy of β in X'. Then $x' > \beta'$ if $x > \beta$ in X.
(3) If x_1 and x_2 are both neither maximal nor minimal, then x_1' and x_2' are incomparable in X'.

This construction gives a finite poset X' with no more than $\#X$ many points, and the third condition restricts the height of the poset to be no bigger than 3. Now we claim that $E(\mathcal{K}(X), x_0)$ is isomorphic to $E(\mathcal{K}(X'), x_0')$.

To show this, let us define a map

$$\phi : E(\mathcal{K}(X'), x_0') \to E(\mathcal{K}(X), x_0)$$
$$\left[(x_0', x_1')(x_1', x_2') \ldots (x_{n-1}', x_0')\right] \mapsto \left[(x_0, x_1)(x_1, x_2) \ldots (x_{n-1}, x_0)\right].$$

This map is well defined for the following reason: if

$$\epsilon_1'\epsilon_2' \ldots (x', y')(y', z') \ldots \epsilon_n' \approx \epsilon_1'\epsilon_2' \ldots (x', z') \ldots \epsilon_n'$$

in $(\mathcal{K}(X'), x_0')$, then x', y' and z' are within a triangle, which implies that x, y and z are also within a triagle since the relation on X' corresponds to a subset of the relation on X. Thus in $(\mathcal{K}(X), x_0)$, we also have

$$\epsilon_1\epsilon_2 \ldots (x, y)(y, z) \ldots \epsilon_n \approx \epsilon_1\epsilon_2 \ldots (x, z) \ldots \epsilon_n.$$

Note that this map is a homomorphism by construction.

To show that the two groups are actually isomorphic, we want to define another map in the reverse direction:

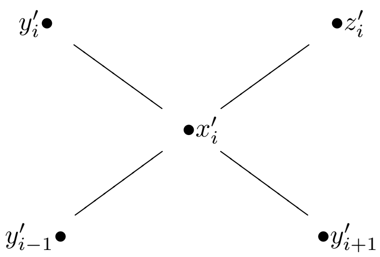
$$\psi : E(\mathcal{K}(X), x_0) \to E(\mathcal{K}(X'), x_0'),$$

Let ψ work as follows: for any edge-loop class $[\sigma] \in E(\mathcal{K}(X), x_0)$, first use the equivalence move from Definition 3.3 to shrink the edge-loop σ to the stage $(x_0, x_1)(x_1, x_2) \ldots (x_{n-1}, x_0)$ where x_i is bigger than both x_{i-1} and x_{i+1} for all odd i's and smaller tahn both x_{i-1} and x_{i+1} for all even i. Now for each x_i with odd i, pick a maximal point $y_i \geq x_i$, and similarly pick a minimal point $y_i \leq x_i$ for all the even i. Collect all the y_i in order into an edge-loop ξ at x_0. It is not hard to see that ξ is equivalent to σ in $(\mathcal{K}(X), x_0)$. Then since ξ is an edge-loop that consists of edges only with maximal and minimal vertices, it has a copy of it in $(\mathcal{K}(X'), x'_0)$, say ξ'. Now we simply set $\psi([\sigma]) = [\xi']$.

We need to show this map is well defined, i.e., the image does not depend on the maximal and minimal that are chosen nor the representative of the edge-loop class in $E(\mathcal{K}(X), x_0)$.

Without loss of generality, we are going to just look at the case when i is odd. Suppose we choose the a different maximal z_i instead of y_i. Then within $(\mathcal{K}(X'), x'_0)$:

$$\begin{aligned}
&(x'_0, y'_1)(y'_1, y'_2)\dots(y'_{i-1}, y'_i)(y'_i, y'_{i+1})\dots(y'_{n-1}, x'_0)\\
\approx&(x'_0, y'_1)(y'_1, y'_2)\dots(y'_{i-1}, x'_i)(x'_i, y'_{i+1})\dots(y'_{n-1}, x'_0)\\
\approx&(x'_0, y'_1)(y'_1, y'_2)\dots(y'_{i-1}, z'_i)(z'_i, y'_{i+1})\dots(y'_{n-1}, x'_0).
\end{aligned}$$

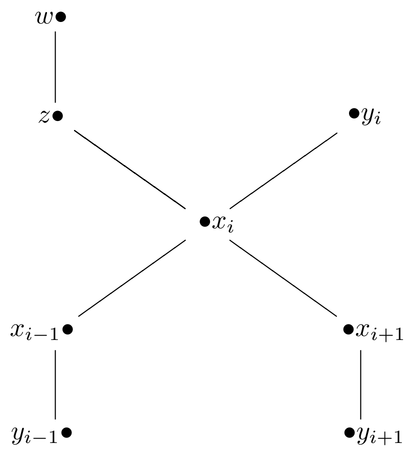


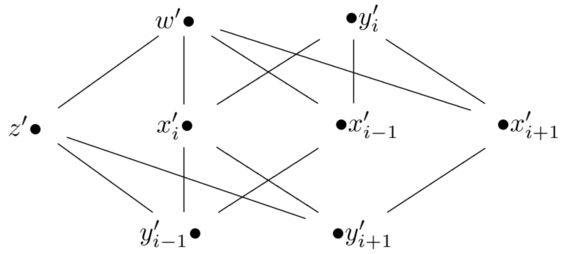
Therefore, the image does not depend on the maximal and minmal points that are chosen.

