TOPOLOGICAL SPACES AND THE FUNDAMENTAL GROUP

MAXWELL XIONG

Abstract. This paper provides a basic introduction to topological spaces. A topological space is less well defined than our every day metric spaces, and includes no notion of strict distances. Instead, it uses the concept of open sets to indicate the nearness of certain points. It also discusses the idea of homeomorphism, and how two topological spaces are essentially the same given a few certain conditions. Finally, this paper describes the fundamental group of a topological space, a characteristic which provides some important information about the space.

Contents

1. What is a topological space?

Definition 1.1. A *topological space* is a set X together with a collection τ of subsets of X that satisfy the following conditions:

(1) $X, \{\} \in \tau$
(2) The union of any sets in τ is in τ
(3) The finite intersection of any sets in τ is in τ

We will refer to the elements of X as *points*. We will also call τ a *topology* on X and we will refer to any element of τ as an *open set*.

Of course, if we are referring to the elements of τ as "open" sets, then what exactly is a "closed" set? As you might imagine, it is related to a set being open.

Definition 1.2. We call a set *closed* if the complement of that set is open.

With this definition, it is easy to see that oddly enough, if a set is not open, it does not mean that it is necessarily closed. It is possible for a set to be neither closed nor open, or even both closed and open at the same time. In fact, we are guaranteed two such sets in the definition of a topology τ. Both X and the empty set are guaranteed to be open, and because they are each other's complements, they are both guaranteed to be closed as well.

Now that we have these notions of a topological space and open sets, let us look at some simple examples of what qualifies as a topological space, and what open and closed sets look like within those spaces. We should all be familiar with the real line, or in other words, the number line consisting of all real numbers. The set X, then, is the set containing all real numbers, and the elements of the collection τ can be defined in the following way. The subset A of X is open if for any point $x \in A$, there exists a ball of radius ϵ (i.e. the set $\{y\ s.t.\ |x-y| < \epsilon\}$) that is a subset of A.

Theorem 1.3. *The above notion of an open set defines a topology on* R^n

Proof. This theorem simply states that our notion of open set satisfies the three conditions listed in definition 1.1. It is easy to see that the first condition is trivially satisfied.

For the second, we must show that the union of any open sets is open. We shall use $B(x, \epsilon)$ to denote a ball of radius ϵ about the point x. Given some family of open sets U, consider any point x in the union of U. This x must be within some individual open set within the family. By our notion of open, there exists an ϵ such that $B(x, \epsilon)$ is contained within that individual open set, and therefore $B(x, \epsilon)$ is also contained within the union. Thus, U is also open.

For the last criterion, we must show that the finite intersection of any open sets is open. Given some finite collection of open sets U_i, consider any point x in the intersection. Thus, for every i, we know that $x \in U_i$, and there exists ϵ_i such that $B(x, \epsilon_i) \subset U_i$. Let ϵ be the minimum of all the ϵ_i. Thus $B(x, \epsilon)$ is in the intersection. Therefore, the intersection is open.

□

Using this topology in one dimension then, an example of an open set would be any open interval, such as the interval $(0, 1)$. Let us check to make sure this is true. If we consider any number between 0 and 1, then in order to find a ball that stays within the interval, we must simply look at the distance from that number to either 0 or 1, and whichever distance is smaller will be the upper bound on the radius of our ball. Since we can do this for any number in the interval, it must be open.

On the other hand, an example of a closed set would be any closed interval. If we now consider the closed interval $[0, 1]$, in order to verify that it is closed, we must verify that its complement is open. So consider any point less than 0 or greater than 1. Again, we can simply look at its distance from the interval, and then make a ball of that radius around our point. We can also see that a closed interval will not be open, because if we consider the point 0, there is no ball of any radius that will remain within the interval.

While this example may have given us an intuitive idea of open and closed sets which we can easily extend to higher dimensional space, there are many examples of topologies that do not lend themselves so easily to these labels. For instance, going back to the real line, if we define our topology such that open sets are any sets that contain all but a finite number of points, we still satisfy the axioms of a topological space.

