THE HOPF DEGREE THEOREM

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ABSTRACT. This paper develops the theory of differential topology, the study of manifolds and smooth maps between manifolds, towards investigating the classification of homotopy classes of smooth maps to the n-sphere S^n. Our primary destination is the Hopf Degree Theorem, which states that two maps of a compact, connected, oriented n-dimensional manifold without boundary into the n-sphere are homotopic if and only if they have the same degree. While providing an introduction to the fundamental notions of differential topology, the exposition proceeds from an assumed background in real analysis and linear algebra.

CONTENTS

1. INTRODUCTION

Differential topology commences with the goal of generalizing the tools of real analysis used to study Euclidean space to subsets which are locally equivalent to Euclidean space, thus allowing the exploration of a wider variety of geometrical objects called manifolds. The first section of this paper introduces manifolds as our fundamental objects of interest. We also examine the relationship between manifolds and smooth functions defined between them, inferring a function's local behavior from its derivative. In particular, we develop the notion of regular values to determine conditions under which the preimage of a point determines a submanifold in its domain. Due to the guaranteed existence of regular values as a consequence of Sard's theorem, we focus on preimage manifolds as the primary avenue towards recovering information about functions. We proceed to further specify manifolds with orientations in order to define the degree of a mapping as an invariant property within smooth homotopy classes. The rest of the paper will be occupied with the converse question of when we can use degree to infer a map's homotopy class. To answer this question when the target manifold is the n-sphere, we identify preimage submanifolds up to framed cobordism and associate to each homotopy class of maps a unique class of framed submanifolds, culminating in the Hopf Degree Theorem.

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2. Manifolds and Maps of Manifolds

As a convention used throughout this paper, let e_1,...,e_k denote the standard basis vectors for $\mathbb{R}^k$, where e_i is the vector with 1 in the ith place and 0 elsewhere. This section is primarily based on chapters 1 and 2 of [3]. We begin by formalizing what it means for a subset of Euclidean space $\mathbb{R}^k$ to locally look like another Euclidean space $\mathbb{R}^n$ by equating spaces up to diffeomorphism.

Definition 2.1. If U is an open subset in $\mathbb{R}^n$, a map $f : U \to \mathbb{R}^m$ is *smooth* if all of its partial derivatives exist and are continuous. If X is an arbitrary subset of $\mathbb{R}^n$, a map $f : X \to \mathbb{R}^m$ is smooth if for each $x \in X$ there exists an open set $U \subset \mathbb{R}^n$ containing x and a smooth map $F : U \to \mathbb{R}^m$ such that $F\mid_{U\cap X} = f$. A bijective map $f : X \to Y$ is a *diffeomorphism* if it is smooth and its inverse is also smooth.

A quick application of the chain rule shows that the composition of smooth functions is smooth. It follows that diffeomorphism is an equivalence relation.

Definition 2.2. We call a subset $X \subset \mathbb{R}^k$ an n-dimensional *manifold* if every $x \in X$ is contained in a set $V \subset X$ open relative to X which is diffeomorphic to an open set $U \subset \mathbb{R}^n$. A diffeomorphism $\phi : U \to V$ is a local *parametrization* of X near x. Sometimes we indicate the dimension of X by $\dim X$.

For the sake of brevity, any reference to a parametrization of an n-dimensional manifold X assumes that its domain and image are open in $\mathbb{R}^n$ and X, respectively. Also, the ambient space $\mathbb{R}^k$ will often be omitted except when directly relevant.

Examples 2.3. One simple manifold is the open unit ball B^n. Define the map $h : B^n \to \mathbb{R}^n$ by $h(x) = \frac{x}{1-\|x\|}$, which is smooth. Since the inverse $h^{-1}(x) = \frac{x}{1+\|x\|}$ is also smooth, h is a diffeomorphism from B^n to $\mathbb{R}^n$.

We shall also show that the n-sphere S^n is locally diffeomorphic to $\mathbb{R}^n$ by introducing stereographic projection. Choose a base point $p \in S^n$ and the n-dimensional subspace E^n perpendicular to the vector p. We define our projection $h : S^n \setminus p \to E^n$ by linearly extending the vector from p to a point $x \in S^n$ until it hits the subspace E^n (see figure (1)). It is easy to check that h is a diffeomorphism from an open set of S^n to an n-dimensional subspace diffeomorphic to $\mathbb{R}^n$. For example, if $p = e_{n+1}$ define the projection of $x = (x_1, ..., x_n)$ by $h(x_1, ..., x_{n+1}) = (\frac{x_1}{1-x_{n+1}}, ..., \frac{x_n}{1-x_{n+1}}, 0)$. The inverse of $x = (x_1, ..., x_n, 0)$ is defined by $h^{-1}(x_1, ..., x_n, 0) = (\frac{2x_1}{1+\|x\|^2}, ..., \frac{2x_n}{1+\|x\|^2}, \frac{-1+\|x\|^2}{1+\|x\|^2})$. By performing stereographic projection from two different base points we acquire a cover of S^n by parametrizations.

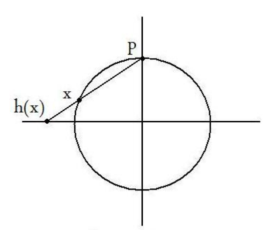

Figure 1. Stereographic projection of S^1

Definition 2.4. If X and Z are manifolds in $\mathbb{R}^k$ and $Z \subset X$, then Z is a *submanifold* of X. The *codimension* of Z in X is the difference $\dim X - \dim Z$.

Example 2.5. An example of a submanifold is the equator of S^2 consisting of a copy of S^1.

We shall also extend these definitions to manifolds with boundaries, such as the closed unit ball in $\mathbb{R}^n$ consisting of the open ball B^n and its boundary S^{n-1}. A prototypical example is the upper half-space $H^n = \{(x_1, ..., x_n) \in \mathbb{R}^n\colon\ x_n \geq 0\}$, whose boundary ∂H^n consists of the points $(x_1, ..., x_n) \in H^n$ with $x_n = 0$. We now simply generalize this case to arbitrary manifolds with boundary via local parametrizations.

Definition 2.6. A subset $X \subset \mathbb{R}^k$ is an n-dimensional *manifold with boundary* if every $x \in X$ is contained in a set $V \subset X$ open relative to X which is diffeomorphic to an open set $U \subset H^n$. The *boundary* ∂X consists of the points $x \in X$ mapped from ∂H^n by any local parametrization. The *interior* $\mathrm{Int}(X)$ consists of the complement $X \setminus \partial X$.

Note that an arbitrary manifold with boundary may have empty boundary, making it a manifold in the sense of definition (2.2). To avoid ambiguity, henceforth we will refer to a manifold with fixed empty boundary as a manifold without boundary.

In the examples it is clear that both the interior and boundary of a manifold with boundary are themselves independently manifolds. It turns out that this holds in general. It is easy to check that any open subset of a manifold with boundary, such as its interior, is also a manifold of the same dimension.

Proposition 2.7. *Let X be an n-dimensional manifold with boundary. Then the boundary ∂X is an $(n-1)$-dimensional manifold without boundary.*

Proof. Given $x \in \partial X$, take a local parametrization $\phi : U \to X$. Since ∂H^n is clearly diffeomorphic to $\mathbb{R}^{n-1}$, it suffices to show that the restriction $\phi\mid_{\partial H^n \cap U}$ maps onto the neighborhood $\partial X \cap \phi(U)$ open relative to ∂X. Given another parametrization $\psi : V \to X$ such that $\psi(V) \subset \phi(U)$, define the transition map $g = \phi^{-1} \circ \psi : V \to U$. Since g is an open map it follows $g(\mathrm{Int}(H^n) \cap V) \subset \mathrm{Int}(H^n) \cap U$. By taking complements we have $\psi(\partial H^n \cap V) \subset \phi(\partial H^n \cap U)$. Since this is true for any parametrization of $X \cap \phi(U)$, we conclude $\phi(\partial H^n \cap U) = \partial X \cap \phi(U)$. ☐

Another basic tool we shall require is the ability to make new manifolds from old ones. For example, it is easy to show that the Cartesian product of an n-dimensional manifold without boundary X and an m-dimensional manifold with boundary Y is a manifold with boundary such that $\partial(X \times Y) = X \times \partial Y$ and $\dim(X \times Y) = n + m$. That is, for a point $(x, y) \in X \times Y$ and local parametrizations $\phi : U \to X$ and $\psi : V \to Y$ satisfying $\phi(0) = x$ and $\psi(0) = y$, the product $\phi \times \psi : U \times V \to X \times Y$ defined by $(\phi \times \psi)(x, y) = (\phi(x), \psi(y))$ is a local parametrization of $X \times Y$ around (x, y) defined on the neighborhood $U \times V$ of H^{n+m}.

We now proceed to describe smooth functions between manifolds, again generalizing from smooth functions defined on open sets in Euclidean space.

