SOME ELEMENTARY RESULTS IN REPRESENTATION THEORY

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Abstract. This paper will prove that given a finite group G, the associated irreducible characters form an orthonormal basis for the set of class functions on G. We start with two different (but related) definitions of a representation, though we will make reference to the definition used in Fulton and Harris. After proving Schur's lemma and a useful corollary, we prove that each representation can be written as the direct sum of finitely many irreducible representations. After defining character functions and describing their importance in relation to irreducible representations, we define a Hermitian inner product on $\mathbb{C}_{class,G}$ (the set of class functions from G into $\mathbb{C}$) and show that a certain set of characters are orthonormal to each other. We then show that this orthonormal set is a basis for $\mathbb{C}_{class,G}$. Finally, we end with the character table for S_5, which can be derived from the theorems this paper contains.

1. Representations of Finite Groups, and Schur's Lemma

Representation Theory arises from the study of group actions. A group action is a function that describes how a group acts on a set. For example, consider D_6 as an abstract group. If we let X be the set of vertices of an equilateral triangle, we can think of D_6 as the group of symmetries on X, whose group elements act as rotations and reflections (indeed, the group D_{2n} is usually thought of in terms of symmetries on the regular n-gon).

Definition 1.1. Let G be a group, and X a set. A **group action** is a function $\phi : G \times X \to X$ with the following properties:

1) If $e \in G$ is the identity, then $\phi(e, x) = x$ for all $x \in X$.
2) Let $g_1, g_2 \in G$. Then $\phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x)$.

One usually writes $g \cdot x$ in place of $\phi(g, x)$.

Now, according to the above definition, the set X need not have any properties at all. If we use a vector space V instead of X as well as a few more conditions, we get the following definition.

Definition 1.2. Let G be a group and V a vector space. The map $\phi : G \times V \to V$ is a **representation** of G on V if ϕ is a group action and if, for every $g \in G$, the function $\phi_g : V \to V$ defined by $\phi_g(v) = \phi(g, v)$ is a linear function.

For the purposes of this paper, we will use the definition of a representation given in Fulton and Harris, which is similar to the definition above. Whereas the above definition defines a representation as a map from $G \times V$ into V, Fulton and Harris define a representation as a map from G into the set of linear transformations of V.

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Definition 1.3. Let G be a finite group, and V an n-dimensional vector space. A **representation** of G on V is a homomorphism $\sigma : G \to GL_n(V)$.

Note that we will always assume our vector space V is taken over the complex number field $\mathbb{C}$. Also, unless otherwise stated, we will always assume that the vector space V is finite dimensional.

Because it is sometimes confusing to remember that a representation is a homomorphism, but often one needs to discuss the associated vector space as well, this paper will use the notation (σ, V) to signify the representation. The σ is the homomorphism, and the V is the representation.

A single finite group G will usually have multiple representations. In preparation for results to come, we shall define the direct sum of two representations here. Let (α, V) and (β, W) be two representations of the finite group G. The direct sum of vector spaces V and W gives rise to another representation, written $(\alpha \oplus \beta, V \oplus W)$ where, for each $v \in V$, $w \in W$, and $g \in G$, we have $(\alpha \oplus \beta)(g)(v \oplus w) = (\alpha(g)v) \oplus (\beta(g)w)$.

Definition 1.4. Let (σ, V) be a representation of the finite group G. We say that (σ, W) is a **sub-representation** of (σ, V) if W is a vector subspace of V and W is invariant under G, that is, if $\sigma(g)w \in W$ for all $w \in W$.

The importance of this definition is only realized in conjunction with the following definition.

Definition 1.5. Let (σ, V) be a representation of the finite group G. Then (σ, V) is **irreducible** if it has no proper, nonzero sub-representations.

As it turns out, all representations can be written as the direct sum of irreducible representations. That result implies that knowing all irreducible representations for a given group (a finite group, for our purposes) allows one to construct all possible representations for that group. In addition, the decomposition of a representation into a direct sum of irreducible representations is unique.

Before proving the above paragraph, we present the statement and proof of Schur's Lemma, an extremely useful result about functions between irreducible representations. In preparation for the proof, we give the definition for G-linear maps.

