COMPOSITE KNOT DETERMINANTS

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Abstract. In this paper, we will introduce the basics of knot theory, with special focus on tricolorability, Fox r-colorings of knots, and knot determinants. We will use these techniques to generalize tricolorability, and discuss how knot determinants behave when we compose two knots.

Contents

1. Introduction

Knot theory is the study of the various properties and behaviors of mathematical knots.

Definition 1.1. A knot is any closed non-self-intersecting loop embedded in three dimensions.

We can visualize a mathematical knot as a knot that we would encounter in our everyday experience, but with the loose ends glued together. Generally, knots in 3-space are presented as their two-dimensional projections, with breaks in the understrand denoting crossings (Figure 1).

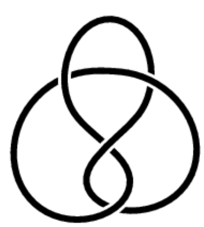

Figure 1. A 2-D projection of the figure-eight knot

Knots can be oriented. This orientation is usually denoted by placing arrows on the strands of the knot projection (Figure 2).

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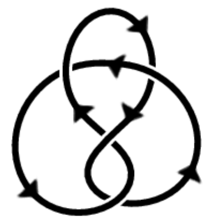


Figure 2. An oriented knot

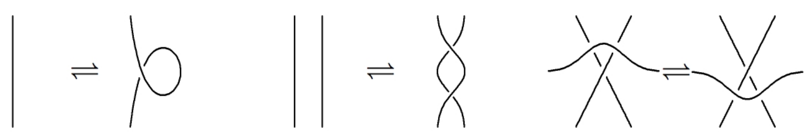
In order to rigorously study knots, we have to have some way of determining whether two projections of a knot are in fact two projections of the same knot. For the purposes of this paper, we will define equivalance up to orientation.

Definition 1.2. Two knots K and J are said to be equivalent if and only if it is possible to transform one into the other via ambient isotopy. That is, $K \sim J$ if K can be be manipulated through 3-space without ever passing through itself until it resembles J. A proof that this is an equivalence relation is obvious.

These manipulations can either be planar isotopies, which do not affect the relations between crossings, and merely change the shape of the knot within the plane of the projection, or one of three Reidemeister moves, which do change the crossing relations, and require movement in the third dimension.

Definition 1.3. A Reidemeister move is a knot manipulation of one of the following three types (Figure 3):

- **Type I.** This type allows us to add or remove a twist in a knot.
- **Type II.** This type allows us to add or remove two crossings in a knot.
- **Type III.** This type allows us to slide one strand of the knot over or under two other strands in the knot.



(a) Type I (b) Type II (c) Type III

Figure 3. The three Reidemeister moves

Planar isotopies and Reidemeister moves are the *only* manipulations of knots that are possible in three dimensions, though a proof of this is outside the scope of this paper. Therefore, we can clarify our definition of equivalent knots by saying that two knots K and J are equivalent if and only if K can be transformed into J via planar isotopies and Reidemeister moves.

2. Invariants and Tricolorability

When discussing the equivalence of knots, it is easiest to think in terms of knot invariants.

Definition 2.1. A **knot invariant** is property of a projection of a knot that is kept constant through any series of planar isotopies or Reidemeister moves.

Warning 2.2. Note that this does not imply a knot invariant can determine that two knots are equivalent (we will see this a little more clearly in our discussion of tricolorability). What invariants do tell us is that if two knots have different values for a given knot invariant, they are necessarily inequivalent.

An example of a knot invariant is tricolorability.

Definition 2.3. A projection of a knot K is tricolorable if each of the strands in the projection can be colored in one of three different colors such that at each crossing either all three colors come together or only one does. Additionally, at least two different colors must be used; a coloring using a single color is a trivial solution (Figure 4).

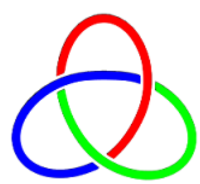


FIGURE 4. The trefoil knot: an example of tricolorability

Theorem 2.4. *Tricolorability is a knot invariant.*

Proof. To prove this, we must show that tricolorability is maintained through any of the three types of Reidemeister moves, as it is obvious that tricolorability is preserved through planar isotopies. The simplest method to do this is via illustration.