Definition 2.8. Suppose f is a map defined on an open set $U \subset \mathbb{R}^n$ into $\mathbb{R}^m$. We say f is *differentiable* at a point $x \in U$ if there exists a linear operator $T \in M_{m\times n}(\mathbb{R})$ such that $\lim_{h\to 0} \frac{f(x+h)-f(x)-Th}{\|h\|} = 0$. In this case denote the *derivative* of f at x as $df_x := T$.

If f is a smooth map defined on an open set $U \subset H^n$ into $\mathbb{R}^m$, then at a point $x \in \partial H^n$ the smoothness of f implies that it extends to a smooth function $F : V \to \mathbb{R}^m$ where V is open in $\mathbb{R}^n$. Define $df_x := dF_x$.

Recall from real analysis that if f is smooth then df_x is defined for all $x \in U \subset \mathbb{R}^n$. Also, it is clear that for a function f defined on H^n, the derivative at a boundary point in ∂H^n is well-defined since the derivative of a smooth extension varies continuously and agrees with the derivative of f at any point of the interior $\mathrm{Int}(H^n)$. As an initial observation, note that if $f : U \to \mathbb{R}^n$ is a diffeomorphism on a neighborhood $U \subset \mathbb{R}^n$, then for each $x \in U$ the chain rule implies that the derivative $df_x : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism.

Since the derivative of a function defined on an open set of Euclidean space is a linear map between vector spaces, we would like the derivative of a map between manifolds to have the same property. We thus associate to every $x \in X$ its tangent space $T_x(X)$.

Definition 2.9. Let $X \subset \mathbb{R}^n$ be an n-dimensional manifold. Given a point $x \in X$, take a local parametrization $\phi : U \to X$, and without loss of generality let $\phi(0) = x$. Then the *tangent space* of X at x is $T_x(X) = d\phi_0(\mathbb{R}^n)$.

We equip $T_x(X)$ with the structure of a vector space by declaring the bijection $d\phi_0$ to be linear. As the definition of the tangent space involves taking a particular parametrization, we must show that it does not depend on which parametrization is taken.

Proposition 2.10. *The tangent space $T_x(X)$ is well-defined.*

Proof. Take two parametrizations $\phi : U \to X$ and $\psi : V \to X$ such that $\phi(0) = \psi(0) = x$ and $\phi(U) = \psi(V)$. Then the transition map $h = \phi^{-1} \circ \psi$ is a diffeomorphism between V and U whose derivative dh_0 is therefore an isomorphism. Since $d\phi_0 = d\psi_0 \circ dh_0^{-1}$, we conclude $d\psi_0$ and $d\phi_0$ impart the same structure to $T_x(X)$. □

One may note that for a product of manifolds $X \times Y$, the tangent space at a point (x, y) can be represented as $T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y)$. To see this, take local parametrizations $\phi : U \to X$ and $\psi : V \to Y$ satisfying $\phi(0) = x$ and $\psi(0) = y$. Then the product parametrization satisfies $d(\phi \times \psi)_{(0,0)} = d\phi_0 \times d\psi_0$. The result follows.

Now we can define the derivative of a map between tangent spaces of manifolds by referring back to familar derivatives between open sets in Euclidean space.

Definition 2.11. Given a smooth function $f : X \to Y$ from an n-dimensional manifold X to an m-dimensional manifold Y and a point $x \in X$, take parametrizations $\phi : U \to X$ and $\psi : V \to Y$ with $\phi(0) = x$ and $\psi(0) = f(x)$, choosing U so that $f(\phi(U)) \subset \psi(V)$. This yields a map $h : U \to V$ defined by $h = \psi^{-1} \circ f \circ \phi$. We can now write f as a composition of maps defined on open sets in Euclidean space, namely $f = \psi \circ h \circ \phi^{-1}$. Define the derivative $df_x : T_x(X) \to T_y(Y)$ to be $d\psi_0 \circ dh_0 \circ d\phi_0^{-1}$.

One can easily check that the derivative does not depend on which parametrizations are chosen by again examining transition maps between parametrizations. As a first application, we would like to understand the relationship between the derivative at a point and the local behavior of f near that point. We begin by noting that the derivative of a map between manifolds inherits the chain rule via local parametrizations. As before, this implies that the derivative of a diffeomorphism at any point is an isomorphism. We may further generalize this result by weakening the condition that f be a diffeomorphism on its whole domain.

Definition 2.12. We say that f is a *local diffeomorphism* around x if there exists an open set $U \subset X$ around x such that $f\mid_U$ is a diffeomorphism onto an open neighborhood of $f(y)$.

Since the derivative is locally defined, it follows that df_x is an isomorphism if f is a local diffeomorphism around x. To obtain the converse, another useful result we borrow from real analysis is the Inverse Function Theorem. One can find a proof in any real analysis textbook. For example, it is theorem 9.24 in [4].

Theorem 2.13 (Inverse Function Theorem)**.** *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map whose derivative df_x at a point x is an isomorphism, then f is a local diffeomorphism at x.*

This local result can be translated for smooth maps of manifolds via local parametrizations. We now shift our attention towards examining preimages of a smooth map of manifolds $f : X \to Y$. That is, given a point $y \in Y$, under what conditions will the preimage $f^{-1}(y) \subset X$ be a submanifold of X? To answer this question we need only examine a simple property of the derivative at points in the preimage.

Definition 2.14. Let $f : X \to Y$ be a smooth map between manifolds. If at a point $x \in X$ the derivative df_x is surjective from $T_x(X)$ onto $T_{f(x)}(Y)$ then x is a *regular point* of f, otherwise x is a *critical point* of f. Given $y \in Y$, if every $x \in f^{-1}(y)$ is a regular point of f, then y is a *regular value* of f.

First observe that the set of regular points of a map f is open in X due to the continuity of df_x as a function of x. Furthermore, we shall demonstrate using the Inverse Function Theorem that the preimage of a regular value is always a submanifold of the domain manifold.

Proposition 2.15. *Let $f : X \to Y$ be a smooth map between manifolds with dimensions n and m, respectively, satisfying $n \geq m$. If y is a regular value of f, then $f^{-1}(y)$ is a submanifold of X with dimension $n - m$.*

Proof. Suppose $X \subset \mathbb{R}^k$. Take $x \in f^{-1}(y)$. Since $df_x : T_x(X) \to T_y(Y)$ is surjective, the rank-nullity theorem determines that df_x has an $(n-m)$-dimensional kernel, which we shall denote as the subspace W. We shall consider W as a $(n-m)$-dimensional subspace of $\mathbb{R}^k$. Extend a basis $\{v_1, ..., v_{n-m}\}$ of W to a basis $\{v_1, ..., v_k\}$ of $\mathbb{R}^k$, then define a linear operator $T : \mathbb{R}^k \to W$ by

$$T(v_i) = \begin{cases} e_i, & i \leq n-m \\ 0, & n-m < i < n \end{cases}.$$

Extend f to a map $F : X \to Y \times W$ defined by

$$F(x) = (f(x), T(x)).$$

Then $dF_x = (df_x, T)$, which maps $T_x(X)$ isomorphically onto $T_y(Y) \times W$. By the Inverse Function Theorem F maps a neighborhood U of x diffeomorphically onto a neighborhood V of $(y, T(x))$ in the product manifold $Y \times W$. In particular F^{-1}

maps a neighborhood $\{y\} \times W \cap V$ of the $(n-m)$-dimensional product $\{y\} \times W$ diffeomorphically onto the neighborhood $f^{-1}(y) \cap U$ of x in $f^{-1}(y)$. □

Example 2.16. Consider the map $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) = \|x\|^2$. Let $x = (x_1, ..., x_n)$. The derivative $df_x = (2x_1, ..., 2x_n)$ is surjective onto $\mathbb{R}$ for $x \neq 0$. This provides another representation of the manifold S^n as the preimage $f^{-1}(1)$.

Proposition 2.17. *Given an n-dimensional submanifold Z in an m-dimensional manifold X, the tangent space of the submanifold $T_x(Z)$ is a linear subspace of the total tangent space $T_x(X)$.*

Moreover, if $f : X \to Y$ is a smooth map of manifolds with regular value $y \in Y$, then then at a point x in the submanifold $f^{-1}(y)$ the tangent space of the submanifold $T_x(f^{-1}(y))$ constitutes the kernel of the derivative $df_x : T_x(X) \to T_y(Y)$.

