METRIC SPACES AND DIFFERENTIAL EQUATIONS

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ABSTRACT. The concepts of metric spaces (complete and incomplete) and associated topics are illustrated and used to prove existence and uniqueness theorems in the theory of differential equations.

1. Metric spaces

The foundation of calculus is certainly the limit, and the conceptual foundation of the limit is the notion of numbers being arbitrarily "close together". That is, it depends on the idea of the distance between points. We can make intuitive sense of distance within certain sets: real numbers are points on a line, and ordered pairs points in a plane, so we can think of the distance between them geometrically as the length of the line connecting them. However, if we wish to generalize this structure to any set, we need to specify precisely what properties are required of it.

Definition 1.1. Let M be a set and $d : M \times M \to \mathbb{R}$. Suppose for all x, y, and z in M:

(1) $d(x, y) \geq 0$ with equality if and only if we have $x = y$,
(2) $d(x, y) = d(y, x)$, and
(3) $d(x, y) \leq d(x, z) + d(z, y)$.

Then we call (M, d) a *metric space*, and the function d a *metric*.

A metric formalizes the idea of the distance between points in a set. The first two conditions are easy to justify conceptually: distance cannot be negative, can only be zero between two identical points, and does not depend upon the order in which you consider points. To see the necessity of the third condition, consider the set of points in the Euclidean plane. Any three points not on a line form a triangle, and any side length of a triangle must be less than the sum of the other two side lengths. As a result, the distance between two points must be less than the sum of the distances between those two points and a third point. (If the second point lies on the line segment connecting the first and the third, then we have equality.) Condition 3 generalizes this notion to all metric spaces, and for this reason is often referred to as the *triangle inequality*.

Example 1.2. The set of real numbers $\mathbb{R}$ is a metric space when paired with the absolute value function, its usual metric. We can extend this to $\mathbb{R}^n$ by defining the distance between two n-tuples $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ as $\sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$.

Example 1.3. We can also create a metric space out of any non-empty set X with the metric f defined as:

$$f(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

This is called the *discrete metric* on X.

Example 1.4. As a less intuitive example, consider the set C of continuous functions on the closed interval $[a, b]$, along with the metric

$$d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|.$$

The function d is well-defined on all of $C \times C$ since continuous functions obtain a maximum and minimum value on a closed interval, and therefore the range of $|f - g|$ must be bounded above. The first two requirements of a metric space follow for C directly from those properties of the usual real metric. The triangle inequality follows as well, since if we ever had

$$\sup_{x \in [a,b]} |f(x) - g(x)| > \sup_{x \in [a,b]} |f(x) - h(x)| + \sup_{x \in [a,b]} |g(x) - h(x)|$$

then at the maximum point x_0 of $|f - g|$, we would have

$$|f(x_0) - g(x_0)| > |f(x_0) - h(x_0)| + |g(x_0) - h(x_0)|.$$

This is a contradiction, since the usual real metric satisfies the triangle inequality.

Given a function between metric spaces, we can then define limits and continuity in a fashion analogous to limits and continuity in calculus.

Definition 1.5. Suppose (M_1, d_1) and (M_2, d_2) are metric spaces, $x_0 \in M_1$, $y_0 \in M_2$, and $f : M_1 \to M_2$. Then we say that f approaches y_0 at x_0 if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that $d_2(f(x), y_0) < \epsilon$ whenever $0 < d_1(x, x_0) < \delta$. Since it can be shown that a function cannot approach more than one value at a point, we call this value the *limit* of f at x_0. If, in addition, we have $f(x_0) = y_0$, then we say that f is *continuous* at x_0.

Likewise, we can define the convergence of a sequence in an arbitrary metric space.

Definition 1.6. Let $\{x_n\}$ be a sequence of points in a metric space (M, d). Then we say that $\{x_n\}$ converges to a value l if, for all $\epsilon > 0$, we have a number K such that $d(x_n, l) < \epsilon$ whenever $n > K$.

2. Cauchy sequences and completeness of metric spaces

Definition 2.1. We call a sequence of points $\{x_n\}$ in M a *Cauchy sequence* if, for every $\epsilon > 0$, there exists some natural number N such that, for all n, $m > N$, we have $d(x_n, x_m) < \epsilon$. Or, more concisely:

$$\lim_{n,m \to \infty} d(a_n, a_m) = 0$$

Example 2.2. In any metric space, every convergent sequence is Cauchy. If $\{y_n\}$ converges to y, then for a given $\epsilon > 0$ we have a number M such that, for any $j, k > M$ we have both $d(y_j, y) < \epsilon/2$ and $d(y_k, y) < \epsilon/2$. By the triangle inequality, we have $d(y_j, y_k) \leq d(y_j, y) + d(y_k, y) < \epsilon$, which shows that $\{y_n\}$ is Cauchy.