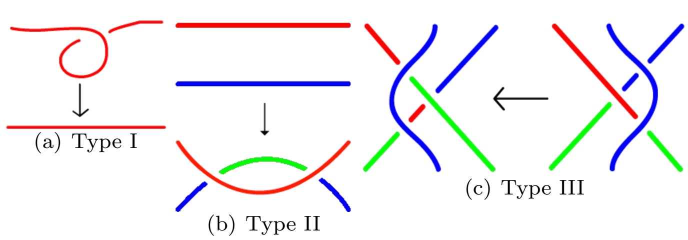


FIGURE 5. A simple illustration showing that tricolorability is invariant. All other cases proceed in a similar fashion.

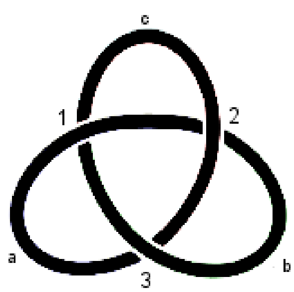
Since tricolorability is maintained through any kind of ambient isotopy, it is a knot invariant. ∎

Tricolorability is useful in classifying several knots. For example, we can see that the unknot is not tricolorable, while the trefoil knot is. Since we proved above that tricolorability is a knot invariant, the tricolorable trefoil knot can never be manipulated via planar isotopies or Reidemeister moves to become non-tricolorable. The trefoil knot, then, must be distinct from the unknot. If we can prove that a knot is tricolorable, we can immediately conclude that that knot is not the unknot. However, this cannot differentiate between any two tricolorable knots or two non-tricolorable knots.

3. Fox r-Colorings and Knot Determinants

In order to further classify knots, we must turn to some other invariant. One such alternative uses the coloring matrix of knots.

Definition 3.1. The coloring matrix, M, of a knot K with n crossings is the $n \times n$ matrix such that each column represents an arc in the 2-dimensional projection of K (a strand), and each row represents a crossing in K. We know that this matrix is square, because every strand ends in two crossings and each crossing marks the end of two strands. In these matrices, the row entry corresponding to the overstrand is a 2, while the entries corresponding to the two understrands are both -1. All other entries, i.e. those corresponding to strands that are not a one of the three at a crossing, are 0. The row sums are all zero and the entries of the rows consist of 2, -1, -1, and $(n-3)$ 0's, in some order. (Figure 6).



$$\begin{array}{c|ccc} & a & b & c \\ \hline 1 & 2 & -1 & -1 \\ 2 & -1 & -1 & 2 \\ 3 & -1 & 2 & -1 \end{array} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix}$$

Figure 6. A trefoil knot and its coloring matrix

Perhaps the most interesting observation to make about these matrices is that they are the coefficient matrices of a homogeneous system of equations, known as the coloring system of equations. The solutions of these equations modulo r are called Fox r-colorings, and are analogous to assigning each strand an integer from 1 to $r-1$ such that at each crossing, twice the overstrand is equal to the sum of both understrands, modulo r. Knots that admit r-colorings are called r-colorable.

The determinant of a given coloring matrix is a seemingly obvious choice for an invariant. However, by our definition, the rows and columns of the coloring matrix of a knot are always linearly dependent, meaning that its determinant is always 0. But, by observing the first minors of these matrices, we come upon a somewhat unique value for each knot, which will be known as the determinant of the knot.

Definition 3.2. The i, j minor of an n x n matrix, A, denoted $M_{ij}(A)$, is the determinant of the $(n-1)$ x $(n-1)$ matrix formed by removing the i^{th} row and the j^{th} column from A. Any smaller $(n-1)\times(n-1)$ matrix formed in this manner from the coloring matrix of a knot K will be denoted M'_K.

Definition 3.3. The determinant of a knot K, denoted $\det(K)$, is the absolute value of any minor of its coloring matrix.

Remark 3.4. The determinant of the unknot cannot be calculated in this manner, because the unknot has no crossings, making a coloring matrix impossible to form. Therefore, we will just define its determinant to be 1. A justification of this choice will become clear once we gain a greater understanding of the behavior of knot compositions.

Theorem 3.5. *The determinant of a knot does not depend on the labeling convention that we impose on the knot, on our choice of minor, nor on which projection of the knot we choose. Therefore, the determinant of a knot is an invariant.*

Proof. We first need to make two observations from linear algebra.

