REGULAR POLYTOPES IN $\mathbb{Z}^n$

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Abstract. In [3], all embeddings of regular polyhedra in the three dimensional integer lattice were characterized. Here, we prove some results toward solving this problem for all higher dimensions. Similarly to [3], we consider a few special polytopes in dimension 4 that do not have analogues in higher dimensions. We then begin a classification of hypercubes, and consequently regular cross polytopes in terms of the generalized duality between the two. Finally, we investigate lattice embeddings of regular simplices, specifically when such simplices can be inscribed in hypercubes of the same dimension.

Contents

1. Introduction

Definition 1.1. A **regular polytope** is a polytope such that its symmetry group action is transitive on its flags.

This definition is not all that important to what we wish to accomplish, as the classification of all such polytopes is well known, and can be found, for example, in [2]. The particulars will be introduced as each case comes up.

Definition 1.2. The **dual** of a polytope of dimension n is the polytope formed by taking the centroids of the $n-1$ cells to be its vertices. The dual of the dual of a regular polytope is homothetic to the original with respect to their mutual center.

We will say that a polytope is "in $\mathbb{Z}^n$" if one can choose a set of points in the lattice of integer points in $\mathbb{R}^n$ such that the points can form the vertices of said polytope.

Definition 1.3. A regular polytope in $\mathbb{Z}^n$ is called **primitive** if, when one vertex is translated to the origin, it cannot be scaled (preserving orientation) to a smaller such polytope, also embedded in $\mathbb{Z}^n$.

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We wish to classify regular polytopes embedded in the standard orthonormal lattice in Euclidean n-space. This task is a relatively simple exercise for all regular polygons in 2-space, and all the regular polyhedra in dimension 3 were classified in [3].

For convenience, we will often consider polytopes where one of the vertices is the origin (these can be used to classify all others by a simple translation). Also, any polytope named should be assumed to be regular unless stated otherwise.

2. Some special cases, and a word on convexity.

Definition 2.1. A **regular simplex** of dimension n is the polytope with the largest number of points that can be embedded in $\mathbb{R}^n$ such that all points are equidistant from each other. Alternatively, it is the polytope formed by adding one point equidistant from all points of a regular simplex of dimension $n-1$, with a 0-dimensional simplex being a point.

Definition 2.2. A **hypercube** of dimension n is the polytope formed by taking two congruent parallel hypercubes of dimension $n-1$ and edge length d and joining pairs of points, so that the distance between them is d. A hypercube of dimension 0 is one point.

Definition 2.3. A **regular cross polytope** of dimension n is the dual polytope of a hypercube of dimension n.

For dimension 5 and up, there are only three regular polytopes possible, the generalized regular simplex, the generalized hypercube, and the generalized cross polytope.

However, dimension 4 has three polytopes that do not generalize to higher dimensions, the 24-cell, the 120-cell, and the 600-cell. The 24-cell is self-dual, while the two others are mutually dual. The 24-cell does have embeddings in $\mathbb{Z}^4$, for example $(\pm 1, \pm 1, 0, 0)$ and all permutations of these coordinates.

The others, however, are too "round" to fit into the "square" Cartesian lattice. We will prove a stronger result that implies this.

Definition 2.4. A **rational lattice** is the $\mathbb{Q}$-span of some collection of basis vectors (As opposed to simply a lattice, which would be the $\mathbb{Z}$-span.).

Lemma 2.5. *The regular pentagon cannot be embedded in any* 2*-dimensional rational lattice.*

Proof. Assume otherwise. Let O be the origin, and label the other vertices as A, B, C, D so that we have pentagon $ABCDO$. If $\vec{OA}, \vec{OB}, \vec{OC}, \vec{OD}$ are in the $\mathbb{Q}$-span of some basis vectors α, β, then $\vec{OB}, \vec{OC}$ are both rational linear combinations of $\vec{OA}, \vec{OD}$, as we can perform a rational change of basis such that OA, OD span the same rational points as α, β. Now, however, we must attempt to express $\vec{OB}$ as a rational linear combination of $\vec{OA}, \vec{OD}$.