The converse, however, is not necessarily true. For example, we know that the sequence $\{1/n\}$ is Cauchy in $\mathbb{R}$ since it converges there; therefore, it must also be Cauchy in the space $\mathbb{R} \setminus \{0\}$, since the metric is the same and the elements of the sequence are entirely contained within it. However, the sequence does not converge there, since we have removed its limit point 0. An even more deficient example is the space $\mathbb{Q}$: the sequence $\left\{ \sum_{k=0}^{n} \frac{1}{k!} \right\}$ consists only of rational numbers, but converges to e, which is irrational. Similarly, we have power series for π, roots, and logarithms, providing infinitely many sequences that do not converge in the space.

Definition 2.3. If every Cauchy sequence in M does converge to an element of M, however, then we say that M is a *complete metric space*.

Examples 2.4. Examples of complete metric spaces include $\mathbb{R}, \mathbb{C}$, and any closed subset thereof. Any set X with the discrete metric is also complete: since f can only take on the values 1 and 0, to get $f(x_j, x_k) < 1$ for $j, k > N$, we would need $f(x_j, x_k) = 0$, which only occurs when $x_j = x_k$ for all $j, k > N$. Thus, any Cauchy sequence $\{x_n\}$ has only one value for values of n greater than N, and thus converges to that value.

Example 2.5. We can also show that C with its metric forms a *complete* metric space. Let $\{f_n\}$ be an arbitrary Cauchy sequence of functions in C, using the same metric d as before. We wish to find a function f in C such that, for some N and all $n > N$, we have:

$$\sup_{x \in [a,b]} |f_n(x) - f(x)| < \epsilon$$

for arbitrary $\epsilon > 0$. This is equivalent to $\{f_n\}$ converging *uniformly* to a function f on $[a, b]$, and since the limit of a uniformly convergent sequence of continuous functions is itself continuous, the limit will be in the space C. Thus, proving uniform convergence of this sequence is sufficient.

So fix an arbitrary $\epsilon > 0$. Then since $\{f_n\}$ is Cauchy, there exists a number N such that for all $n, m > N$, we have:

$$d(f_n, f_m) < \epsilon.$$

In particular, this means that for any fixed x_0 in $[a, b]$, we can define a sequence of real numbers $\{f_n(x_0)\}$. Since the left-hand side of the above inequality is, by definition, an upper-bound for $|f_n(x) - f_m(x)|$ on $[a, b]$, $|f_n(x_0) - f_m(x_0)|$ will also be less than ϵ for all n and m greater than the same N. In other words, $\{f_n(x_0)\}$ is Cauchy in $\mathbb{R}$, and since $\mathbb{R}$ is complete, $\{f_n(x_0)\}$ converges to some real number; call it y_{x_0}. Since x_0 is arbitrary, we can find such a convergent sequence for all x in $[a, b]$. If we define a function f by $f(x) = y_x$, then $\{f_n\}$ converges pointwise to f.

To show that this convergence is uniform, use the triangle inequality to write, for all x in $[a, b]$:

$$|f_{n'}(x) - f(x)| - |f_{m'}(x) - f(x)| \leq |f_{n'}(x) - f_{m'}(x)|$$

for any n' and m'. The sequence $\{f_n\}$ is Cauchy in C, so the right-hand side is less than an arbitrary $\epsilon' > 0$ for sufficiently large n', m', giving us:

$$|f_{n'}(x) - f(x)| - |f_{m'}(x) - f(x)| \leq \epsilon'.$$

Since $\{f_n\}$ converges pointwise to f on all of $[a, b]$, the limit of the left-hand side as m' approaches infinity is $|f_{n'}(x) - f(x)|$. Taking limits preserves inequalities, so we will still have, for sufficiently large n',

$$|f_{n'}(x) - f(x)| \leq \epsilon',$$

Since the choice of n' did not depend on a choice of x, it works for all x in $[a, b]$. Thus, $\{f_n\}$ converges uniformly to f, as desired.