Lemma 3.6. *If we add a row and a column to a matrix that contains all 0's with the exception of a single 1 on the diagonal, the determinant remains unchanged.*

Proof. We can use an identity of matrix determinants known as the cofactor expansion along a row to prove this. This identity says that for a matrix $A = (a_{i,j})$ and a fixed i^{th} row:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \cdot a_{i,j} \cdot \det(A_{\hat{i},\hat{j}}).$$

where $A_{\hat{i},\hat{j}}$ is the matrix formed by removing the i^{th} row and the j^{th} column.

Let A be an $n \times n$ matrix and B be the matrix formed by adding a row and a column that contain all 0's with the exception of a single 1 on the diagonal. If we expand B along this new row (called i), we have:

$$\det(B) = \sum_{j=1}^{n+1} (-1)^{i+j} \cdot b_{i,j} \cdot \det(A) = \det(A).$$

□

Lemma 3.7. *If, in a square matrix, the sum of the rows and the sum of the columns is always 0, then if we remove one row and one column, the determinant of the resulting matrix (a minor of the original matrix) is independent of our choice of row and column.*

Proof. Let A be any $n \times n$ matrix whose rows and columns all sum to 0, and let B be an $n \times n$ matrix whose entries are all 1. We can calculate the $\det(A + B)$ in terms of the (i, j) minor of A in the following manner:

(1) Add all the other rows of $A + B$ to the i^{th} row. Because of every row and column of A must add to 0, every entry in the i^{th} row of this new matrix will be n, and all other entries will remain unchanged.
(2) Add all the other columns to the j^{th} column. Now, the (i, j) entry will be n^2, all other entries in the i^{th} row and the j^{th} column will be n, and the remaining entries are not affected.

(3) Factor out n from the i^{th} row.
(4) Subtract the i^{th} row from the remaining rows. The (i,j) entry will be n, the other entries in the j^{th} column will all be zero, and the remaining entries will be as they are in A.

Therefore, $\det(A+B) = n^2 A_{i,j}$, where $A_{i,j}$ is the (i,j) minor of A, implying that all minors, and thus the absolute values of these minors, of A are equal. Thus, the absolute value of the first minor of A is independent of our row and column choice. □

By multiplying certain rows in any coloring matrix by -1, we can force the matrix to be such that all the rows and columns sum to 0, without changing the absolute value of its determinant [1]. Thus, by the previous lemma, knot determinants are independent of our labeling convention.

The rest of the proof checks the effects of each type of Reidemeister move on the coloring matrix of a knot to prove that the knot determinant is independent of a knot's diagram.

- **Type I**: Type I moves require that either the original knot diagram or the resulting one have at least one strand that acts as both the understrand and overstrand at a single crossing. Therefore, we have no way to assign a value to an entry of the coloring matrix corresponding to this crossing and strand. If we "unravel" the Type I configurations that are present in the knot, we remove this problem, and have a way of creating a proper coloring matrix.
- **Type II**: Type II crossings add two strands and two crossings to the knot diagram. In the case where this move is equivalent to performing a series of two Type I moves, we have the same problem that we did with Type I moves: one strand acts as both the under- and over-strand at a single crossing. In this case, we can remove the problem the same way we did earlier. However, if this is not the case, then the new matrix can be transformed into the old matrix with the addition of two rows containing a single 1 on the diagonal. By Lemmas 3.6 and 3.7 then, the knot determinant remains unaffected, as long as the rows and columns removed before evaluating the determinant are not the new rows or columns (i.e. the ones containing a single entry of 1 on the diagonal). The rows and columns that we can remove are precisely the ones that are present in the projection of the knot without the Type II configurations present.

Example 3.8. Consider the following projection of the trefoil knot:

[1]A more precise explanation of exactly how we may multiply rows by -1 to get a matrix with rows and columns summing to 0 can be found on pg. 44 of [4]

The coloring matrix of this diagram is:

$$\begin{bmatrix} -1 & -1 & 2 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 \\ 2 & -1 & -1 & 0 & 0 \\ -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 & -1 \end{bmatrix}$$

We can first multiply the fifth row by -1 to obtain a matrix with all of the rows and columns summing to 0. We then want to check that we can transform this matrix into the following matrix, that of the trefoil before we performed the Type II move, but with two rows and columns added, each containing only a 1 on the diagonal.

$$\begin{bmatrix} -1 & -1 & 2 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The following steps constitute one way of doing this.
(1) Add the fourth column to the first.
(2) Add the fifth row to the fourth.
(3) Add the fourth row to the second.
(4) Add the second and third rows to the fifth.
(5) Add the fifth column to the second.
(6) Add the fourth row to the fifth.

