SHORT ABSTRACTS: 2011 SUMMER VIGRE REU

All concepts mentioned in the abstracts will be carefully defined

APPRENTICE PROGRAM

Apprentices attend courses offered in the full program during week 1. A morning program of introductory material will be given in weeks 2–5. Apprentices are also welcome to attend all other courses in weeks 2–5, AND LATER. Several of the courses in the full program are designed to be accessible to apprentice participants.

Weeks 2–5: László Babai

Linear algebra and combinatorics

The course will develop the usual topics of linear algebra and illustrate them on highly unusual and often striking applications to combinatorics, discrete geometry, and discrete probability. Emphasis will be on creative problem solving and discovery.

The basic topics include permutations, determinants, linear transformations, the characteristic polynomial, Euclidean spaces, orthogonalization, the Spectral Theorem, linear algebra over finite fields. Application areas to be highlighted include, time permitting, extremal set theory, the spectral theory of graphs (expansion, mixing of random walks, independence number, Shannon capacity, etc.), k-wise independence of events, counting zero-patterns of polynomial maps, and more.

FULL PROGRAM

Whatever faculty have posted in past abstracts, the material actually presented has generally diverged from that originally planned so far in advance. Therefore, only quite abbreviated indications are posted. Tentative week by week schedules have also been posted on the web page.

1a. Benson Farb: Weeks 1-2: geometry
1b. David Constantine and Ben McReynolds: Weeks 1–8: geometry
2a. László Babai: Weeks 2–5: problems
2b. Maryanthe Malliaris: Weeks 1–2: logic
2c. Matthew Morrow: Week 1: number theory
3a. Gregory Lawler: Weeks 1–2: probability
3b. Jonathan Weare: Weeks 3–4: probability
3c. Eva Strawbridge: Weeks 1-2: nonlinear dynamics
4. Peter May and Emily Riehl: Weeks 1-8: algebra, topology, category theory

1. Topics in geometry

1a. Weeks 1–2: Benson Farb, geometry

Title: Surfaces: geometry, topology and dynamics

Abstract: Surfaces are fundamental objects of mathematics. In this series of lectures I will discuss a few aspects of geometry, topology and dynamics on surfaces. The main goal will be to explain the remarkable fact that these three seemingly completely different viewpoints are intimately connected. Only a knowledge of multivariable calculus will be assumed.

1b. Weeks 1–8: David Constantine and Ben McReynolds, geometry

Title: The circle: A (surprisingly) rich and fundamental cross road of mathematics.

Abstract: The circle is perhaps the most important geometric/topological example in mathematics. We will explore algebraic/dynamical/geometric properties on higher dimensional analogs of circles called n-tori. As a set, an n-torus is the direct product of n circles with the 1-torus being simply a circle.

Why study the n-torus? Well the best reason is because the n-torus is awesome. However, if you require a less subjective reason, here are two. First, n-tori are simple objects but are complicated enough to support rich mathematical theories. We will be led to several fundamental concepts in mathematics: Topological/Riemannian manifolds, dynamics of group actions, dynamics of the geodesic flow, Teichmuller and moduli spaces, spectral geometry, arithmetic lattices and spaces, symmetric and locally symmetric spaces, .etc. Quite plainly, via tori we can touch on analytic/algebraic number theory, algebraic geometry, Riemannian geometry, geometric analysis, (geometric) group theory, dynamics without breaking a sweat. The torus is thus a perfect host fully capable of entertaining the algebraically, geometrically, or analytically minded guest. Second, despite their simple nature, there are still many important open questions for n-tori with regard to the above topics that undergraduates can think deeply about after a short period of time.

Aside from laying down the basic foundational material required for further exploration, there will be quite a bit of freedom in the paths we elect to meander down. In particular, students will certainly have input on such matters.

2. Miscellaneous topics with an algebraic flavor

2a. Weeks 2–5: László Babai

A few annoying problems

In this course, you will be presented with a set of elementary problems in a variety of disciplines, ranging from math puzzles an 8th-grader could understand, to problems on polynomials, complex numbers, number theory, geometry, combinatorics, linear algebra, group theory, logic, set theory, measure theory, topology, and more. You will find the problems annoying enough to rob you of sleep. Students will present their solutions at the blackboard. Each solution earns the solver a blue point; particularly elegant solutions earn green points. At the end of the course, blue and green points can be traded for red points. Motto: "However contracted, that solution is the result of expanded meditation." First puzzle: this motto paraphrases a sentence from what literary work? Only one word was changed; what was the original?