Now to show that ψ does not depend on the choice of representative, we can just check that the image does not change after the equivalence move from Definition 3.3. Notice that σ is already assumed to be shrunk to the least, and therefore it is impossible to use one move to combine two edges into one. On the other hand, if we replace the edge (x_{i-1}, x_i) by two other edges within a triangle, say $(x_{i-1}, z)(z, x_i)$, and assume without loss of generality that $x_{i-1} < x_i$, then one the following three is going to be true: $z < x_{i-1}$, $x_{i-1} \leq z \leq x_i$ or $x_i < z$.

If it is the case where $x_{i-1} \leq z \leq x_i$, then it does not change the image at all, since after shrinking we would get back σ.

The other two cases are analogous, and we are just going to consider the case where $x_i < z$. (Note that even in this situation, we have not mess up the alternating order of maximal points and minimal points, since z will be the new maximal element instead of x_i, and the representative, after shrinking, will be $\sigma = x_0 x_1 \dots x_{i-1} z x_{i+1} \dots x_0$.) Now if the y_i that we chose before is also greater than or equal to z, then we are set, because we can use y_i again for the maximal that extends z. The only "bad" situation is the left hand side of the following diagram, when z is incomparable with y_i:





But this is actually not bad at all, since as we can see in $(\mathcal{K}(X'), x_0')$,

$$\begin{aligned}&(x_0', y_1')(y_1', y_2') \dots (y_{i-1}', y_i')(y_i', y_{i+1}') \dots (y_{n-1}', x_0')\\ \approx&(x_0', y_1')(y_1', y_2') \dots (y_{i-1}', x_i')(x_i', y_{i+1}') \dots (y_{n-1}', x_0')\\ \approx&(x_0', y_1')(y_1', y_2') \dots (y_{i-1}', w')(w', y_{i+1}') \dots (y_{n-1}', x_0').\end{aligned}$$

Therefore, the image does not depend on the representative of the edge-loop class in $E(\mathcal{K}(X), x_0)$ either, and we conclude that the map ψ is well defined.

Note that ψ is also a homomorphism by construction. Moreover, if we take any edge-loop class $[\sigma] \in E(\mathcal{K}(X), x_0)$, we can choose the representative σ to contain only maximal and minimal points, say $\sigma = (x_0, y_1)(y_1, y_2) \dots (y_{n-1}, x_0)$. Then we have

$$\phi \circ \psi([\sigma]) = \phi\left([(x_0', y_1')(y_1', y_2') \dots (y_{n-1}', x_0')]\right) = [(x_0, y_1)(y_1, y_2) \dots (y_{n-1}, x_0)].$$

Conversely, if we take any edge-loop class $[\xi'] \in E(\mathcal{K}(X'), x_0')$, we can also choose the representative ξ to contain only maximal and minimal points, say $\xi' = (x_0', y_1')(y_1', y_2') \dots (y_{n-1}', x_0')$. Then we have

$$\psi \circ \phi([\xi']) = \psi\left([(x_0, y_1)(y_1, y_2) \dots (y_{n-1}, x_0)]\right) = [(x_0', y_1')(y_1', y_2') \dots (y_{n-1}', x_0')].$$

From these evaluations, we see that ϕ and ψ are actually inverses of each other, and hence $E(\mathcal{K}(X), x_0)$ is isomorphic to $E(\mathcal{K}(X'), x_0')$. By Corollary 3.6, we deduce immediately that (X, x_0) and (X', x_0') have isomorphic fundamental group. □

Since we have fully explored the case of finitely generated free groups in the previous section, in this section we will only focus on non-free groups with finite presentations. Note that the fundamental group of a poset with height no more than 2 is free, thus we are only focusing on finite realizations with a height of exactly 3.

One good thing about a finite poset with a height of 3 is that all the middle points now are unrelated to each other. Recall from Proposition 4.1 that, for a finite based poset (X, x_0) (x_0 is assumed to be a minimal) with the subspace of extremals Y, there is an epimorphism from $\pi_1(Y, x_0)$ onto $\pi_1(X, x_0)$, and since $h(Y) \leq 2$, $\pi_1(Y, x_0)$ is free. Hence the subspace Y gives us the generators of the fundamental group, and the middle points induce relations on $\pi_1(Y, x_0)$ to make it into $\pi_1(X, x_0)$, or equivalently, making $E(\mathcal{K}(Y), x_0)$ into $E(\mathcal{K}(X), x_0)$.