2.1. **Application: Existence and Uniqueness of Solutions to Linear Systems of ODEs.** The completeness of C guarantees the existence of a unique solution to a linear system of initial value problems. The proof proceeds by recursively constructing a sequence of continuous vector-valued functions on a closed interval. Since the sequence is Cauchy, it must converge to another continuous function on that closed interval. This, along with the recursive formula used to define the sequence, is sufficient to show that the limit function of the sequence satisfies the initial value problem.

Lemma 2.6. *Let C_n be the space of continuous functions $g : [a, b] \to \mathbb{R}^n$, and for $f = (f_1, \cdots, f_n), g = (g_1, \cdots, g_n) \in C_n$, define the metric*

$$d_n = \sup_{1 \le i \le n} d(f_i, g_i).$$

Then C_n is a complete metric space.

Proof. The argument that C_n is a metric space is exactly similar to the argument for C. To see that it is complete, notice that a Cauchy sequence in C_n is an n-tuple of Cauchy sequences in C. To find the limit of the Cauchy sequence in C_n, take the entrywise limits of these sequences in C; these limits are guaranteed to exist because C is complete. □

Note that the above proof also applies to the space of matrix-valued functions with a similar metric; we will use the same notation for both metrics and spaces.

Lemma 2.7. *Let the norm $\|A\|$ of a real matrix A be defined as the sum of the absolute values of its entries. Then if we have $\|A(t) - B(t)\| \le M$ on all of a closed interval, then we have $d_n(A(t), B(t)) \le M$ as well.*

Proof. Note that for any given t and element a_{ij} of A and b_{ij} of B, we have $|a_{ij} - b_{ij}| \le \|A - B\|$, since $\|A - B\|$ is simply the sum of nonnegative terms, among them the left-hand side. Note also that the quantity $d_n(A(t), B(t))$ is by definition the value of $|a_{kl}(t_0) - b_{kl}(t_0)|$, for some pair (k, l) and some t_0. So at this t_0 we have $d_n(A(t), B(t)) = |a_{kl}(t_0) - b_{kl}(t_0)| \le \|A(t_0) - B(t_0)\|$; by assumption, we have $\|A(t_0) - B(t_0)\| \le M$, so we also have $d_n(A(t), B(t)) \le M$. □

Theorem 2.8. *Let a be a real number, and let J be a closed interval in $\mathbb{R}$ containing a. Furthermore, suppose A is an $n \times n$ real-valued matrix function defined and continuous on J, and that B is some element of $\mathbb{R}^n$. Then there exists a once-differentiable function $f : J \to \mathbb{R}^n$ that satisfies the initial value problem*

$$\frac{df}{dx} = A(x)f(x) \qquad f(a) = B.$$

Proof. Taking a hint from the differential equation, we construct a sequence of functions $\{f_n\}$ on J by letting $f_0 = B$ and defining each subsequent term by recursion:

$$f_{k+1}(x) = B + \int_a^x A(t)f_k(t)dt.$$

The function f_0 is clearly continuous, and the Fundamental Theorem of Calculus ensures that if f_k is continuous, f_{k+1} is differentiable and therefore continuous, so by induction every f_n is continuous on J.

We wish to show that $\{f_n\}$ is Cauchy. To begin, we show that for all k, we have

$$\|f_{k+1}(x) - f_k(x)\| \leq \|B\| \frac{M^k |x-a|^k}{k!}$$

where $\|A(t)\| \leq M$ on all of J. (The continuity of A guarantees that $\|A\|$ is bounded.) We proceed by induction: for $k = 0$, the difference between terms becomes

$$\|f_1(x) - f_0(x)\| = \left\| \int_a^x A(t)Bdt \right\| \leq \left| \int_a^x \|A(t)\| \|B\| dt \right| \leq \left| \|B\| \int_a^x M dt \right| = \|B\| \frac{M^1 |x-a|^1}{1!}.$$

Now assume that for some m, we have

$$\|f_m(x) - f_{m-1}(x)\| \leq \|B\| \frac{M^m |x-a|^m}{m!}.$$

Noting that

$$f_{m+1}(x) - f_m(x) = \int_a^x A(t) \Big[f_m(t) - f_{m-1}(t) \Big] dt$$

we have

$$\|f_{m+1}(x) - f_m(x)\| = \left\| \int_a^x A(t) \Big[f_m(t) - f_{m-1}(t) \Big] dt \right\| \leq \left| \int_a^x M \|f_m(t) - f_{m-1}(t)\| dt \right|.$$