2b. Weeks 1–2: Maryanthe Malliaris, logic

Title: Order and randomness

Abstract: Dichotomies between order and randomness arise in many areas of mathematics. I will motivate and prove several examples of this phenomenon drawn primarily from model theory and graph theory. This will include both classical results and very recent work.

I intend this course to be accessible to anyone with an interest in classification – broadly stated – and a fondness for combinatorial proofs; the issues we'll discuss will often come down to interesting combinatorics. In particular, no prior knowledge of model theory or graph theory will be assumed.

2c. Week 1: Matthew Morrow, number theory

Title: Reciprocity laws and polynomials

Given a polynomial with integer coefficients, it is natural to ask for which prime numbers p our polynomial has a root modulo p. This simple question leads to so-called reciprocity laws. We will completely prove the Law of Quadratic Reciprocity to describe the situation for quadratic polynomials, and hint at where the theory goes to from there.

3. Topics in probability and applied mathematics

3a. Weeks 1–2: Gregory Lawler, probability

Markov chains

Markov chains are one of the most fundamental of random processes. I will give introductory lectures in this area — no previous background in probability is needed. In the first week we will study finite chains for which the main tool is linear algebra. In the second week we consider chains with infinitely many states such as random walk on the integer lattice. The material we will cover comes from the first few chapters of my book, Introduction to Stochastic Processes.

Previous exposure to probability is not required.

3b. Weeks 3–4: Jonathan Weare, probability

Title: Monte Carlo sampling methods

Many important problems in the physical, biological, and social sciences require generating multiple copies of a complicated system in order to infer average characteristics of the system. In this mini-course we will introduce the general class of random algorithms used to solve these problems (Monte Carlo methods) as well as some of the basic strategies used to build efficient sampling tools including importance sampling and Markov chain Monte Carlo techniques. No background is required though attending Lawler's course on Markov chains would be useful.

3c. Weeks 1–2: Eva Strawbridge, fluid dynamics

Title: An Introduction to Elementary Fluid Dynamics

Abstract: Questions revolving around the dynamics and mechanics of fluids arise very naturally from physical observation. Substances which range from the very air we breath to the oceans which cover seventy percent of the earth behave as fluids. Shockingly, the behavior of such a grand continuum as the ocean (or the syrup you put on your pancakes this morning) can often be packaged into intuitive and compact equations. We will develop the basic concepts of fluid dynamics following the general procedure of D. J. Acheson, briefly covering special topics in invisid (or non-viscous) fluids before introducing elementary viscous flow. No previous experience with differential equations or vector calculus will be assumed.

4. Weeks 1–8: Peter May and Emily Riehl; algebra, topology, category theory

In an influential philosophical paper, Sir Michael Atiyah axiomatized "Topological Quantum Field Theories". While motivated by physics, these are really quite elementary mathematical structures. However, although they are easy to define, they are very hard to construct rigorously in dimensions greater than 2. However, in dimension 2, even their construction is elementary.

The idea in dimension 2 is to think about closed strings (perhaps many of them) moving in time and tracing out surfaces with boundaries. We think of such a surface as giving a transition from a state space of strings at time 0 to a state space of strings at time 1.

One can classify all such surfaces with boundaries up to a suitable notion of equivalence (homeomorphism or diffeomorphism, in dimension 2 it makes no difference), and one can put algebraic structure on the set of equivalence classes.

As we shall explain, this construction gives us a category. Categories give a precise way to define mathematical structures specified by objects and maps. Functors give a precise way to compare two such mathematical structures. A 2D TQFT is a functor from our topologically defined category of surfaces to a purely algebraic category, usually taken to be the category of vector spaces over a field, such as the field of real or complex numbers. The algebraic structure on the source category corresponds to algebraic structure on the target category. When one works this all out, one reaches the conclusion that 2D TQFT's are equivalent to certain kinds of algebras, called commutative Frobenius algebras.

It is doable fun to actually work out concrete examples of commutative Frobenius algebras. Participants might well find some new ones.

We will develop from scratch everything that is needed to make this completely rigorous, introducing lots of algebra, topology, and category theory along the way, in a slow and meandering fashion.

The essential point is the beautiful and surprising interaction between various branches of mathematics, and the language that allows mathematicians to turn comparisons by analogy into theorems of comparison.

There is a helpful and accessible little book on the main topic:

Joachim Kock. Frobenius algebras and 2D topological quantum field theories.