So far, we have only considered Euclidean spaces equipped with different topologies, but it is important to understand that the underlying set can take on essentially any form. For example, we can look at topological spaces on any sort of surface, such as a sphere, a mobius strip, a torus, or a cylinder. For each of these spaces, we can give them an induced toplogy from normal Euclidean space by intersecting each open set in Euclidean space with our new space. For instance, on the torus, we can think of it as embedded in 3 dimensional Euclidean space, where a "ball" centered at point x on the torus is really the intersection of the surface and the ball centered around x in Euclidean space.

2. Continuity, Connectedness, and Compactness

Definition 2.1. Let X and Y be topological spaces, equipped with the topologies τ_X and τ_Y respectively. A function $f : X \to Y$ is *continuous* if and only if for every open set $O \in \tau_Y$, $f^{-1}(O) \in \tau_X$.

Definition 2.2. Given two topological spaces X and Y, a function $f : X \to Y$ is a *homeomorphism* if f is continuous, a bijection, and has a continuous inverse f^{-1}.

If a homeomorphism exists between two spaces, they are called homeomorphic, and can be considered essentially the same topologically.

Note that a map may be bijective and continuous, but still not a homeomorphism. Consider the map $f : [0, 1) \to S^1$ where $f(x) = e^{2\pi ix}$. It is both bijective and continuous, but there exist open sets in $[0, 1)$ that are not open in S^1, such as $[0, \frac{1}{3})$.

Definition 2.3. A topological space X is *path connected* if for any two points $x, y \in X$, there exists a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

Intuitively, it is just as it sounds. A space is path connected if you can draw a path between any two points in the space. This notion will be important later when we discuss the fundamental group and homotopy classes.

Given three points a, b, c in a topological space X, and two paths, one from a to b, and the other from b to c, it makes sense to be able to adjoin the two paths to create a new path from a to c. This is a new continuous function defined on the same domain as the first two by simply following the first path by the second. If the first path is f_1 and the second is f_2, then the new path $f(x)$ would be defined as $f_1(2x)$ for $0 \leq x \leq \frac{1}{2}$ and $f_2(2x - 1)$ for $\frac{1}{2} \leq x \leq 1$.

Definition 2.4. Let X be a topological space and let $A \subseteq X$. A collection G of open sets G_a is an *open cover* of S if

$$S \subseteq \bigcup_G G_a.$$

Definition 2.5. S is *compact* if and only if every open cover of S has a finite subcover.

If X is compact, we call X a *compact topological space.*

Theorem 2.6. *Let X and Y be topological spaces, and let $A \subset X$ be compact. If $f : X \to Y$ is continuous, then $f(A)$ is compact.*

Proof. In other words, the continuous image of a compact set is compact. Let G be some open cover of $f(A)$. For any open set $B \in G$, consider the inverse $f^{-1}(B)$. This inverse must be open because f is continuous. The union of these inverses covers A, and because A is compact, there must be a finite subcover of A, $\{f^{-1}(B_i)\}$. Thus, the set of $\{B_i\}$ must be a finite subcover of $f(A)$, and therefore $f(A)$ is compact.

□

3. Product space

Definition 3.1. Given topological spaces X and Y, the *product space* is the cartesian product of those spaces equipped with the *product topology*, whose open sets are the unions of subsets of $A \times B$, where A and B are open subsets of X and Y, respectively.

For example, the product space of $[0, 1]$ with S^1 equipped with their usual topologies is the cylinder.

4. Quotient Space

Definition 4.1. Suppose X is a topological space and $\sim$ is an equivalence relation on X. We define a topology on the quotient set $X/\sim$ (the set consisting of all equivalence classes of $\sim$) as follows: a set of equivalence classes in $X/\sim$ is open if and only if their union is open in X. This is called the *quotient topology* on the quotient set $X/\sim$.