The induction hypothesis gives us

$$\left| M \int_a^x \|f_m(t) - f_{m-1}(t)\| dt \right| \leq \left| \int_a^x \|B\| \frac{M^m |x-a|^m}{m!} \right| = \|B\| \frac{M^{m+1} |x-a|^{m+1}}{(m+1)!}$$

as desired, so by induction we have this for all m. If we take L to be the length of J, we have

$$\|f_{m+1}(x) - f_m(x)\| \leq \|B\| \frac{(ML)^m}{m!}$$

and therefore, by the lemma,

$$d_n(f_{k+1}, f_k) \leq \|B\| \frac{(ML)^k}{k!}.$$

To finally show that $\{f_k\}$ is Cauchy, let $\epsilon > 0$ be arbitrary. Note that since $\sum (ML)^k / k!$ converges to $e^{ML} - 1$, the sequence of partial sums converges and therefore is Cauchy in $\mathbb{R}$. Thus, for an arbitrary $\epsilon > 0$, we can find an N such that for $p, q > N$,

$$\sum_{i=p}^{q-1} \frac{(ML)^i}{i!} < \epsilon.$$

Note also that for any p, q, the triangle inequality gives us:

$$d_n(f_p, f_q) \leq d(f_p, f_{p+1}) + d(f_{p+1}, f_{p+2}) + \cdots + d(f_{q-1}, f_q) \leq \sum_{i=p}^{q-1} \frac{(ML)^i}{i!}.$$

So we have

$$d_n(f_p, f_q) < \epsilon.$$

Thus, $\{f_k\}$ is Cauchy in C_n and therefore converges to a function in C_n. Call this limit function f. All that remains to be proven is that f is a solution to the differential equation. To see this, recall the recursion formula for the sequence

$$f_{k+1}(x) = B + \int_a^x A(t) f_k(t)dt.$$

Letting k go to infinity, we have

$$f(x) = B + \lim_{k\to\infty} \int_a^x A(t) f_k(t)dt.$$

Since the convergence of $\{f_k\}$ is uniform, we can pull the limit inside the integral, giving

$$f(x) = B + \int_a^x A(t) f(t)dt$$

on all of J. ☐

Uniqueness of solutions is easier to prove.

Theorem 2.9. *Suppose that both $g(x)$ and $h(x)$ are both solutions to the differential equation*

$$\frac{df}{dx} = A(x)f(x) \qquad f(a) = B$$

on a closed interval J containing a as before. Then $g = h$.

Proof. Since both g and h satisfy the differential equation, we have, for all x in J, both

$$g(x) = B + \int_a^x A(t)g(t)dt$$

and

$$h(x) = B + \int_a^x A(t)g(t)dt,$$

which together imply

$$g(x) - h(x) = \int_a^x A(t)[g(t) - h(t)]dt.$$

Since A and $g - h$ are continuous, their norms are bounded, say by M and N respectively. We wish to show that for all x in J and for all m, $\|g(x) - h(x)\| \leq N\frac{(M|x-a|)^m}{m!}$. We clearly have this for $m = 0$, since $\|g(x) - h(x)\| \leq M$ by construction. Assuming this for arbitrary k, then we have

$$\|g(x)-h(x)\| \leq \left| \int_a^x \big\|A(t)[g(t)-h(t)]\big\| dt \right| \leq \left| \int_a^x M(N\frac{(M|x-a|)^k}{k!} dt \right| = N\frac{(M|x-a|)^{k+1}}{(k+1)!}.$$

Thus, by induction, we have $\|g(x) - h(x)\| \leq M\frac{(N|x-a|)^m}{m!}$ for all m. Since factorial growth outstrips polynomial growth, for any positive number we can pick an m to make the right-hand side less than it. Thus, $\|g(x) - h(x)\|$ cannot be positive for any x. It cannot be negative, either, since the norm is the sum of nonnegative terms, so $\|g(x) - h(x)\|$ must be identically zero, and therefore g must be identically equal to h. ☐

3. Lipschitz continuity and contraction mappings

We now turn our attention to functions on metric spaces.