Definition 1.6. Let σ_1 and σ_2 be representations of the finite group G on vector spaces V and W respectively. A **G-linear map**, also called a **G-module homomorphism**, is a map $\alpha : V \to W$ such that $\alpha(\sigma_1(g)v) = \sigma_2(g)\alpha(v)$.

Lemma 1.7. ***(Schur's Lemma)*** *Let G be a finite group with nontrivial irreducible representations (σ_1, V) and (σ_2, W). Suppose $\phi : V \to W$ is a G-linear map. Then either ϕ is an isomorphism, or $\phi(v) = 0$ for all $v \in V$.*

Proof. By the G-linearity of ϕ, we have

$$\phi(\sigma_1(g)v) = \sigma_2(g)\phi(v) = 0$$

for al $v \in \ker(\phi)$. Therefore, $\sigma_1(g)v \in \ker(\phi)$ for all $g \in G$. Consequently, $\ker(\phi)$ is invariant under G, so $(\sigma_1, \ker(\phi))$ is a sub-representation of (σ_1, V).

Before moving on, it is worth showing that $(\sigma_2, \phi(V))$ is a sub-representation of (σ_2, W). Firstly, $\phi(V)$ is a vector subspace of W. Secondly, for all $w \in \phi(V)$ (where we write $w = \phi(v)$ for some $v \in V$), we have the following:

$$\sigma_2(g)w = \sigma_2(g)\phi(v) = \phi(\sigma_1(g)v).$$

Therefore $\sigma_2(g)w \in \phi(V)$ for all $g \in G$. Consequently, $\phi(V)$ is invariant under G, so $(\sigma_2, \phi(V))$ is a sub-representation of (σ_2, W).

Collecting these two facts will complete the proof. Since V is irreducible, $\ker(\phi) = 0$ or $\ker(\phi) = V$. If $\ker(\phi) = 0$ then ϕ is an injection. Since $(\sigma_2, \phi(V))$ is a sub-representation of (σ_2, W), it must be all of W since neither W nor V are the trivial vector space. Therefore, ϕ is a bijective G-linear map, and hence is an isomorphism.

Finally, if $\ker(\phi) = V$, then $\phi(v) = 0$ for all $v \in V$. This completes the proof.

□

We now state an important corollary to this lemma, which we will use in later sections.

Corollary 1.8. *Let G be a finite group with the nontrivial irreducible representation (σ, V). Suppose $\phi : V \to V$ is a G-linear map. Then there exists a $\lambda \in \mathbb{C}$ such that $\phi(v) = \lambda v$ for all $v \in V$.*

Proof. The field $\mathbb{C}$ is algebraically closed. Therefore, each linear transformation on V, in this case, $\phi : V \to V$, has an eigenvalue λ with eigenvector x. Consider the function $f : V \to V$ defined by

$$f(v) = \phi(v) - \lambda I(v),$$

where I is the identity transformation. Then f is a G-linear map, so by Schur's lemma, f is an isomorphism or f is the zero map. But since $\phi(x) = \lambda I(x)$, we have that $x \in \ker(f)$. Consequently, $\ker(f) \neq 0$. Thus, f is the zero map and so $\phi(v) = \lambda I(v)$ for all $v \in V$.

□

We are now prepared to prove the following theorem.

Theorem 1.9. *Let (σ, V) be a representation of the finite group G. Then V can be written as the direct sum of finitely many irreducible representations.*

Proof. We define a recursive procedure for splitting V into sub-representations, and show that this process eventually terminates.

Step 1) *Check if V is irreducible:* If V is irreducible we're done, since then obviously V can be written as the direct sum of finitely many irreducible representations.

Step 2) *Decomposition of V into complementary sub-representation:* If V is not irreducible, then let (σ, W) be a (nontrivial) sub-representation of V. We will show that there exists a sub-representation (σ, W') of (σ, V) where W' is a complement of W and $V = W' \oplus W$. To do this, let X be an arbitrary subspace of V such that $V = W \oplus X$.