Intuitively this can be thought of as the gluing together of points, so that they may be considered equivalent, glued together as one point. For example, if we take the closed unit disc consider all boundary points to be within the same equivalence class, then this space mod this equivalence relation will be homeomorphic to the unit sphere. Or for another example, we can consider the interval $[0, 1]$ with only $0 \sim 1$. Then $[0, 1]/\sim$ will be homeomorphic to the unit circle.

5. The Fundamental Group

The fundamental group is a characteristic of a topological space which intuitively can be said to record information on the shape of the space, or the holes that a space has. A key idea when discussing the fundamental group is the concept of homotopy.

Definition 5.1. A *homotopy* between two continuous functions f and g from a topological space X to a topological space Y is a continuous function $H : X \times [0, 1] \to Y$ such that if $x \in X$ then $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. If such a homotopy exists, then f and g *homotopic*.

We can think of a homotopy as a continuous deformation from one function to another function, while thinking of the interval $[0, 1]$ as the time that the deformation is at, with 0 being the starting time, when the function is still the original f, and with 1 being the ending time, when the function is the final g.

For example, in the real plane, consider functions f and g that map from the unit interval so that $f(x) = (0, x)$ and $g(x) = (1, x)$. A simple homotopy between these two functions then, would be H(x,t) = (t,x), so that at time 0, it would be equal to $(0, x)$, the same as $f(x)$, and at time 1, it would be equal to $(1, x)$, the same as $g(x)$.

Now then, if we have a topological space X, and some point x_0 in the space, then we can consider the set of all continuous functions $f : [0, 1] \to X$ that satisfy $f(0) = f(1) = x_0$. Or in other words, the set of loops that start and end at the same base point x_0.

With this set, and the idea of homotopy in mind, we can now consider the equivalence classes of homotopic loops relative to the point x_0. We will call these *homotopy classes*. It should be apparent though, that in a path connected space these homotopy classes will be essentially independent of which base point we are considering, because we can move the base point of a loop through a homotopy by a direct path from one base point to the other.

Let us consider then, the homotopy classes present within the annulus (the space between two concentric circles). The most basic being the set of those functions homotopic to a fixed point. With a little consideration, one can see that any loop will be homotopic to a fixed point unless it goes around the hole in the center of the annulus. However, a loop can go around the hole in many different ways. Recall that a loop is a continuous map from $[0, 1]$ to the space, so that a clockwise loop will not be homotopic to a counterclockwise loop.

The product of two loops is a fairly intuitive notion, and is very similar to the idea of adjoining paths. Both loops start and end at the same base point, and it makes sense to simply "run" one loop after the other, a twice the "speed".

Definition 5.2. The product $(f * g)(t)$ of two loops f and g is defined as $f(2t)$ for $0 \leq t \leq \frac{1}{2}$ and $g(2t-1)$ for $\frac{1}{2} \leq t \leq 1$

If we give this definition a little thought, we can see that this binary operation will not be associative when directly applied to loops. By applying this operation to 3 loops, we will have two of the loops run four times faster, and one the remaining loop will run twice as fast, depending on the order in which we apply the operation.

However, if we instead apply the operation to homotopy classes, rather than to the loops themselves, we can see that the operation becomes associative.

Definition 5.3. Let X be a topological space, and let x_0 be a point in X. The *fundamental group* $\pi(X, x_0)$ consists of the set of all homotopy classes of loops with the base point x_0, with the product defined as above.

As we noted before, however, in a path-connected space, the homotopy classes are essentially independent of the choice of base point. As it turns out, this choice also will not affect the fundamental group up to isomorphism. Therefore, it makes sense to talk about $\pi(X)$, or the fundamental group of a space, when that space is path-connected.

Acknowledgments. Many thanks to my mentor, Benjamin Fehrman. He carefully took the time to explain countless ideas to me, and patiently made sure I understood everything. Without him, this paper would not exist.