Proof. At a point $x \in Z$, any $v \in T_x(Z)$ can be written as the velocity vector for a path in Z. That is, given a parametrization $\phi : U \to Z$ with $\phi(0) = x$, if $v = d\phi_0(w)$ for some $w \in \mathbb{R}^n$ we can take a path $c : (-\epsilon, \epsilon) \to \mathbb{R}^n$ defined by $c(t) = tw$, where ϵ is chosen to be small enough so that the path is contained in U. Then $g = \phi \circ c$ defines a path in Z such that $dg_0 = d\phi_0 \circ dc_0 = d\phi_0(w) = v$. But g is also a path in X. Hence given a local parametrization $\psi : V \to X$ with $\psi(0) = x$, by shrinking ϵ if necessary we construct a path $\psi^{-1} \circ g$ in $\mathbb{R}^m$ satisying $d(\psi^{-1} \circ g)_0 = d\psi^{-1} \circ dg_0 = d\psi^{-1}(v) \in \mathbb{R}^m$. Hence $v \in T_x(X)$.

Given a map f with the stated conditions, the fact that f is constant on $f^{-1}(y) = Z$ implies that df_x maps $T_x(Z)$ into $T_y(\{y\}) = 0$. Since df_x is surjective the rank-nullity theorem determines that the dimension of the kernel is $\dim T_x(X) - \dim T_x(X) = \dim T_z(Z)$. Thus the kernel of df_x is exactly $T_x(Z)$. □

With an added condition on the behavior of the map on the boundary we may generalize this result to manifolds with boundary. For convenience, if f is a smooth map whose domain X is a manifold with boundary, denote by ∂f the restriction of f to the boundary ∂X.

Proposition 2.18. *Let f be a smooth map of an n-dimensional manifold with boundary X into an m-dimensional manifold without boundary Y with y a regular value of both f and ∂f. Then the preimage $f^{-1}(y)$ is a manifold with boundary $\partial(f^{-1}(y)) = (\partial f)^{-1}(y) \cap \partial X$ whose codimension in X is $\dim Y$.*

Proof. Take $x \in f^{-1}(y)$. For $x \in \mathrm{Int}(X)$ we may locally refer to the proof of proposition (2.15). If $x \in \partial X$, first take a parametrization $\phi : U \to X$ with $\phi(0) = x$ and consider the map $g = f \circ \phi : U \to Y$. Since y is a regular value of g, then y is a regular value for an extension G of g defined on an open set $V \subset \mathbb{R}^n$. Since the set of regular points is open, we can choose V small enough to contain only regular points of G. Then y is a regular value of G, which implies $G^{-1}(y)$ is a submanifold of $\mathbb{R}^n$.

Let $\pi : G^{-1}(y) \to \mathbb{R}$ denote the projection $\pi(x_1, ..., x_k) = x_k$. We show that 0 is a regular value of π. At a point $x \in G^{-1}(y) \cap \partial H^n = \pi^{-1}(0)$ the map $dg_x : \mathbb{R}^n \to T_y(Y)$ is surjective with kernel $T_x(G^{-1}(y))$, and the map $d(\partial g)_x : \partial H^n \to T_y(Y)$ is also surjective. To determine the kernel of $d(\partial g)_x$ note that since $\pi : \mathbb{R}^n \to \mathbb{R}$ is linear, we have $\pi = d\pi_u$; hence if x is not a regular point then $T_x(G^{-1}(y)) \subset \partial H^n$. Since the kernel of dg_x lies in ∂H^n, this implies dg_x and $d(\partial g)_x$ have the same kernel, which contradicts the fact that both maps have the same image but domains of different dimensions. Thus 0 is a regular value of π.

We see $g^{-1}(y) = \{x \in G^{-1}(y) : \pi(x) \geq 0\}$. Clearly the set of points $x \in G^{-1}(y)$ satisfying $\pi(x) > 0$ is open and thus a submanifold of the same dimension. If $\pi(x) = 0$, then since $d\pi_x$ is surjective, we can find a suitable linear map T as in proposition (2.15) and extend it to a map $\pi' = (T, \pi)$ whose derivative $d\pi'_x = (T(x), \pi(x))$ is an isomorphism. Thus π' maps a neighborhood U of x diffeomorphically onto a neighborhood V of $(T(x), 0)$. In particular π' maps $U \cap H^n$ diffeomorphically onto the neighborhood $V \cap H^n$, mapping exactly the points u such that $\pi(u) = 0$ to ∂H^n. Hence $f^{-1}(y) \cap H^n$ is a manifold with boundary $f^{-1}(y) \cap \partial H^n$. □

To ensure the existence of regular values we invoke Sard's theorem, which states that the set of critical values has Lebesgue measure zero. Since the proof of Sard's theorem is removed from the main ideas of this paper, we state the theorem without proof. A proof can be found in chapter 3 of [3].

Theorem 2.19 (Sard's Theorem)**.** *If $f : X \to Y$ is a smooth map of manifolds, then almost every $y \in Y$ is a regular value of f.*

It quickly follows that the set of common regular values of an at most countable collection of smooth functions $\{f_i\}_{i \in I}$ from X to Y is dense in Y. This allows us to readily choose regular values in a wide variety of circumstances. As Sard's theorem will be used repeatedly in the course of this paper we shall invoke it implicitly when selecting regular values.

Another critical property pertaining maps of manifolds is smooth homotopy.

Definition 2.20. Two smooth maps of manifolds $f, g : X \to Y$ are called *smoothly homotopic* if there exists a smooth map $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. Such a map F is called a *homotopy* between f and g.

Proposition 2.21. *Homotopy is an equivalence relation on the set of smooth maps from X to Y.*

Proof. Let $f, g, h : X \to Y$ be smooth maps of manifolds. Reflexivity follows from taking the constant homotopy $F(x, t) = f(x)$. If F is a homotopy from f to g, symmetry holds by taking the reverse homotopy $G(x, t) = F(x, 1 - t)$. If F is a homotopy between f and g and G is a homotopy between g and h, then smoothly concatenating F and G yields a homotopy between f to h. □

A homotopy class consists of smooth maps between manifolds which are smoothly deformable into each other. We shall currently postpone our discussion of homotopy to introduce orientation, which will eventually allow us to define degree as a homotopy invariant.

3. Orientation and Degree

This section is primarily based on material from chapter 3 of [1]. We begin by defining an orientation for a finite-dimensional vector-space V. Given two ordered bases $\beta = \{v_1, ..., v_n\}$ and $\beta' = \{v'_1, ..., v'_n\}$ there exists a unique linear isomorphism $A : V \to V$ such that $\beta' = A\beta$. Rendering A as a matrix with respect to any basis of V, we say that β and β' are equivalently oriented if $\det(A) > 0$. Using the fact that the identity matrix has positive determinant and employing from linear algebra the product rule $\det(AB) = \det(A)\det(B)$, it is clear that this determines an equivalence relation partitioning the set of bases into two classes.

Definition 3.1. An *orientation* of a finite and positive dimensional vector space V is an assignment of $+1$ to one class of equivalently ordered bases and -1 to the other class. If V is zero-dimensional then define an orientation as an assignment of $+1$ or -1 to the vector space V. The orientation of a basis β is sometimes denoted by $\operatorname{sgn}(\beta)$.

As a matter of convention, the standard basis for $\mathbb{R}^n$ is taken to have positive orientation $+1$. For any oriented n-dimensional vector space V, we can choose a positively oriented basis β and associate to a basis β' the isomorphism A such that $\beta' = A\beta$. We can thus use a few elementary linear algebra facts to calculate orientations.

Lemma 3.2. *Let $\beta = \{v_1, ..., v_n\}$ be an ordered basis for a vector space V. Then subtracting from one v_i a linear combination of the others yields an equivalently ordered basis, replacing one v_i with a multiple cv_i yields an equivalently ordered basis if and only $c > 0$, and transposing two elements yields an oppositely oriented basis.*

Proof. The statement follows from the fact that the corresponding elementary matrix operations change the determinant in likewise fashion. $\square$

We can now define an orientation of a manifold X by orienting the tangent spaces $T_x(X)$ in a smooth way.

Definition 3.3. An *orientation* of an n-dimensional manifold X with $n \geq 1$ is a choice of orientation for the tangent spaces $T_x(X)$ such that for each $x \in X$ there exists a a neighborhood V of x and a local parametrization $\phi : U \to V$ for which at each $u \in U$ the map $d\phi_u : \mathbb{R}^n \to T_x(X)$ preserves orientation. That is, $d\phi_u$ sends the standard basis of $\mathbb{R}^n$ to a positively oriented basis of $T_x(X)$. The manifold X is said to be orientable if it admits an orientation. If X is zero-dimensional, then it is orientable and we define the orientation by assigning to each $x \in X$ an orientation number $+1$ or -1.

One can check whether two orientation-preserving diffeomorphisms agree by calculating the determinant of their transition function. That is, for any $x \in X$ with two local parametrizations $\phi : U \to X$ and $\psi : U' \to X$ satisfying $\phi(0) = x = \psi(0)$ and $\phi(U) = \psi(U')$, the induced orientations on the tangent space $T_x(X)$ of a point $x \in U$ agree if and only if $d(\phi^{-1})_0 \circ d(\psi)_0 = d(\phi^{-1} \circ \psi)_0 : \mathbb{R}^n \to \mathbb{R}^n$ preserves orientation and thus has a positive determinant. In particular, since the determinant is continuous, this shows that a connected orientable manifold admits exactly two orientations.