Define the projection map $p : V \to W$ by $p(v) = p(v_1, v_2) = v_1$. Then define the map

$$f(v) = \frac{1}{|G|} \sum_{g \in G} \sigma(g) p(\sigma(g^{-1})v).$$

Now, for $w \in W$, note that

$$f(w) = \frac{1}{|G|}\sum_{g\in G}\sigma(g)p(\sigma(g^{-1})w) = \frac{1}{|G|}\sum_{g\in G}\sigma(g)\sigma(g^{-1})w = \frac{1}{|G|}\sum_{g\in G} w = w$$

because (σ, W) is a sub-representation of (σ, V), and consequently the vector space W is invariant under the action of G. Thus, f fixes W, and $f(V) = W$. We now show that we may take $ker(f) = W'$ as our invariant subspace of V which is complementary to W.

Clearly, W' is complementary to W because when viewed as elements of V, the elements of W' are of the form $(0, w')$. It remains to be shown that W' is fixed under the action of G. To see that this is the case, note that

$$\sigma(h)\cdot f\cdot\sigma(h^{-1}) = \frac{1}{|G|}\sum_{g\in G}\sigma(h)\sigma(g)\cdot p\cdot\sigma(g^{-1})\sigma(h^{-1}) = \frac{1}{|G|}\sum_{g\in G}\sigma(hg)\cdot p\cdot\sigma((hg)^{-1})$$

$$= \frac{1}{|G|}\sum_{g\in G}\sigma(g)\cdot p\cdot\sigma(g^{-1}) = f.$$

In other words, $\sigma(h)\cdot f = f\cdot\sigma(h)$. As a consequence of this, for each $x \in W'$ we have $f(\sigma(h)x) = \sigma(h)f(x) = 0$, and therefore $\sigma(h)x \in W'$. Therefore, W' is the complementary sub-representation of W we wanted.

Step 3) *Return to Step 1) with W and W':* We can apply a similar procedure to both (σ, W) and (σ, W'), writing each of them as the sum of complementary sub-representations. For example, consider (σ, W). If (σ, W) is irreducible, we're done. If not, then applying step 2 shows that we can write $W = U' \oplus U$, where U' is complementary to U and (σ, U') and (σ, U') are sub-representations of (σ, W).

Note that $\dim(V) = \dim(W) + \dim(W')$, which implies that step 2 involves writing the representation as the sum of two representations of smaller dimension. If the dimension of a vector space is 0, then that vector space is trivial. Hence, this process eventually terminates at step 1. When this process is completed, we can write $V = \sum_{i=1}^{k} V_i^{\alpha_i}$, where (σ, V_i) is a distinct, irreducible sub-representation of V, and $V_i^{\alpha_i}$ denotes the direct sum of V_i with itself α_i times.

□

We call α_i the multiplicity of V_i in the decomposition of V. To close this section, we prove that this decomposition is unique.

Theorem 1.10. *Suppose (σ, V) is a representation of the finite group G, and suppose $V = \sum_{i=1}^{k} V_i^{v_i}$ where each (σ, V_i) is a distinct, irreducible sub-representation of V (we know such decomposition exists by the theorem above). Then this decomposition is unique.*

Proof. Suppose V may also be written as the direct sum

$$V = \sum_{i=1}^{n} W_1^{\beta_1}$$

where (σ, W_i) is an irreducible representation.

Consider the identity map $I : V \to V$. As proved in Schur's lemma, any nonzero map between irreducible representations is an isomorphism. Therefore, the map I

takes each subspace W_i to a subspace V_j (that is, $W_i \cong V_j$), and therefore I takes $W_i^{\beta_i}$ to $V_j^{\alpha_j}$. Thus, $\beta_i = \alpha_j$, which proves that the decomposition of V is unique. □

2. Characters and Irreducibility

We have shown that every representation (σ, V) of a finite group G may be written as the direct sum of irreducible representations. Thus, in order to find all representations of a given group, one need only find the irreducible representations. The study of characters (a term we define below) is conducted in order to describe all irreducible representations of a given group G.

Definition 2.1. Let (σ, V) be a representation of the finite group G. The **character** of that representation is a function $\chi_{\sigma,V} : G \to \mathbb{C}$ defined as $\chi_{\sigma,V}(g) = Trace(\sigma(g))$. Furthermore, if (σ, V) is an irreducible representation, then we will call $\chi_{\sigma,V}$ an **irreducible character**.