OA is parallel to DB, so $\vec{OB} = \vec{OD} + \vec{OX}$, for some X on the ray OA. However, if Y is the base of the perpendicular from D onto ray OA, then Y is the midpoint of segment AX. Thus

$$\vec{OX} = (1 + 2\sin\frac{\pi}{10})\vec{OA}$$

and

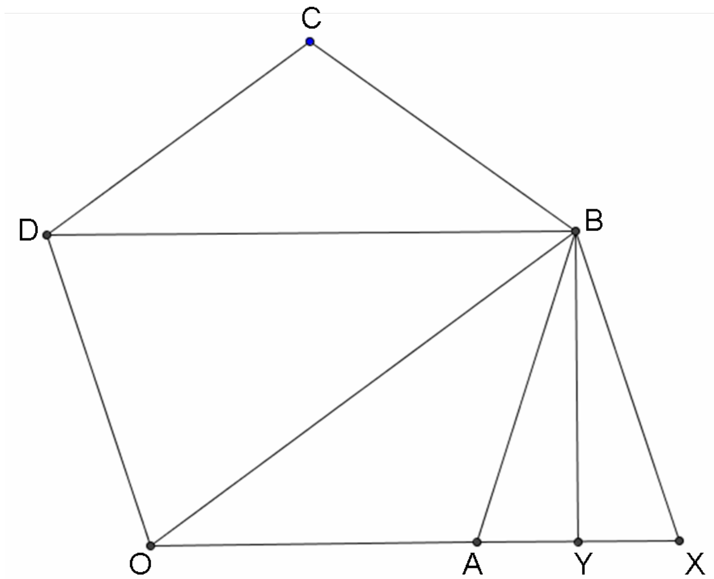

FIGURE 1. Regular Pentagon ABCDO

$$\vec{OB} = \vec{OD} + (1 + 2\sin\frac{\pi}{10})\vec{OA}$$

But $\sin\pi/10$ is irrational, so $\vec{OB}$ is not in the rational lattice spanned by α, β. □

Lemma 2.6. *The regular icosahedron and dodecahedron cannot be embedded in any 3-dimensional rational lattice.*

Proof. Again, assume otherwise. Attempt to embed the regular dodecahedron, which has regular pentagonal faces. As per the previous argument, translate one vertex to the origin and perform a rational basis change. Then the whole dodecahedron must lie in the rational points of the lattice spanned by the three edges emanating from one vertex. In particular, the pentagonal face containing two of these vectors must lie in a two-dimensional rational lattice, and this is impossible by the previous lemma.

The regular icosahedron contains regular pentagons as vertex figures (the set of vertices connected to one chosen vertex), so an identical argument applies. □

Theorem 2.7. *The regular* 120*-cell and* 600*-cell cannot be embedded in any* 4*-dimensional rational lattice.*

Proof. Assume otherwise. Attempt to embed the regular 120-cell, which has regular dodecahedral cells. Again, translate one vertex to the origin and perform a rational change of basis. Then, if the entire polytope can be embedded in a 4-dimensional rational lattice, then, in particular, one cell spanned by three such basis vectors must lie in a 3-dimensional rational lattice, and this is impossible by the previous lemma.

The regular 600-cell has regular icosahedra as vertex figures, so the argument is again identical. □

There are some polytopes which can be considered both non-convex and regular in dimensions 3 and 4. However, all contain as subsets of their vertices some of the aforementioned "bad" polyhedra, so they have no lattice embeddings to speak of.