Examples 3.4. It is trivial to show that we can orient the open unit ball B^n or any manifold which has a global parametrization.

We may also orient S^n using two stereographic projection parametrizations. That is, given two parametrizations $h : S^n \setminus \{e_{n+1}\} \to \mathbb{R}^n$ and $h' : S^n \setminus \{-e_{n+1}\} \to \mathbb{R}^n$, one can calculate that at any point the derivative of the transition function described by $f(x) = h' \circ h^{-1}(x) = \frac{x}{\|x\|^2}$ has positive determinant.

The Möbius strip is an example of a non-orientable manifold. Take the unit square $[0,1] \times [0,1]$ and identify boundary points by the relation $(0, x) \sim (1, -x)$ and $(x, 0) \sim (x, 1)$. Then the Möbius strip is the quotient $[0,1] \times [0,1]/\sim$. Suppose that the Möbius strip is orientable. Then it has a covering by local parametrizations such that all of them, as well as their transition functions, are orientation-preserving. However, it is not hard to show that the orientation is reversed as you go around the center circle, resulting in a contradiction.

Henceforth we will be concerned primarily with oriented manifolds. Before proceeding we need a way to orient different pieces and compositions of manifolds. First we focus on orienting the boundary of a manifold with boundary by using the outward normal vector of a boundary point.

If $X \subset \mathbb{R}^k$ is an n-dimensional manifold with boundary, then at a point $x \in \partial X$ the tangent space of the boundary $T_x(\partial X)$ is a $(n-1)$-dimensional subspace of the total tangent space $T_x(X)$. By the Gram-Schmidt algorithm we can find a unit vector w in $T_x(X)$ orthogonal to $T_x(\partial X)$. Given a local parametrization of X by $\phi : U \to X$ with $\phi(x) = x$ we have shown in proposition (2.7) that the restriction of ϕ to ∂H^n is a parametrization of ∂X, which implies that the restriction of the derivative $d\phi_0$ to ∂H^n is an isomorphism onto $T_x(\partial X)$. Thus $(d\phi_0)^{-1}(w)$ lies in one of $H^n \setminus \partial H^n$ or $-H^n \setminus \partial H^n$. Denote the outward normal vector n_x whichever of w or $-w$ is mapped from $-H^n$.

Lemma 3.5. *The outward normal vector $n_x \in T_x(X)$ is independent of parametrization.*

Proof. Take a local parametrization of X by $\phi : U \to X$ with $\phi(0) = x$. Extend this parametrization to a function ϕ' defined on a neighborhood U' of $\mathbb{R}^n$. Consider a vector $v \in H^n \setminus \partial H^n$. Define a path $c : (-\epsilon, \epsilon) \to U$ by $c(t) = tv$, where ϵ is chosen to be small enough so that the path is contained in U'. Then the restriction of $g = \phi \circ c : (-\epsilon, \epsilon) \to X$ to nonnegative values is a path in X with $dg_0 = d\phi_0 \circ dc_0 = d\phi_0(v)$. Thus $d\phi_0(v)$ must point into the manifold. We can just as easily reverse these steps to show that if $w \in T_x(X)$ is the velocity vector of some path in X then $d\phi_0^{-1}(w)$ lies in H^n. It follows that $d\phi_0$ sends H^n exactly to velocity vectors of paths in X. By taking complements $d\phi_0$ sends $-H^n$ to vectors in the tangent space which are not velocity vectors of paths in X, or to vectors which point out of X. The image of $-H^n$ is thus independent of parametrization. $\square$

Now that the normal vector n_x is well-defined, we can orient the tangent space of a point in the boundary $T_x(\partial X)$ by giving a basis $\beta = \{v_1, ..., v_n\}$ a positive orientation if the basis $\{n_x, \beta\}$ is positively oriented in $T_x(X)$. First note that the boundary orientation smoothly orients ∂H^n in H^n. To generalize to arbitrary manifolds with boundary, recall from proposition (2.7) that at any point x the restriction of a local parametrization $\phi : U \to X$ around x to the boundary $U \cap \partial H^n$ yields a parametrization of $\partial\phi$ of ∂X. One can check that if ϕ is orientation-preserving for X, then $\partial\phi$ is orientation-preserving for ∂X with the boundary orientation. Hence the boundary orientation is smooth.

Example 3.6. The closed interval $[0, 1]$ is a manifold with boundary. At 1 the outward normal vector of 1 determines a positive orientation, whereas at 0 the outward normal vector of -1 determines a negative orientation. Thus the orientation numbers of the boundary sum to 0.

Given an n-dimensional manifold without boundary X and an m-dimensional manifold with boundary Y, we orient the product manifold $X \times Y$ at a point (x, y) by selecting a basis α for $T_x(X)$ and a basis β for $T_y(Y)$. This defines a basis $(\alpha \times 0, 0 \times \beta)$ for the product tangent space $T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y)$. Define the orientation of $T_x(X) \times T_y(Y)$ by $\operatorname{sgn}(\alpha \times 0, 0 \times \beta) = \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta)$. Smoothness follows from the fact that given orientation-preserving parametrizations $\phi : U \to X$ and $\psi : V \to Y$, the product parametrization $\phi \times \psi$ is orientation-preserving on $U \times V$. Also, it is easy to show that this choice of orientation is independent of the choice of bases α and β.

Example 3.7. Let X be a manifold without boundary and consider the manifold with boundary $[0,1] \times X$. For $(1, x)$ in the boundary $\{1\} \times X$ note that the tangent space of the boundary submanifold is given by $T_1(\{1\}) \times T_x(X) = 0 \times T_x(X)$. Taking a basis β for $T_x(X)$, orient a basis $0 \times \beta$ of $0 \times T_x(X)$ by giving it the orientation of $\{n_{(1,x)}, 0 \times \beta\}$, where $n_{(1,x)}$ is the outward normal $n_{(1,x)} = (1,0) \in T_1([0,1]) \times T_x(X)$. The product orientation determines $\operatorname{sgn}(1 \times 0, 0 \times \beta) = \operatorname{sgn}(1)\operatorname{sgn}(\beta) = \operatorname{sgn}(\beta)$, which implies $1 \times X$ is a copy of X with the correct orientation. By an analogous argument $0 \times X$ is a copy of X with the opposite orientation.

Lastly, we seek to derive the orientation for the preimage of a regular value. The goal of intersection theory is to understand the relationship between a smooth map and the orientation it induces on its preimage manifolds. Since we would like to describe this relationship simply in terms of one number, we shall engineer the conditions so that the preimage consists of only finitely many points. Let X and Y be boundaryless manifolds such that X is compact, Y is connected, and $\dim X = \dim Y$. Given a smooth function $f : X \to Y$ with a regular value $y \in Y$, the preimage $f^{-1}(y)$ is a compact zero-dimensional submanifold of X consisting of finitely many points.

To determine the orientation of a point $x \in f^{-1}(y)$ we use the fact that df_x is an isomorphism between the oriented vector spaces $T_x(X)$ and $T_{f(x)}(Y)$. Orient x by giving it a positive orientation if df_x preserves orientation or a negative orientation if df_x reverses orientation.

Definition 3.8. Let X and Y be manifolds without boundary such that X is compact, Y is connected, and $\dim X = \dim Y$. The *degree* $\deg(f, y)$ of a map f at a point $y \in Y$ is the sum of the orientation numbers of $x \in f^{-1}(y)$.

Proposition 3.9. *The degree* $\deg(f, y)$ *is locally constant as a function of regular values* $y \in Y$*.*

Proof. Enumerate $f^{-1}(y) = \{x_1, ..., x_n\}$. By the Inverse Function Theorem, the surjectivity of df_{x_i} implies that f is a local diffeomorphism at each x_i, which means f maps a neighborhood V_i of x_i diffeomorphically onto a neighborhood U_i of y. Using the continuity of df_x shrink the V_i so that $\det(df_x) = \det(df_{x_i})$ for $x \in V_i$ and so that the V_i are pairwise disjoint. Then $X \setminus \bigcup_{1 \le i \le n} V_i$ is a closed and therefore compact subset of X which is mapped to a compact subset of Y not containing y. Hence $W = \bigcap_{1 \le i \le n} U_i \setminus f(X \setminus \bigcup_{1 \le i \le n} V_i)$ is an open neighborhood of y such that $f^{-1}(W) \subset \bigcup_{1 \le i \le n} V_i$. For each y' in W the preimage $f^{-1}(y')$ consists of one point from each V_i having the same sign as x_i. Hence $\deg(f, y') = \deg(f, y)$. $\square$

Proposition 3.10. *The degree* $\deg(f, y)$ *is globally constant.*

Proof. For each $y \in Y$ take a neighborhood U_y such that $\deg(f, z) = \deg(f, z')$ for regular values $z, z' \in U_y$. Since the set of regular values is dense in Y it follows that each U_y has an associated degree for all the regular values it contains. Thus given a regular value z, the set of points z' such that $\deg(f, z) = \deg(f, z')$ can be written as the union of the U_y with the same associated degree as z. Likewise, the set of points z' such that $\deg(f,z) \neq \deg(f,z')$ can be written as the union of the U_y with different associated degrees as z. Since these are both open subsets whose union is the connected manifold Y, one of them, the latter in particular, must be empty. $\square$

Since the degree $\deg(f,y)$ is independent of y, we denote it simply as $\deg(f)$. The first notable property of the degree of a function is that it is invariant with respect to homotopy classes. The examples in this section have prepared the preliminaries of the proof. So as not to become distracted from our main goal, we state without proof the classification of compact, connected, one-dimensional manifolds. Consult appendix 2 of [1] for a proof.