As the rest of the paper will show, these character functions contain all the information one needs to classify irreducible representations. It is the trace of the linear transformation $\sigma(g)$ that contains all the necessary data; one need not know the eigenvalues of $\sigma(g)$, just their sum. Quite an elegant result.

Definition 2.2. A **class function** is a function f on G such that $f(g) = f(hgh^{-1})$. For our purposes, the class functions will go into $\mathbb{C}$, that is, $f : G \to \mathbb{C}$. Notationally, we write $\mathbb{C}_{class}(G)$ for the set of all class functions into $\mathbb{C}$.

Note that characters are class functions, since

$$\begin{aligned}\chi_{\sigma,V}(hgh^{-1}) &= Trace(\sigma(h)\sigma(g)\sigma(h^{-1}) = Trace(\sigma(h^{-1})\sigma(h)\sigma(g))\\ &= Trace(\sigma(h^{-1}h)\sigma(g)) = Trace(\sigma(g)) = \chi_{\sigma,V}(g).\end{aligned}$$

Before moving on, we need to establish several properties concerning characters and representations. Firstly, if (α, V) and (β, W) are two representations of the finite group G, the tensor product of the vector spaces gives rise to a representation, written $(\alpha \otimes \beta, V \otimes W)$ where, if $v \in V$, $w \in W$, and $g \in G$, we have $(\alpha \otimes \beta)(g)(v \otimes w) = (\alpha(g)v) \otimes (\beta(g)w)$ (this is entirely similar to the direct sum of two representations).

For the purposes of the upcoming theorem, it is equally necessary, though slightly more involved, to define the dual of a representation as a representation. Suppose (σ, V) is a representation of the finite group G. The dual of V, denoted $V^\star$, is the set $Hom(V, \mathbb{C})$.

Let $\sigma^\star : G \to GL(V^\star)$ be defined by $\sigma^\star(g) = {}^t(\sigma(g^{-1}))$. Then $(\sigma^\star, V^\star)$ is indeed a representation for the group G. Furthermore, it respects analogous pairing of a vector space to its dual space: if $v \in V$, then $v^\star \in V^\star$ is the associated linear functional in the dual space. Likewise, $\sigma(g)(v)$ is associated with $\sigma^\star(g)(v)$ for each $g \in G, v \in V$.

These different kinds of representations translate into operations on characters, as the following proposition illuminates.

Proposition 2.3. *Let (α, V) and (β, W) be representations of the finite group G. Then the following equivalences hold:*

1) $\chi_{\alpha\oplus\beta,V\oplus W} = \chi_{\alpha,V} + \chi_{\beta,W}$.
2) $\chi_{\alpha\otimes\beta,V\otimes W} = \chi_{\alpha,V} \cdot \chi_{\beta,W}$.
3) $\chi_{\alpha^\star,V^\star} = \overline{\chi_{\alpha,V}}$.

Proof.

1) For all $g \in G$, consider $Trace((\alpha \oplus \beta)(g))$. This trace is equal to the sum of the eigenvalues of the matrix $(\alpha \oplus \beta)(g)$. The eigenvalues of that matrix are the sum of the eigenvalues of $\alpha(g)$ and $\beta(g)$. That is, if $\{\lambda_i\}_{i=1}^{k_1}$ is the set of eigenvalues of $\alpha(g)$, and $\{\gamma_i\}_{j=1}^{k_2}$ is the set of eigenvalues for $\beta(g)$, then for $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$, the set $\{\lambda_i + \gamma_j\}$ is the set of eigenvalues for $(\alpha \oplus \beta)(g)$. Therefore, $Trace((\alpha \oplus \beta)(g)) = Trace(\alpha(g)) + Trace(\beta(g))$, which completes the proof of the first equality.

2) The proof here is similar to the proof in part one. The only difference is to note that each eigenvalue of the matrix $(\alpha \otimes \beta)(g)$ may be written as the product of an eigenvalue of $\alpha(g)$ and an eigenvalue of $\beta(g)$. Also, given any eigenvalue of $\alpha(g)$ and any eigenvalue of $\beta(g)$, their product is an eigenvalue of $(\alpha \otimes \beta)(g)$. Therefore, the second equation follows.