We end up with the matrix:

$$\begin{bmatrix} -1 & -1 & 2 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Other cases follow similarly.

- **Type III**: Type III moves maintain the same number of strands and crossings, and it can be checked that the matrix before the move is performed can be transformed into the matrix after the move is performed by elementary row and columns operations that do not change the determinant.

□

We can now use this new concept to view tricolorability as 3-colorability. If a knot is tricolorable, we can assign a color to each strand such that at each crossing, each strand has a different color. But we can also notice that if we equate each color with 0, 1, or 2, at each crossing, we must find all three numbers, or 3 occurences of the same number. No matter what the case, twice the overstrand is equal to the sum of the understrands, modulo 3. Thus, tricolorable knots are 3-colorable. Interestingly enough, if we study the trefoil knot (which we showed earlier to be tricolorable, and thus 3-colorable), we will find that its determinant is 3. A similar phenomenon is found with the figure 8 and cinquefoil knot, which are both 5-colorable and have determinants equal to 5. It seems natural, then, to venture a guess that for any knot K, K is r-colorable if and only if $r = \det(K)$. However, as we will see in the next section, some knots are colorable for several different r's. The composition of the trefoil knot and the figure-eight knot, shown below, is both 3-colorable and 5-colorable, and the determinant of this knot is 15.

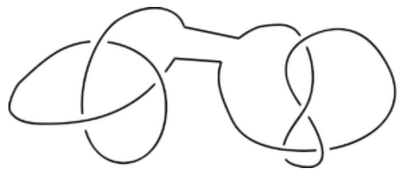


FIGURE 7. The knot sum of the trefoil knot and the figure 8 knot

This is obviously a counterexample to our previous conjecture, so we must alter its claim. We now have a new theorem:

Theorem 3.9. *For any knot K, K is p-colorable if and only if p divides* $\det(K)$, *where p is some odd prime.*

Proof. The coloring matrix of K is the $n \times n$ coefficient matrix of the coloring system of equations for K. Note that in order for us to have a nonconstant solution to this system of equations, the rank of the matrix must be at most $n-2$. This forces the minors of the matrix to be 0. Because we are working mod p, the minors of the coloring matrix need to be congruent to 0 mod p in order for the coloring system of equations to have a nonconstant solution (i.e. for there to exist a nontrivial p-coloring of K). Thus, K is p-colorable if and only if $\det(K) \equiv 0 \bmod p$ or $p | \det(K)$. □

Notice that every knot whose determinant is not 1 (not the unknot, for example) has at least one p for which it is p-colorable, and for a knot K such that $\det(K) = p_1 \cdot p_2$, K is both p_1-colorable and p_2-colorable. Every knot is r-colorable by any r that is a prime factor of its determinant.

4. Compositions of Knots

In this section we will explore what happens to the knot determinant when we compose two knots.

Definition 4.1. The knot sum of two knots K and L is denoted $K \# L$ and represents the knot formed by removing a piece of an outer arc from both knots and connecting the loose ends such that there are no additional crossings made. Without loss of generality, we will also place the following two conditions on the knot:

(1) The cut strands cannot be oriented in the same direction. That is, if i is the strand that will be cut in K and j is the strand that will be cut in L, whatever direction i is pointing, j must be pointing in the opposite direction. This ensures that the resultant knot is orientable.
(2) The connecting strands must go from being the overstrand in their original knot to the understrand in the opposite knot or vice versa. That is, the connecting strand coming from K must be an understrand in K and an overstrand in L, and similarly for the connecting strand from L. This ensures that the resulting coloring matrix is of the form that will be discussed later.

These conditions are provided to simplify matters, and proving the general case requires just slightly more sophistication.

By observing the behavior of the coloring matrices, we can come to a few conclusions about knot sums. First, let's have a look at the simplest composite knot, formed by composing two trefoil knots that we will label K and L.

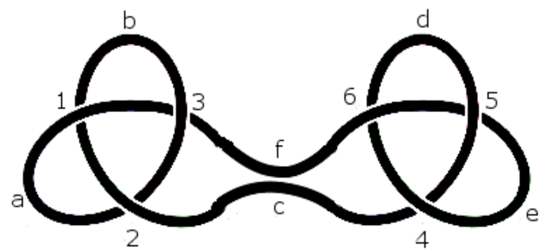


FIGURE 8. The composition of two trefoil knots.