Proposition 3.11. *Every compact, connected, one-dimensional manifold with boundary is diffeomorphic to* $[0,1]$ *or* S^1.

Corollary 3.12. *The sum of the orientation numbers of the boundary of a compact one-dimensional manifold* X *is* 0.

Proof. Since example (3.6) shows that the statement is true for every connected component, it must be true for the entire manifold. $\square$

Lemma 3.13. *Suppose* X *is the boundary of a compact oriented manifold* W. *If a smooth map* $f : X \to Y$ *extends to a smooth map* $F : W \to Y$ *then* $\deg(f) = 0$.

Proof. Select $y \in Y$ which is a regular value of both f and F. Then $F^{-1}(y)$ is a compact one-dimensional manifold. Thus the orientation numbers of the boundary sum to 0. But the boundary is just $\partial(F^{-1}(y)) = \partial W \cap F^{-1}(y) = X \cap F^{-1}(y) = f^{-1}(y)$. $\square$

Theorem 3.14. *If* f *and* g *are homotopic maps from* X *to* Y *then* $\deg(f) = \deg(g)$.

Proof. Take a homotopy $F : [0,1] \times X \to Y$ between f and g such that $F(0,x) = f(x)$ and $F(1,x) = g(x)$, and select $y \in Y$ which is a regular value for F,f, and g. By lemma (3.13), $\deg(\partial F) = 0$. But $\partial F \mid_{0\times X} = f$ and $\partial F \mid_{1\times X} = g$. Recalling from example (3.7) that $1 \times X$ is a copy of X with the correct orientation and $0 \times X$ is a copy of X with the opposite orientation, we conclude $0 = \deg(\partial F) = \deg(f) - \deg(g)$. $\square$

4. Homotopy Classes and Framed Cobordisms

After defining the degree of a smooth map of manifolds and demonstrating that it is constant within homotopy classes, the next natural question, which defines the central investigation of this paper, concerns how we may conversely use degree to infer homotopy classes. It turns out that this reverse project is notably more difficult, as two maps having the same degree are not necessarily homotopic in arbitrary circumstances.

Example 4.1. The 2-torus T^2 consists of the product $S^1 \times S^1$. It can also be represented as the quotient $\mathbb{R}^2/\mathbb{Z}^2$, where two points $a, b \in \mathbb{R}^2$ are identified if $a - b \in \mathbb{Z}^2$. For some positive integer n, consider the linear maps

$$A = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$$

from $\mathbb{R}^2$ to itself. Since A and B are linear and send $\mathbb{Z}^2$ to itself, it follows that they are well-defined on the equivalence classes in $\mathbb{R}^2/\mathbb{Z}^2$; thus we can take them to be mappings from T^2 to itself. In particular, the map A can be thought of as making n revolutions around $S^1 \times y$ for each $y \in S^1$, and similarly B makes n revolutions around $x \times S^1$ for each $x \in S^1$. Note that at any point x in T^2 the determinant of the derivative $\det(dA_x) = \det(A) = n > 0$, and likewise for B. Thus for either map the preimage of any point consists of n points at which the determinant of the derivative is positive, so we conclude $\deg(A) = n = \deg(B)$.

However, by informally invoking some elementary algebraic topology (see chapter 1 of [2]) we can show that A and B are not homotopic. The fundamental group $\pi(X)$ of a path-connected space X consists of the homotopy classes of maps from S^1 to X, or loops on X from any designated basepoint $x_0 \in X$; the group product of two homotopy classes is determined by taking a representative loop from each class and concatenating them. In particular, the fundamental group of $S^1 \times S^1$ is the product $\mathbb{Z}^2$. It is clear by the description that A induces a map $A_* : \mathbb{Z}^2 \to \mathbb{Z}^2$ which sends (a, b) to (na, b), whereas B induces map B_* which sends (a, b) to (a, nb). Since the associated group homomorphisms A_* and B_* differ, the maps A and B cannot be homotopic.

While the example shows that two maps of the same degree to the torus may not be homotopic, we shall expand on the differential techniques introduced in prior sections to investigate homotopy classes in the particular case of maps to the n-dimensional sphere based on a construction developed by Pontryagin. This section is primarily based on chapter 7 of [3].

Definition 4.2. Let X be a compact manifold. A *framing* of a submanifold $Z \subset X$ with codimension n in X is a smooth map α assigning to every $x \in X$ a basis $\alpha(x)$ for the orthogonal complement of its tangent space $T_x(Z)^\perp \subset T_x(X)$. The pair (Z, α) is a *framed submanifold* of X.

A smooth map $f : X \to S^n$ with regular value $y \in S^n$ induces a framing on its preimage submanifold $f^{-1}(y)$: for each $x \in f^{-1}(y)$ the derivative $df_x : T_x(X) \to T_y(S^n)$ is surjective with kernel $T_x(f^{-1}(y))$. Hence df_x maps the orthogonal complement $T_x(f^{-1}(y))^\perp$ isomorphically onto $T_y(S^n)$. Given a positively oriented basis $\beta = \{v_1, ..., v_n\}$ for $T_y(S^n)$, construct a framing $\alpha(x) = \{w_1(x), ..., w_n(x)\}$ of $f^{-1}(y)$ by letting $w_i(x)$ be the unique vector in $T_x(f^{-1}(y))^\perp$ mapped to v_i by df_x. The induced framing, denoted by $f^*(\beta)$, is smooth due to the smoothness of df_x as a function of x.

Definition 4.3. Let X be a compact manifold. Given a map $f : X \to S^n$, a regular value y, and a positive basis β for the tangent space $T_y(S^n)$, we call the *Pontryagin manifold* of f with respect to y and β to be the framed preimage submanifold $(f^{-1}(y), f^*(\beta))$.

We partition framed submanifolds of X by identifying any submanifolds Z and Z' which can be connected by a framed manifold with boundary in $X \times [0, 1]$ whose boundary is Z on one side and Z' on the other:

Definition 4.4. Two compact framed submanifolds Z and Z' of X having codimension n are *cobordant* if for some positive ϵ the subset $Z \times [0, \epsilon) \cup Z' \times (1-\epsilon, 1]$ of $X \times [0,1]$ can be extended to a compact framed submanifold $Y \subset X \times [0,1]$ such that $\partial Y = Z \times 0 \cup Z' \times 1$ and Y does not intersect $X \times 0 \cup X \times 1$ except at points in ∂X. If (Z, α) and (Z', β) are framed submanifolds of X then they are *framed cobordant* if there exists a framed cobordism (Y, η) such that the framing η agrees with the framings α and β on the ϵ-extensions $Z \times [0, \epsilon)$ and $Z' \times (1-\epsilon, 1]$ of the respective boundaries.

By concatenating two framed cobordisms which both agree on one boundary, it is easy to see that framed cobordism is an equivalence relation.

Proposition 4.5. *If β and β' are two different positively bases for the tangent space of a regular value y, then the Pontryagin manifolds $(f^{-1}(y), f^*(\beta))$ and $(f^{-1}(y), f^*(\beta'))$ are framed cobordant.*

Proof. We may identify the space of invertible matrices over $T_y(S^n)$ with $GL_n(\mathbb{R})$, or the space of invertible matrices over $\mathbb{R}^n$. Let A denote the matrix with positive determinant satisfying $\beta' = A\beta$. Since the space of matrices with positive determinant is path-connected we can choose a smooth path A_t from the identity matrix to A. Then defining the maps $f_t : X \times \{t\} \to S^n$ by $f_t(x,t) = f(x)$, it is clear that $f_t^*(A_t\beta)$ for each t provides a framing for the cobordism $f^{-1}(y) \times [0,1]$. □

For convenience, given a regular value y and a positive basis β we may refer to the Pontryagin manifold $(f^{-1}(y), f^*(\beta))$ as simply $f^{-1}(y)$ unless otherwise indicated.