3) Again, this proof follows the logic of the two above it. If $\{\gamma_i\}_{i=1}^n$ are the eigenvalues for $\alpha(g) \in G$, then the eigenvalues for $\alpha^\star(g)$ are $\{\overline{\gamma_i}\}_{i=1}^n = \{\gamma_i^{-1}\}_{i=1}^n$ since the γ_i have absolute value equal to 1.

□

Let (α, V) and (β, W) be two representations of the finite group G. Then $(\alpha^\star \otimes \beta, V^\star \otimes W)$ is a representation, which we identify by $(\alpha^\star \otimes \beta, Hom(V, W))$. This is because if $\phi \in H(V, W)$ then $\big((\alpha^\star \otimes \beta)(g)\phi\big)(v) = \big(\alpha^\star \otimes \beta(g)\big)\phi(\alpha^\star \otimes \beta(g^{-1})v)$.

We now prove a lemma in preparation for the upcoming theorem.

Lemma 2.4. *Let (α, V) and (β, W) be two representations of the finite group G. Then the vector space of G-linear maps from V to W, written as $Hom_G(V, W)$, is a subspace of $Hom(V, W)$ invariant under G.*

Proof. Clearly, $Hom_G(V, W)$ is a subspace of $Hom(V, W)$, since the sum of two linear functions, and the product of a scalar times a linear function, are both linear functions. To show invariance under the action of G, take $g \in G$ and ϕ a linear map. Then for all $v \in V$, we have $\beta(g)\phi(v) = \phi(\alpha(g)v)$, so obviously each G-linear function is fixed under the action of g.

Let H be the subspace of $Hom(V, W)$ fixed under the action of G, and take some $g \in G$. Then for each $h \in H$ and $v \in V$ we have $\beta(g)h(v) = h(\alpha(g)v)$, and therefore h is a G-linear function.

□

In preparation for the following theorem, we define a Hermitian inner product on the set of all class functions. Let $\mu, \nu \in \mathbb{C}_{class}(G)$. Their inner product is

$$(\mu, \nu) = \frac{1}{|G|} \sum_{g \in G} \overline{\mu(g)} \nu(g).$$

We are now ready for an important theorem concerning irreducible representations and their characters.

Theorem 2.5. *For each irreducible representation of G, the corresponding irreducible characters are orthonormal with respect to the Hermitian inner product defined above.*

Proof. Let (α, V) and (β, W) be irreducible representations of the finite group G. We want to show that if $V \cong W$,

$$\frac{1}{|G|}\sum_{g\in G}\overline{\chi_{\alpha,V}(g)}\chi_{\beta,W}(g) = 1,$$

but if $V \ncong W$,

$$\frac{1}{|G|}\sum_{g\in G}\overline{\chi_{\alpha,V}(g)}\chi_{\beta,W}(g) = 0.$$

Letting $\sigma = \alpha^{\star} \otimes \beta$ and $Y = Hom(V, W)$ for convenience, we get

$$\overline{\chi_{\alpha,V}(g)}\chi_{\beta,W}(g) = \chi_{\sigma,Y}.$$

Thus, we want to evaluate the sum

$$\frac{1}{|G|}\sum_{g\in G}\chi_{\sigma,Y}.$$

We now define the following sub-representation of (σ, Y) as

$$(\sigma, Y_G)$$

where $Y_G = \{\phi \in Y \mid \sigma(g)\phi = \phi \text{ for all } g \in G\}$. Note that by the lemma above, Y_G is the set of all G-linear homomorphisms from V to W. This will allow us to make assumptions about the nature of the function ϕ, defined as

$$\phi = \frac{1}{|G|}\sum_{g\in G}\sigma(g).$$

Note that ϕ is a G-linear homomorphism, since $\frac{1}{|G|}\sum_{g\in G}\sigma(g) = \frac{1}{|G|}\sum_{g\in G}\sigma(hgh^{-1})$. In a moment, we will prove that ϕ is a projection into Y_G, but first we give the reasons why such a fact is useful.

Now, ϕ might seem pretty arbitrary. But notice what happens when we take the trace of ϕ:

$$Trace(\phi) = \frac{1}{|G|}\sum_{g\in G}Trace(\sigma(g)) = \frac{1}{|G|}\sum_{g\in G}\chi_{\sigma,Y}(g).$$

The result is precisely sum we are trying to find. Thus, we have reduced our problem to finding the trace of ϕ.