Definition 3.1. Let (M_1, d_1) and (M_2, d_2) be metric spaces, and K some positive constant. Furthermore, suppose ϕ is a function that maps from M_1 to M_2 such that for all x, y in M_1, we have $d_2(\phi(x), \phi(y)) \leq K d_1(x, y)$. Then say that ϕ is *Lipschitz continuous* on M_1, and K a *Lipschitz constant* of ϕ. If $K < 1$, then ϕ is called a *contraction mapping.*

Remark 3.2. As suggested by the name, Lipschitz continuity implies continuity. Suppose $\phi : M_1 \to M_2$ is Lipschitz continuous with constant K. Let $\epsilon > 0$ and $x_1 \in M_1$ be arbitrary and $\delta = \epsilon/K$. Then if, for any other $x \in M_1$ we have $d_1(x, x_1) < \delta$, then $K d_1(x, x_1) < \epsilon$ and, by the Lipschitz continuity of ϕ, $d_2(\phi(x), \phi(x_1)) < \epsilon$, so ϕ is continuous at x_1. Since we could have picked any x_1 in M_1, ϕ is continuous on all of M_1 as well.

The converse is not necessarily true; the function $1/x$ is continuous on the open interval $0 < x < \infty$, but is not Lipschitz continuous, since for any positive K, we can pick distinct y and z such that

$$\frac{1}{|zy|} = \frac{|\frac{1}{y} - \frac{1}{z}|}{|y - z|} > K.$$

(For example, let $y = 1/\sqrt{K}, z = 1/2\sqrt{K}$.) Thus, Lipschitz continuity is a stronger condition than continuity.

Definition 3.3. Let A be a set and $f : A \to A$. Then we call a point x_0 in A a *fixed point* of f if $f(x_0) = x_0$.

Theorem 3.4 (Banach fixed point theorem)**.** *Suppose (A, d) is a nonempty complete metric space and $f : A \to A$ is a contraction mapping. Then f has a unique fixed point in A.*

Proof. The proof proceeds by using the map f to construct a sequence of points in A. By showing that this sequence is Cauchy, we can guarantee that it has a limit point in A. We then show that this limit point is a fixed point in A.

Let x_0 be an arbitrary point in A, and define $\{x_n\}$ by the formula $x_{k+1} = f(x_k)$. We wish to show that for all n, we have:

$$d(x_{n+1}, x_n) \leq K^n d(x_0, x_1) \tag{3.5}$$

where $K < 1$ is a Lipschitz constant for f. We show this by induction. We have (3.5) for $n = 1$, since $d(x_2, x_1) = d(f(x_1), f(x_0))$ by the definition of our sequence, and $d(f(x_1), f(x_0) \leq K^1 d(x_1, x_0)$ because f is a contraction mapping. For the inductive step, assume that $d(x_{k+1}, x_k) \leq K^k d(x_1, x_0)$ for some fixed k. Then $d(x_{(k+1)+1}, x_{k+1}) = d(f(x_{k+1}), f(x_k)) \leq K d(x_{k+1}, x_k) \leq K(K^k d(x_1, x_0))$, as desired. Thus, (3.5) is true for all n.

Now, we wish to show that $\{x_n\}$ is Cauchy. For arbitrary p and q, the triangle inequality and (3.5) give

$$d(x_p, x_q) \leq d(x_p, x_{p+1}) + \cdots + d(x_{q-1}, x_q) \leq d(x_1, x_0)(K^p + K^{p+1} + \cdots + K^{q-1})$$

This latter sum is the difference of the pth and $q-1$th partial sums of $\sum K^n$. Since $0 \leq K < 1$, this series converges, and therefore the sequence of partial sums is Cauchy, meaning that the right-hand side of the above inequality can be made arbitrarily small by ensuring p and q are sufficiently large. Thus, the same is true of the left-hand side of the inequality, and so $\{x_n\}$ is Cauchy.

Since $\{x_n\}$ is Cauchy and A is complete, $\{x_n\}$ has a limit point x in A. To see that x is a fixed point of f, notice that

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f\big(\lim_{n\to\infty} x_{n-1}\big) = f(x).$$

(Bringing the limit into the argument of f is justified since f is Lipschitz continuous and therefore continuous on all of A.)

That the fixed point is unique is a direct consequence of the contraction condition on f. Suppose y and z are fixed points for f. Since f is a contraction map, we have:

$$d(f(y), f(z)) \leq K d(y, z)$$

Since y and z are fixed points, $f(y) = y$ and $f(z) = z$, so

$$d(y, z) \leq K d(y, z)$$

which is equivalent to

$$(1 - K) d(y, z) \leq 0.$$

Because $1 - K > 0$, we have $d(y, z) \leq 0$ which can only happen if $d(y, z) = 0$, since (M, d) is a metric space, and therefore only if $y = z$. □

Note that a function is only a contraction mapping with respect to a certain metric space and therefore with respect to a certain metric. This theorem then guarantees that a map has a fixed point as long as there exists at least one metric on its domain under which its domain is complete and the map is a contraction.