Example 4.6. Suppose an n-dimensional manifold X is connected and has two 0-dimensional submanifolds $\{y\}$ and $\{z\}$. A framing of a 0-dimensional submanifold is simply a choice basis for the tangent space at every point. Let β_y be the framed basis for $T_y(X)$ and β_z the framed basis for $T_z(X)$. Suppose β_y and β_z have the same orientation number for their respective tangent spaces $T_y(X)$ and $T_z(X)$; without loss of generality we can suppose this orientation number is $+1$. We shall construct a framed cobordism between the submanifolds $\{y\}$ and $\{z\}$.

Since X is connected, there exists a smooth path between y and z. Then it is easy to see there exists a smooth path $c : [0,1] \to X \times [0,1]$ between $(y, 0) \in X \times 0$ and $(z, 1) \in X \times 1$, thereby providing a cobordism between these two points.

We shall outline a construction of a framing for c. Suppose the space $X \times [0,1]$ is a subset of $\mathbb{R}^k$. At any point $x \in X \times [0,1]$ we can view the tangent space $T_x(X \times [0,1])$ as a smoothly varying $(n+1)$-dimensional subspace of $\mathbb{R}^{k+1}$. First determine a basis of $T_{(y,0)}(X \times [0,1]) = T_y(X) \times T_0([0,1])$ by $\{(\beta_y, 0), (0, 1)\}$, which is positively oriented by the product orientation. We can smoothly rotate this basis along the path so that at any point $c(t)$ the last basis vector v_t is always tangent to the path and orthogonal to the subspace spanned by the first n vectors (see figure (2) left for an example where $X = S^1$). Note that the removal of v_t from each basis determines a smooth framing on the path. Also, one can check using local parametrizations that these rotations preserve the orientation. Thus there exists a basis γ of $T_z(X)$ such that the resulting positively oriented basis for $T_{(z,1)}(X) \times T_0([0,1])$ is given by $\{(\gamma, 0), (0, 1)\}$. By the product orientation γ must be positively oriented. Using a procedure analogous to the one in proposition (4.5), we may extend this framed cobordism to rotate the basis γ of $T_z(X)$ to β_z.

By an analogous argument one can construct a framed cobordism between the empty set and the union of two points y and z whose framed bases are oppositely oriented (see figure (2) right).

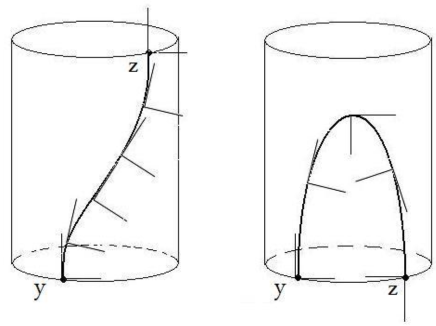

FIGURE 2. A construction of a framed cobordism.

As we did analogously with degree, the first goal is to show that we can associate to each smooth map a single corresponding framed cobordism class which is independent of the choice of regular value and positively oriented basis.

Lemma 4.7. *If y is a regular value of f and z is sufficiently close to y then $f^{-1}(y)$ is framed cobordant to $f^{-1}(z)$.*

Proof. The set of critical points is closed and therefore compact in X; hence the image is compact and therefore closed in Y. We can select a positive ϵ such that the ball of radius ϵ around y contains only regular values. For $z \in B_\epsilon(y)$ choose a smooth family of rotations $r_t : S^n \to S^n$ around the axis perpendicular to the great circle containing y and z such that $r_t =$ identity for $t \in [0, \epsilon')$ and $r_t(y) = z$ for $t \in (1 - \epsilon', 1]$ for some positive ϵ'. Observe that for any t the value $r_t^{-1}(z)$ lies on the great circle from y to z and hence is a regular value of f. Define a homotopy $F : X \times [0, 1] \to S^n$ by

$$F(x, t) = r_t \circ f(x).$$

Since $r_t^{-1}(z)$ is a regular value of f, it follows that z is a regular value of $r_t \circ f$ for each t, which implies z is a regular value of F. Thus $F^{-1}(z) \subset X \times [0, 1]$ is a framed cobordism between $f^{-1}(z)$ and $(r_1 \circ f)^{-1}(z) = f^{-1}(y)$. □

Lemma 4.8. *If y and z are two regular values of f, then $f^{-1}(y)$ is framed cobordant to $f^{-1}(z)$.*

Proof. The proof is exactly analogous to the proof of proposition (3.10). □

We conclude that every smooth map has a unique framed cobordism class. As with degree, we shall also check that the framed cobordism class is invariant with respect to homotopy classes.

Theorem 4.9. *If f and g are smoothly homotopic and y is a regular value for both, then $f^{-1}(y)$ is framed cobordant to $g^{-1}(y)$.*

Proof. Choose a homotopy F with $F(x,t) = f(x)$ for $t \in [0,\epsilon)$ and $F(x,t) = g(x)$ for $t \in (1-\epsilon, 1]$ for some positive ϵ. Choose a regular value z of F close enough to y so that $f^{-1}(z)$ is framed cobordant to $f^{-1}(y)$ and $g^{-1}(z)$ is framed cobordant to $g^{-1}(y)$. Then $F^{-1}(z)$ provides a framed cobordism between $f^{-1}(z)$ and $g^{-1}(z)$. By transitivity $f^{-1}(y)$ is framed cobordant to $g^{-1}(y)$. □

Conversely, we shall find that the framed cobordism class associated to a map determines its homotopy class. To prove this result we begin with the simpler case of when two Pontryagin manifolds exactly coincide. We first simplify the neighborhood of a framed submanifold, commencing with a lemma which generalizes the Inverse Function Theorem:

Lemma 4.10. *Let $f : X \to Y$ be a smooth map of manifolds such that the derivative df_x is an isomorphism whenever x lies in a compact submanifold $Z \subset X$ and that f maps Z diffeomorphically onto $f(Z)$. Then f maps a neighborhood of Z diffeomorphically onto a neighborhood of $f(Z)$.*

Proof. By the Inverse Function Theorem f is a local diffeomorphism around every $x \in X$. Thus it suffices to show that f is injective on a neighborhood of Z to conclude that it maps this neighborhood diffeomorphically onto a neighborhood of $f(Z)$. If not, take sequences $\{a_i\}$ and $\{b_i\}$ approaching Z such that $f(a_i) = f(b_i)$. Since Z is compact there exists a subsequence $\{a_{i_k}\}$ converging to a point z_a in Z. Similarly there exists a subseqence $\{b_{i_k}\}$ converging to z_b. A quick application of the triangle inequality shows that $z_a = z_b$, which contradicts the the fact that f is a diffeomorphism in a neighborhood of z_a. □

Theorem 4.11 (Product Neighborhood Theorem)**.** *If $(Z, \alpha) \subset X$ is a compact framed submanifold whose codimension in X is n, it has a neighborhood diffeomorphic to the product $Z \times \mathbb{R}^n$. The diffeomorphism can be chosen so that each $x \in Z$ corresponds to $(x, 0) \in Z \times \mathbb{R}^n$ and so that each framing basis $\alpha(x)$ corresponds to the standard basis of $\mathbb{R}^n$.*

To avoid extended digression we shall expicitly prove the theorem for the special case $X = \mathbb{R}^m$ while providing an outline of the proof for arbitrary X.

Proof. First, suppose $X = \mathbb{R}^m$. Define a map $f : Z \times \mathbb{R}^n \to \mathbb{R}^m$ by

$$f(z, t_1, ..., t_n) = z + \sum_{i=1}^{n} t_i \beta_i(z).$$

We first show that $df_{z;0}$ is nonsingular. Note that a point $(z, 0)$ is in the intersection of the submanifolds $Z \times 0$ and $\{z\} \times \mathbb{R}^n$, so we can evaluate f as a map restricted to either submanifold. In the first case, it is easy to check that df_x maps the tangent space $T_z(Z \times 0)$ onto $T_z(Z)$ by sending $(v, 0)$ to v. In the second case, df_x is the isomorphism which sends the standard basis of $\mathbb{R}^n$ to the framed basis $\beta(z)$ of the normal space. Thus at a point $(z, t_1, ..., t_n)$ the derivative df_x maps onto $T_z(Z) + T_z(Z)^{\perp} = \mathbb{R}^m$.

By lemma (4.10), f maps a neighborhood $Z \times U$ of $Z \times 0$ diffeomorphically onto a neighborhood V of $Z \subset \mathbb{R}^n$. By shrinking we can choose U to be a ball of radius ϵ around 0. To finish the theorem, note that U is diffeomorphic to $\mathbb{R}^n$ by a map similar to the one in examples (2.3). Thus the diffeomorphism f extends to a diffeomorphism defined on $Z \times \mathbb{R}^n$.