We now show that $Trace(\phi) = \dim(Y_G)$, which will follow from the fact that ϕ is a projection into Y_G.

We now show that ϕ is a projection. First, consider $\phi(v) = \frac{1}{|G|}\sum_{g\in G}\sigma(g)v$. Then for each $h \in G$ we have $\sigma(h)\phi(v) = \frac{1}{|G|}\sum_{g\in G}\sigma(h)\sigma(g)v = \frac{1}{|G|}\sum_{g\in G}\sigma(g)v = \phi(v)$, so therefore $\phi(v) \in Y_G$ for all $v \in V$. Secondly, if $x \in Y_G$ then $\phi(x) = \frac{1}{|G|}\sum_{g\in G}\sigma(g)x = \frac{1}{|G|}\sum_{g\in G}x = x$. Therefore, ϕ is a projection.

Consequently, the trace of ϕ is equal to $\dim(Y_G)$.

Therefore, we only need find $\dim(Y_G)$. But this is simple. If $V \cong W$, then obviously $\dim(Y_G) = 1$. If, however, $V \ncong W$, then since both (α, V) and (β, W) are irreducible, they do not appear in each other's decomposition. Consequently, there are no homomorphisms between them, and thus $\dim(Y_G) = 0$.

Because this theorem holds for arbitrary irreducible representations (α, V) and (β, W), the proof is complete.

□

3. Irreducible Characters Form A Basis

The final part of this paper builds upon the final theorem of the previous section. Before proceeding, we must define the regular representation of a group.

Definition 3.1. Let G be a finite group. We call the representation (r, R) the **regular representation** if R is a finite dimensional vector space whose basis can be indexed by the elements of G. That is, $\{e_g \mid g \in G\}$ is a basis for R. Letting $a_h \in \mathbb{C}$ for $h \in G$, the homomorphism $r : G \to GL_n(R)$ acts on R by

$$r(g) \sum_{h=1}^{|G|} a_h e_h = \sum_{h=1}^{|G|} a_h e_{gh}.$$

We will use this regular representation to prove properties about the irreducible characters of a finite group. It turns out that not only are irreducible characters orthonormal, but they form a basis for $\mathbb{C}_{class}(G)$. Hence the following theorem:

Theorem 3.2. *Let G be a finite group with irreducible characters $\chi_{\sigma_1,V_1}, \cdots, \chi_{\sigma_n,V_n}$. Let $\mathbb{C}_{class,G}$ be the set of class functions on G. Then the χ_{σ_i,V_i} form an orthonormal basis for $\mathbb{C}_{class,G}$.*

Proof. We have already shown that the χ_{σ_i,V_i} are orthonormal, so it remains to show that they span $\mathbb{C}_{class,G}$. To do this, suppose that $\mu \in \mathbb{C}_{class,G}$ is orthogonal to each χ_{σ_i,V_i}, that is, $(\mu, \chi_{\sigma_i,V_i}) = 0$ for all i with $1 \leq i \leq n$. We want to show that μ must be the zero function.

Given a representation (β, W) of G, let

$$\beta_\mu = \sum_{g \in G} \mu(g)\beta(g).$$

Now suppose that (β, W) is an irreducible representation (that is, $\beta = \sigma_i$ for some i). We show that $\beta_\mu = 0$. Because each β can be written as a direct sum of irreducible representations, any β (irreducible or not) must be the zero map.

To show that β being irreducible implies $\beta_\mu = 0$, we show that β_μ is a G-linear map and then apply Schur's lemma an Corollary 1.8. For each $h \in G$, we have

$$\beta_\mu \beta(h) = \sum_{g \in G} \mu(g)\beta(g)\beta(h) = \sum_{g \in G} \mu(g)\beta(gh).$$