3.1. **Application: The Picard Theorem.** We once again turn our attention to function spaces on a closed interval. By restricting a function to a domain around a certain point, and finding an appropriate contraction map between continuous functions on that interval, we can guarantee the existence and uniqueness of a solution to certain nonlinear differential equations.

Theorem 3.6. Picard *Let $a, B_1 \cdots B_n \in \mathbb{R}$. Furthermore, suppose we have n continuous real-valued functions f_i defined on an $n+1$-dimensional closed rectangle around $(a, B_1, \cdots, B_n)$, and that each function is Lipschitz continuous in each coordinate except the first. That is, for each f_i,*

$$|f_i(x, y_1, y_2, \cdots, y_k, \cdots, y_n) - f_i(x, y_1, y_2, \cdots, y_{k'}, \cdots, y_n)| \leq K|y_k - y_{k'}|$$

for any $1 \leq k \leq n$. Then for some $\delta > 0$ there exists a unique set of n functions y_i satisfying the system of differential equations

$$y_1'(x) = f_1(x, y_1(x), y_2(x), \cdots, y_n(x))$$

$$y_2'(x) = f_2(x, y_1(x), y_2(x), \cdots, y_n(x))$$

$$\vdots$$

$$y_n'(x) = f_n(x, y_1(x), y_2(x), \cdots, y_n(x))$$

on the interval $a - \delta \leq x \leq a + \delta$ along with the initial conditions $y_k(a) = B_k$.

Proof. We begin by finding the appropriate δ; we then construct a function space on this interval, and find a contraction mapping on this space whose fixed point is a solution to the system of differential equations.

Since each f_i is continuous, they each achieve a maximum value on their domain; let M be the maximum of all these maxima. Let δ be such that $[a-\delta, a+\delta] \times [B_1 - M\delta, B_1 + M\delta] \times \cdots \times [B_n - M\delta, B_n + M\delta]$ is fully contained within the domain of each f_i. Furthermore, decreasing δ if necessary, let $K\delta < 1$.

Define the space C_n^δ as the set of n-tuples g of continuous real functions g_i on the interval $[a-\delta, a+\delta]$ such that $|g(x) - B_i| \leq M\delta$ on $[a-\delta, a+\delta]$. Using the same metric d_n as before, C_n^δ is a complete metric space; the additional boundedness stipulation does not affect completeness since the value at a point of the limit function of a Cauchy sequence in C_n^δ is itself the limit of a Cauchy sequence in $[B_i - M\delta, B_i + \delta]$, which is complete.

Now consider the map A which takes each $g_i(x)$ of $g(x) \in C_n^\delta$ to $B_i + \int_a^x f_i(t, g_1(t), \cdots, g_n(t))dt$. The map takes C_n^δ to C_n^δ, since

$$|B_i - (B_i + \int_a^x f_i(t, g_1(t), \cdots, g_n(t))dt)| = |\int_a^x f_i(t, g_1(t), \cdots, g_n(t))dt| \leq \int_a^x K dt \leq K\delta$$

Finally, the Lipschitz continuity of each f_i guarantees that A is a contraction mapping; given any two $g, h \in C_n^\delta$, for each component function g_i, h_i, we have

$$|g_i(x) - h_i(x)| \leq \int_a^x |f_i(t, g_1(t), \cdots, g_n(t)) - f_i(t, h_1(t), \cdots, h_n(t))| dt \leq K\delta \max_i |g_i(x) - h_i(x)|.$$

Since this is true for all i and x in $[a-\delta, a+\delta]$, it must be also true when maximized across all i and x, giving

$$d_n(A(g), A(h)) \leq K\delta d_n(g, h).$$

However, since $K\delta < 1$ by construction, A is a contraction mapping. Having fulfilled all the conditions of Banach Fixed Point Theorem, we can ensure that there exists a unique function $y \in C_n^\delta$ satisfying the condition that each component function y_i satisfies the integral equation

$$y_i(x) = B_i + \int_a^x f_i(x, y_1(x), \cdots, y_n(x))$$

on all of $[a-\delta, a+\delta]$, which is equivalent to the desired system of differential equations. □

References

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[2] Kolmogorov, A.N. and S.V. Fomin. Introductory Real Analysis. Trans. Richard A. Silverman. New York: Dover, 1970.