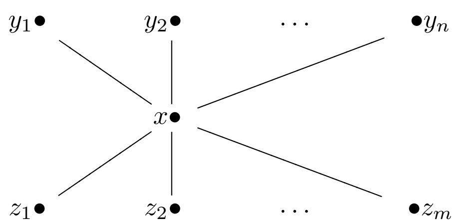
If we recall the notion of upbeat and downbeat points from Definition 1.6, we see that if a middle point is an upbeat or downbeat point, it can be removed without changing the weak homotopy type, and in particular the fundamental group. Therefore, when we try to realize certain group with as few points as possible, all middle points in the realization should be connected to at least minimal points and two maximal points. But then this implies that in $\mathcal{K}(X)$, any edge that contains a middle point must belong to a triangle, and adding a middle point is the same as gluing triangles onto $\mathcal{K}(Y)$.

Now suppose that there are two equivalent edge-loops ξ and ξ' in $\mathcal{K}(X)$ at x_0 that consist of edges between extremals only (i.e. edge-loops that are originally in $\mathcal{K}(Y)$). By definition of edge-loop equivalence, there exists a finite sequence of edge-loops $\{\xi_i \mid 0 \leq i \leq n\}$ at x_0 such that $\xi = \xi_0$ and $\xi' = \xi_n$, and ξ_{i+1} is obtained by applying the equivalence move we defined in Definition 3.3 to ξ_i. Since we know that each move must take place within one triangle, thus each move, which can be viewed as a relation, is induced by only one triangle. But since each triangle only has vertices of one maximal, one minimal and one middle point x only, it also exists in $Y \cup \{x\}$. Therefore, we have the following proposition:

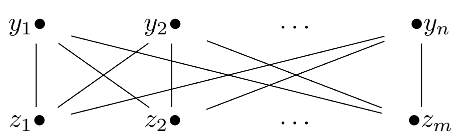
Proposition 5.2. *Let (X, x_0) be a finite poset with $h(X) \leq 3$ and let (Y, x_0) be the subspace of extremals (assuming x_0 is a minimal). Suppose $x_1, x_2, \ldots, x_n$ are the middle points in X. Then we can look at the subspace $Y \cup \{x_i\}$ for each $1 \leq i \leq n$ and consider the relations that x_i induces on $\pi_1(Y, x_0)$. Let $r^i_1, r^i_2 \ldots r^i_{m_i}$ be these relations. Then*

$$\pi_1(X, x_0) \cong \pi_1(Y, x_0) / \bigcup_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m_i}} \{r^i_j\}.$$

So what relations does one individual middle point induce on $\pi_1(Y, x_0)$, or equivalently, $E(\mathcal{K}(Y), x_0)$? Note that we are only interested in middle points that are connected to at least two maximals, which looks like the following Hasse diagram:



If we consider the corresponding part in the subspace Y, the Hasse diagram is the following:



As we can see, if we apply the $\mathcal{K}$ functor to this part of the subspace Y, we obtain a complete bipartite graph $(\{y_i\}, \{z_j\})$, and any two maximals with any two minimals form a loop. After we include the middle point x, all these loops become trivial. Therefore, the relations that the middle point x induces on $E(\mathcal{K}(Y), x_0)$ are just

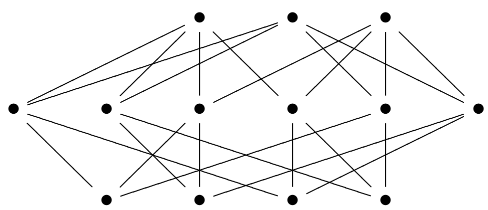
$$y_{i_1} z_{j_1} y_{i_2} z_{j_2} y_{i_1} = e, \quad 1 \leq i_1, i_2 \leq n \text{ and } 1 \leq j_1, j_2 \leq m$$

This provides another way to compute the fundamental group of a finite space X with $h(X) = 3$ (one way that we mentioned was by computing the H-loop group). Theoretically, one can write a program according to this method to compute the smallest size of a height 3 poset that we need to realize certain group with finite presentation. The reason why this is important is the following final remark:

Remark 5.3. For any group G with a finite group presentation, if X is one of the smallest finite posets of height 3 that has G as a fundamental group, then X is a minimal finite model of $|\mathcal{K}(X)|$. This is because if Z is another minimal finite model of $|\mathcal{K}(X)|$, then we can first reduce Z to Z' according to the previous theorem. Note that Z' also has G as its fundamental group, and by assumption we know that $\#X \leq \#Z' \leq \#Z$. Therefore, X is a minimal finite model of $|\mathcal{K}(X)|$.

At the end, we propose the following conjecture about minimal finite models of $\mathbb{R}P^2$, which can be proven with some computation and the remark above:

Conjecture 5.4. *The smallest finite posets of height 3 that realize* $\mathbb{Z}_2$ *have cardinality 13, and the following one is a minimal finite model of* $\mathbb{R}P^2$.



Acknowledgments. It is a pleasure to thank my mentor, Jared Bass, for devoting his time and effort in working with me, helping me overcome many difficulties, and giving a lot of great and professional advice.

I would also like to thank Professor May for the inspiration in this topic and the great lectures on finite topological spaces.

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