We generalize the diffeomorphism to geodesics of an arbitrary manifold X using the normal bundle

$$N(Z,X) = \{(z,v) \mid z \in Z, v \in T_z(X) \text{ and } v \perp T_z(Z)\}.$$

Consult theorem 20 of chapter 9 of [5] for a proof that Z has an open neighborhood U in X diffeomorphic to the normal bundle $N(Z,X)$, where $Z \subset X$ corresponds to the submanifold $Z \times 0 \subset N(Z,X)$. We construct the product neighborhood using a map $f : Z \times \mathbb{R}^n \to N(Z,X)$ defined by

$$f(z,t_1,...,t_n) = (z, \sum_{i=1}^{n} t_i\beta_i(z)),$$

which then corresponds to the appropriate geodesic by the identification of $N(Z,X)$ with U. □

Equipped with the Product Neighborhood Theorem, we first consider the case where two Pontryagin manifolds exactly coincide.

Lemma 4.12. *If the n-dimensional framed preimage $(f^{-1}(y), f^*(\beta))$ is equal to $(g^{-1}(y), g^*(\beta))$, then f is smoothly homotopic to g.*

Proof. Let $Z = f^{-1}(y)$. If f coincides with g on a neighborhood V of Z, then let $h : S^n \setminus y \to \mathbb{R}^n$ be stereographic projection and define a homotopy F by

$$F(x,t) = \begin{cases} f(x), & x \in V \\ h^{-1}[t \cdot h(f(x)) + (1-t) \cdot h(g(x))], & x \notin V \end{cases}.$$

That is, we use the projection to define a linear homotopy in $\mathbb{R}^n$ which is then transformed into the corresponding geodesic on S^n.

If f does not coincide with g on an entire neighborhood then F is no longer guaranteed to be smooth around Z. In this case we shall construct a map f' homotopic to f that does coincide with g on a neighborhood of X and that maps no new points to y. Choose a product representation $\phi : Z \times \mathbb{R}^n \to V \subset X$, with V chosen small enough so that $f(V)$ and $g(V)$ do not contain the antipode $-y$ of y. Let h be stereographic projection from $-y$. Then we can define a map $F : Z \times \mathbb{R}^n \to \mathbb{R}^n$ by $F = h \circ f \circ \phi$, and similarly we define a map $G = h \circ g \circ \phi$.

Since f and g induce the same framing it follows they must have the same derivative dg_x for points $x \in Z$, which by the chain rule implies

$$dF_{(x,0)} = dG_{(x,0)} = dh_y \circ dg_x \circ d\phi_{(x,0)}.$$

Since F and G agree up to second-order terms, we seek a neighborhood U around $Z \times 0$ such that the line-segment between $F(x,u)$ and $G(x,u)$ misses 0 for $x \in Z$ and $u \in U$, allowing us to apply a linear homotopy $(1-t)F(x,u)+tG(x,u)$ without moving new points into 0 at any point along the deformation.

We may suppose β is the basis of $T_y(S^n)$ that dh_y sends to the standard basis of $\mathbb{R}^n$. Recall that $d\phi_{(x,0)} : T_x(Z) \times \mathbb{R}^n \to T_{\phi(x,0)}X$ maps the tangent space $T_x(Z) \times 0$ identically onto $T_x(Z)$ and maps the standard basis of $\mathbb{R}^n$ to the framed basis $f^*(\beta)(x)$ of $T_x(Z)^\perp$. Then df_x maps $T_x(Z)$ to 0 and maps $T_x(Z)^\perp$ isomorphically onto $T_x(S^n)$, sending the framed basis $f^*(\beta)(x)$ to the positively oriented basis β. Finally dh_y sends the basis β of $T_y(S^n)$ to the standard basis of $T_0(\mathbb{R}^n) = \mathbb{R}^n$. Hence $dF_{(x,0)}$ and $dG_{(x,0)} : T_x(Z) \times \mathbb{R}^n \to \mathbb{R}^n$ are projections onto $\mathbb{R}^n$.

We shall use this to find a constant c so that $F(x,u)\cdot u > 0$ and $G(x,u)\cdot u > 0$ for $x \in Z$ and $0 < \|u\| < c$, implying that $F(x,u)$ and $G(x,u)$ lie in the same open half-space and thus do not bound a line-segment which passes through 0. First we find such a c_1 for F. For an arbitrary $x \in Z$ define $F_x : \mathbb{R}^n \to \mathbb{R}^n$ by $F_x(u) = F(x,u)$. Since df_x maps $T_x(Z) = d\phi_{(x,0)}(T_x(Z)\times 0)$ to 0, the restriction of F does not change $dF_{(x,0)}$ as a map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Using the fact that $dF_{(x,0)}$ is a projection and $F_x(0) = 0$, we invoke Taylor's theorem from real analysis to approximate

$$\|F_x(u) - u\| < c_x \cdot \|u\|^2$$

for $\|u\| \le 1$. Thus

$$\begin{aligned} 2F(x,u)\cdot u &= \|F(x,u)\|^2 + \|u\|^2 - \|F(x,u) - u\|^2 \\ &> \|F(x,u)\|^2 + \|u\|^2 - c_x^2\|u\|^4 \\ &> \|u\|^2 - c_x^2\|u\|^4 > 0 \end{aligned}$$

for $0 < \|u\| < \min(c_x^{-1}, 1)$. Since the dot product is continuous by the Cauchy-Schwarz inequality, it follows $F(x,u)\cdot u > 0$ for $0 < \|u\| < \min(c_x^{-1}, 1)$ and a neighborhood $U_x \subset Z$ of x. Doing this for each $x \in Z$ provides an open cover of Z. We thus take a finite subcover and let c_1 be the minimum of the c_x^{-1}. By a similar argument find c_2 for G, then let $c = \min(c_1, c_2)$.

Now that we have taken care of the case for sufficiently small u, we avoid moving distant points by selecting a smooth cut-off function $\lambda : \mathbb{R}^n \to \mathbb{R}$ with

$$\lambda(u) = \begin{cases} 1, & \|u\| < \frac{c}{2} \\ 0, & \|u\| \ge c \end{cases}.$$

The importance of requiring no intermediate maps to move points into 0 now becomes apparent in this extension of the homotopy for the entire domain. That is, for u with $\frac{c}{2} \le \|u\| < c$ some values of $F(x,u)$ only move partially through the line-segment to $G(x,u)$.

This allows us to define a homotopy

$$F_t(x,u) = [1 - \lambda(u)t]F(x,u) + \lambda(u)tG(x,u)$$

between F and a map F_1 which coincides with G for $\|u\| < \frac{c}{2}$ and maps no new points to 0. This provides a homotopy $H_t = h^{-1}\circ F_t \circ \phi^{-1}$ between $f = H_0$ and a map $f' = H_1$ which coincides with g in a neighborhood of Z and maps no new points to y. Thus f' is homotopic to g. By transitivity f is homotopic to g. $\square$

We now generalize this result to maps whose Pontryagin manifolds are framed cobordant. We first show that any compact framed submanifold occurs as the preimage of some smooth function, which in particular will enable us to transform any framed cobordism into a homotopy. A suitable manipulation of this abstract homotopy will then provide a homotopy between two given functions with the same class of Pontryagin manifolds.

Proposition 4.13. *Any compact framed submanifold (Z,α) of codimension n in a compact manifold without boundary X occurs as the Pontryagin manifold for some smooth map $f : X \to S^n$.*

Proof. By the Product Neighborhood Theorem there exists a diffeomorphism $\phi : Z \times \mathbb{R}^n \to V \subset X$ for some neighborhood V of Z. Let $\psi : Z \times \mathbb{R}^n \to \mathbb{R}^n$ be the projection $\psi(x, y) = y$. Then define a projection $\pi : V \to \mathbb{R}^n$ by

$$\pi = \psi \circ \phi^{-1}.$$

It is clear $\pi^{-1}(0) = Z$. Furthermore, at a point $z \in Z$ the derivative $d\pi_z = d\psi_{(z,0)} \circ d\phi_z^{-1}$ associates the framed basis of α of Z to the standard basis β of $\mathbb{R}^n$, so the Pontryagin manifold $(\pi^{-1}(0), \pi^*(\beta))$ is the framed manifold (Z, α).