Because μ is a class function, $\mu(xy) = \mu(yx)$. Therefore,

$$\sum_{g \in G} \mu(g)\beta(gh) = \sum_{r \in G} \mu(rh^{-1})\beta(r) = \sum_{r \in G} \mu(h^{-1}r)\beta(r) = \sum_{u \in G} \mu(u)\beta(hu)$$

$$= \sum_{u \in G} \mu(u)\beta(h)\beta(u) = \beta(h)\beta_\mu$$

for $r = gh$ and $u = h^{-1}r$. Therefore, β_μ is a G-linear function. By Corollary 1.8, this means that $\beta_\mu = \lambda \cdot I$ for some eigenvalue λ. Now, $Trace(\lambda \cdot I) = k\lambda$, and $Trace(\beta_\mu) = \sum_{g \in G} \mu(g) Trace(\beta(g))$. Therefore,

$$\lambda = \frac{1}{k} \sum_{g \in G} \mu(g)\chi_{\beta,W}(g) = \frac{|G|}{k}(\mu, \overline{\chi_{\beta,W}}) = 0.$$

So $\beta_\mu = 0$.

As was explained above, we now know that $\beta_\mu = 0$ for each representation (β, W).

Specifically, if (β, W) is the regular representation and $\{e_g \mid g \in G\}$ is a basis for W, then

$$0 = 0 \cdot e_1 = \beta_\mu e_1 = \sum_{g \in G} \mu(g)\beta(g)e_1 = \sum_{g \in G} \mu(g)e_g.$$

Since the e_g are linearly independent, $\mu(g) = 0$ for all $g \in G$.

Therefore, $\{\chi_{\sigma_i,V_i} \mid 1 \leq i \leq n\}$ is an orthonormal basis for $\mathbb{C}_{class,G}$.

□

As a consequence of this theorem, we get the following corollary, which shows that there is one irreducible character for each conjugacy class of G.

Corollary 3.3. *Let G be a finite group, and $\{\chi_{\sigma_i,V_i}\}_{i=1}^n$ an orthonormal basis for $\mathbb{C}_{class,G}$. Then there are n conjugacy classes of G.*

Proof. For each conjugacy class C_i of G, the function $f_i : G \to \mathbb{C}$, defined by $f_i(x) = 1$ if $x \in C_i$, and $f_i(x) = 0$ if $x \notin C_i$, is a class function. The set of all these f_i are clearly a basis for $\mathbb{C}_{class,G}$. Therefore, by the above theorem (and since every basis of a vector space has the same number of elements) there are as many f_i as there are χ_{σ_i,V_i}, and therefore there are n conjugacy classes of G.

□

4. Character Table for S_5

We end this paper with a complete description of the irreducible characters on the group S_5. That information is organized into a **character table**, in which the irreducible characters are listed along the left-most column, and the conjugacy classes of the group are listed along the top row. Each box has a number, and that number is the value of the character on the respective conjugacy class. Notationally, the conjugacy class represented by 3 is just the set of all 3-cycles, the conjugacy class represented by $2-2$ is the set two 2-cycles.

The function I is the identity function. The function S is the sign function. The function R is equal to the regular representation minus 1. The functions P and Q are other, more archaic irreducible characters, and the functions $S \otimes R$ and $S \otimes V$ is a shorthand for the tensor product of the representations which S and R, and S and V respectively, stand for.

Character	1	2	3	4	5	2-2	2-3
I	1	1	1	1	1	1	1
S	1	-1	1	-1	1	1	-1
R	4	2	1	0	-1	0	-1
$S \otimes R$	4	-2	1	0	-1	0	1
P	5	-1	-1	1	0	1	-1
$S \otimes V$	5	1	-1	-1	0	1	1
Q	6	0	0	0	1	-2	0

To read the table, pick a character and a conjugacy class. The value of that character on that conjugacy class is listed in the appropriate place of the character table. To state a quick example, if $a \in S_5$ is a 4-cycle, then $R(a) = 0$.

We neglect to show the entire calculation for how this character table was derived, because that derivation is lengthy and not very illuminating. The primary tools we used were that the irreducible characters of a group G form an orthonormal basis for $\mathbb{C}_{class,G}$. We also use the fact that

$$\sum_{\chi} \chi_{\sigma,V}(g)\overline{\chi_{\sigma,V}(g)} = \frac{|G|}{|C(g)|},$$

where $C(g)$ is the conjugacy class of g. Of course, this fact is a quick consequence of Theorem 3.2.

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