The coloring matrices of these two knots are each:

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 2 & -1 \\ 2 & -1 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

respectively, and $\det(K) = \det(L) = 3$.

The coloring matrix of this composite knot, with the labelling shown in the figure is:

$$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

and $\det(K\#L) = 9$.

There are two things to notice here:

(1) The smaller 3x3 matrix in the upper left quadrant of the coloring matrix of $K\#L$ looks very similar to the coloring matrix of K, and the smaller 3x3 matrix in the lower right quadrant of the coloring matrix of $K\#L$ looks very similar to the coloring matrix for L. We will call these smaller similar matrices M''_K and M''_L respectively.

(2) $\det(K\#L) = \det(K) \cdot \det(L)$

The first observation is a direct result of the way we defined knot composition. If the condition seen above is not satisfied, there is always some way to relabel the strands and crossings so that the coloring matrix takes this special form.

The second observation, however, requires more justification.

Theorem 4.2. *For any two knots K and L,* $\det(K\#L) = \det(K) \cdot \det(L)$.

Proof. Let the coloring matrix of K be an $n \times n$ matrix called M_K. Let the coloring matrix of L be an $m \times m$ matrix called M_L. The coloring matrix of the composite knot $K\#L$, $M_{K\#L}$ will have the form discussed in our first observation above, that is, this matrix is of the form:

$$\begin{pmatrix} M_K & 0 \\ 0 & M_L \end{pmatrix}$$

but with the entry $a_{n,n}$ swapped with the entry $a_{n,n+m}$ and $a_{n+1,n}$ swapped with $a_{n+1,n+m}$.
In order to find the determinant of $K\#L$, we must first remove exactly one row and one column from $M_{K\#L}$. Since the value of the determinant of a knot is independent of our choice of which row or column, we can conveniently choose to remove the $(n+1)^{th}$ row and the $(n+m)^{th}$ column. When we do this, we are left with a $(m+n-1) \times (m+n-1)$ matrix of the following form:

$$\begin{pmatrix} M''_K & 0 \\ 0 & M'_L \end{pmatrix}$$

Recall that M''_K refers to the smaller matrix within the coloring matrix of the composite knot that is similar to the coloring matrix of K , and M'_L refers to the coloring matrix of L with any one row and one column removed.

Lemma 4.3. *For any knot K,* $\det(M''_K) = \det(K)$.

Proof. From our definition of the determinant of a knot, $\det(K) = \det(M'_K)$. We can use the cofactor expansion along a row to find $\det(M''_K)$ in terms of this known value, $\det(M'_K)$.
The matrix M''_K will be in the following form:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & 0 \end{bmatrix}$$

(Note that the final 0 is a result of how we defined knot composition.)
We can expand the $n \times n$ matrix $M''_K = (a_{i,j})$ along the n^{th} row, getting:

$$\det(M''_K) = \sum_{j=1}^{n} (-1)^{n+j} \cdot a_{n,j} \cdot \det(M''_{K,\hat{n},\hat{j}})$$

$M_{K,\hat{n},\hat{j}}$ is a first minor of M_K, so by Lemma 3.7, for all j,

$$\det(M''_{K,\hat{n},\hat{j}}) = \pm 1 \cdot \det(K)$$

Note that the value of the ± 1 is independent of our choice of column j. Therefore, we have:

$$\det(M''_K) = \pm \sum_{j=1}^{n} a_{n,j} \cdot \det(K).$$

Since M''_K is the same as the coloring matrix of K, with the exception of the entry in the n^{th} row and the n^{th} column, which is changed from a -1 to a 0 (a result of our requiring that the two knots be connected so that the connecting strands go from being an understrand to an overstrand). Therefore, the entries in the n^{th} row are 2,-1, and (n-2) 0's, and using our formula above, we have:

$$\begin{aligned} |\det(M''_K)| &= |(-1)\cdot\det(K) + 2\cdot\det(K)| \\ &= \det(K). \end{aligned}$$

□

A property of determinants tells us that for a matrix

$$M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where A and B are smaller square matrices,

$$\det(M) = \det(A)\cdot\det(B)$$

Since $|\det(M'_L)| = det(L)$ by definition, and the Lemma above shows that $|det(M''_K)| = det(K)$, we have the result:

$$\det(K\#L) = \det(K)\cdot\det(L)$$

□

This result shows that for any prime number p, the composition of two non-p-colorable knots is itself non-p-colorable.

5. Acknowledgements

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