We now simply alter π to a function defined on all of X whose image is contained in S^n. Begin by choosing a basepoint p and the stereographic projection map h. Define a map $\eta : \mathbb{R}^n \to S^n$ which maps the open unit ball in $\mathbb{R}^n$ diffeomorphically into S^n while mapping the complement to the basepoint. That is,

$$\eta(x) = \begin{cases} h^{-1}\left(\frac{x}{\lambda(\|x\|)}\right), & \|x\| < 1 \\ p, & \|x\| \geq 1 \end{cases},$$

where λ is a smooth monotone decreasing cut-off function with $\lambda(t) > 0$ for $t < 1$ and $\lambda(t) = 0$ for $t \geq 1$. Define a map $f : X \to S^n$ by

$$f(x) = \begin{cases} \eta(\pi(x)), & x \in V \\ p, & x \notin V \end{cases}.$$

Since points in V which are further away from Z are associated to vectors with larger magnitude in the product representation, it is clear that f is smooth. It is easy to see $\eta(0) = -p$ is a regular value. Furthermore, if we denote by μ the positively oriented basis of $T_{\eta(0)}(S^n)$ associated to the standard basis β of $\mathbb{R}^n$ by the stereographic projection $dh_{\eta(0)}$, then we conclude

$$(f^{-1}(\eta(0)), f^*(\mu)) = (\pi^{-1}(0), \pi^*(\beta)) = (Z, \alpha).$$

□

First note that by proper choice of the basepoint p one can choose (Z, α) to be the preimage of any arbitrary point in S^n. Additionally, using a boundary extension technique similar to the proof of proposition (2.18) we can extend the Product Neighborhood Theorem and thus proposition (4.13) to apply to a framed cobordism $(Y, \alpha) \subset X \times [0, 1]$.

Corollary 4.14. *Any framed cobordism (Y, α) of codimension n in the compact manifold with boundary $X \times [0, 1]$ occurs as the preimage manifold for some smooth map $F : X \times [0, 1] \to S^n$.*

Proof. If $X = \mathbb{R}^m$, then since the normal space for points near the boundary of Y is parallel in the boundary $\mathbb{R}^m \times 0 \cup \mathbb{R}^m \times 1$, we can shrink $\mathbb{R}^n$ to some ϵ-neighborhood U^ϵ so that the product map $f : Y \times U^\epsilon \to \mathbb{R}^m \times [0, 1]$ defined in theorem (4.11) is well-defined. Additionally, given a point $(x, t) \in \partial(X \times [0, 1])$, it is clear that $(x, t) = f(y, 0)$ for some $y \in \partial Y$. We can take a local parametrization $\phi : V \to Y \times U^\epsilon$ defined on a neighborhood V of H^n such that $\phi(0) = (y, 0)$. The map $g = f \circ \phi$ thus extends to a map $G : V' \to X \times [0, 1]$ defined on a neighborhood V' of $\mathbb{R}^n$ such that dG_0 is an isomorphism. Hence G is a local diffeomorphism on a neighborhood W around 0, so the restriction $G \mid W \cap H^n$ maps $W \cap H^n$ diffeomorphically onto a neighborhood of (x, t) in $X \times [0, 1]$. Having generalized lemma (4.10), the rest of the proof is exactly as before. The generalization to arbitrary X follows analogously. $\square$

Theorem 4.15. *Two smooth maps from X to S^n are smoothly homotopic if and only if their Pontryagin manifolds correspond to the same framed cobordism class.*

Proof. We have already taken care of half of the statement in theorem (4.9). If f and g have the same framed cobordism class select a regular value $y \in S^n$ for both maps and the positive basis that the projection from the antipodal point $-y$ associates with the standard basis of $\mathbb{R}^n$. Then there exists a framed cobordism (Y, α) between $f^{-1}(y)$ and $g^{-1}(y)$. Let $F : X \times [0, 1]$ be the function constructed from corollary (4.14). Evidently F provides a homotopy between $F_0 = F \mid_{t=0}$ and $F_1 = F \mid_{t=1}$. Since $(F_0^{-1}(y), F^*(\beta)) = (f^{-1}(y), f^*(\beta))$ it follows F_0 is smoothly homotopic to f, and likewise F_1 is smoothly homotopic to g. Hence by transivity f is smoothly homotopic to g. $\square$

With this construction we prove the Hopf Degree Theorem, which describes conditions for which the degree of a map does exactly determine its homotopy class:

Theorem 4.16 (Hopf Degree Theorem)**.** *If X is a a compact, connected, oriented, n-dimensional manifold without boundary, then two maps f and g from X to S^n are smoothly homotopic if and only if they have the same degree.*

As an immediate result we can use the Hopf Degree Theorem to completely characterize homotopy classes of maps under the given conditions. That is, the Hopf Degree Theorem shows that degree, taken as a map from homotopy classes of maps from X to S^n to the integers, is well-defined and injective. Additionally, for any non-zero integer k we can choose a framed manifold consisting of $|k|$ points each with a positively oriented basis or a negatively oriented basis depending on the sign of k. Then proposition (4.13) shows that this is the preimage manifold of some map that must therefore have degree k. Also, to obtain a map of degree zero we use a constant function. Hence the degree map is bijective.

To prove the Hopf Degree Theorem we require a lemma that allows us to push intersecting curves off of each other in a manifold of dimension three or greater.

Lemma 4.17. *Suppose two 1-dimensional compact submanifolds Y and Z of an n-dimensional ($n \geq 3$) manifold X intersect. Then for any open set U of X containing $Y \cap Z$ there exists a curve Y' that does not intersect Z and which agrees with X outside of U.*

We shall omit the proof of the lemma since it covers only a minor technicality in the proof of the Hopf Degree Theorem yet involves several ideas that are tangential to the primary focus of this paper. It is a straightforward consequence of the stability of embeddings, which is proven in section 6 of chapter 1 of [1], and the Transversality Homotopy Theorem, which is proven in section 3 of chapter 2 of [1]. We now proceed to the proof of the Hopf Degree Theorem:

Proof (Hopf Degree Theorem). We need only prove the converse of theorem (3.14). Select a common regular value $y \in S^n$ and a positively oriented basis β for the tangent space at y. Recall that the degree of a map f is equal to the sum of the orientation numbers of its preimage $\sum_{x \in f^{-1}(y)} \operatorname{sgn}(x)$. Note also that the induced framing provides each $x \in f^{-1}(y)$ a basis for the tangent space $T_x(X)$ determined by $df_x^{-1}(\beta)$ for some positive basis β. Whether or not this induced basis has positive orientation depends on if df_x preserves orientation. Without loss of generality suppose $\deg(f) = \deg(g) = k \geq 0$.

For $n \geq 2$, pair off points of opposite orientation number in $f^{-1}(y)$ with cobordisms to the empty set until there remain points of a single orientation number, then do the same for $g^{-1}(y)$. The degree is precisely equal to the remaining orientation numbers, so we may connect each remaining point in $f^{-1}(y)$ to a remaining point in $g^{-1}(y)$ by a cobordism. By lemma (4.17) we may assume these cobordisms do not intersect. Thus their union provides a cobordism between $f^{-1}(y)$ and $g^{-1}(y)$. By example (4.6) we may put a framing on each connected component to yield a framed cobordism between the framed manifolds $(f^{-1}(y), f^*(\beta))$ and $(g^{-1}(y), g^*(\beta))$. By theorem (4.15), f and g are smoothly homotopic.

For $n = 1$ we implement the same procedure but with slightly more care since we can no longer push curves off of each other. If X is one-dimensional then it is diffeomorphic to S^1 by proposition (3.11), so we may assume $X = S^1$. The problem now amounts to proving that degree maps homotopy classes of self-maps of S^1 bijectively onto $\mathbb{Z}$. This is well-known by elementary methods, but I shall provide a proof using the methods of this paper to keep it self-contained. Choose a point $x \notin f^{-1}(y) \cup g^{-1}(y)$ and take the projection $\phi : S^1 \setminus \{x\} \to \mathbb{R}$. We can list $f^{-1}(y) = \{x_1, ..., x_m\}$ by letting x_i be the ith element of $f^{-1}(y)$ encountered by traveling clockwise around a full rotation of S^1 starting from x. Take a negatively oriented point x_i sequentially adjacent to a positively oriented point in the enumeration of $f^{-1}(y)$; without loss of generality we can suppose the positively oriented point is x_{i+1}. Using the stereographic projection we can construct a framed cobordism between $\{x_i, x_{i+1}\}$ and the empty set that does not intersect the constant-path cobordisms of the other points. We continue doing this procedure until only k positively oriented points remain in a manifold $Z = \{x_1, ..., x_k\}$ framed cobordant to $f^{-1}(y)$. Do the same for $g^{-1}(y)$ to attain a manifold $Z' = \{x'_1, ..., x'_k\}$ framed cobordant to $g^{-1}(y)$. We can construct a framed cobordism between x_i and x'_i for $1 \leq i \leq k$; additionally, using the stereographic projection we can construct the smooth paths to be pairwise disjoint. The union of each of these framed cobordisms provides a framed cobordism between Z and Z'. Hence $(f^{-1}(y), f^*(\beta))$ and $(g^{-1}(y), g^*(\beta))$ are framed cobordant, so f and g are smoothly homotopic. $\square$

We end by remarking that Pontryagin used this construction to characterize other homotopy classes as well, such as of maps from S^{n+1} to S^n. These results illustrate the scope of differential techniques before homotopy classification became primarily a project of algebraic topology.

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