3. Hypercubes and Cross Polytopes

Lemma 3.1. *The n-hypercube and n-cross polytope are always mutually dual.*

Proof. Embed both in $\mathbb{Z}^n$ as follows. Take the vertices of the cross polytope to be the terminal points of the vectors $\pm e_i$, where the e_i are the standard basis vectors of $\mathbb{R}^n$, and take the vertices of the hypercube to be $(\pm 1, \pm 1, ..., \pm 1)$. The centroids of the constituent $n-1$ cells of the hypercube are then exactly the vertices of the cross polytope, as desired. □

Now we have given particular embeddings of both polytopes in $\mathbb{Z}^n$ for all n. Now let us consider how such embeddings can be transformed into others.

A characterization of the side lengths possible for a hypercube of dimension n is given in [5]. Here we consider one consequence of that characterization. In particular, we use the fact that the side length must be an integer for odd dimension.

This gives us quite a convenient way of looking at primitive hypercubes.

Theorem 3.2. *All embeddings in $\mathbb{Z}^n$, with n odd, of a regular hypercube with one vertex at the origin are in 1-to-$n!$ correspondence with elements of the group $O_n(\mathbb{Q})$ (and exactly half of these are elements of $SO_n(\mathbb{Q})$).*

Proof. First consider the usual embedding of the unit hypercube of dimension n, with one vertex at the origin, and all the edges emanating from it to be the usual basis vectors e_i. Now consider any other primitive hypercube with one vertex at the origin, and consider the edges emanating from the origin. These can be taken to describe a set of n orthogonal vectors spanning our space. When the side length l is integral, as it is for all odd dimension, the dilation of these vectors by $1/l$ projects them onto rational points of the unit sphere.

Furthermore, whenever a set of n rational points on the unit sphere exists such that the vectors terminating at these points are mutually orthogonal, these points can be scaled to lie in the lattice such that they can be completed to a primitive hypercube.

Thus there is a transformation in $O_n(\mathbb{Q})$ that takes the e_i to these new vectors. Moreover, there are exactly $n!$ of them, as every permutation of an orthonormal set of vectors is also in $O_n(\mathbb{Q})$ (exactly half of them have determinant 1 and half have determinant -1).

□

4. Simplices

In dimension 3, there was a useful way to inscribe a tetrahedron in a cube such that a subset of the set of vertices of the cube formed the tetrahedron. Thus it would be important to see in which dimensions a similar embedding is possible.

First, some definitions:

Definition 4.1. A **bit string** of length l is an ordered sequence of 1's and 0's such that there are l bits present in the sequence.

Definition 4.2. The **Hamming distance** $H(a, b)$ between two strings a, b of length l is the number $d \leq l$ such that the strings differ in d of their l positions.

Definition 4.3. A **Hamming path** between two bit strings a, b separated by Hamming distance d is a sequence of d steps that takes a to b such that at each step exactly one bit is switched.

Definition 4.4. A **string simplex** S of dimension n is a matrix of $n+1$ bit strings of length n such that the bit strings form rows and they are equidistant to each other in terms of Hamming distance.

Note that finding a string simplex of dimension n with constant Hamming distance d is equivalent to inscribing a regular simplex of dimension n in a unit hypercube (with the usual coordinates) of dimension n such that the length of each edge of the simplex is $\sqrt{d}$.

Now, we come to a very nice recursive result.

Theorem 4.5. *A regular simplex of dimension n can be inscribed in a hypercube of dimension n if and only if $n = 2^m - 1$ for some m.*

Proof. To see why these are the only dimensions possible, consider the group H of symmetries of our hypercube, and assume that there is a simplex inscribed in the hypercube. Let G be the group of symmetries of this simplex. Every symmetry action on vertices of the inscribed simplex is also a symmetry action on the path between these vertices and thus is a symmetry of the hypercube, so G is isomorphic to a subgroup of H. Then the order of G divides the order of H, or

$$(n+1)!|2^n n!$$

This is only possible if $n + 1$ is a power of two.

Now, we give an inductive construction which shows that all of these dimensions have possible embeddings.

To start, a 0-dimensional simplex and a 0-dimensional hypercube are identical, just the one point. The string simplex which corresponds to this situation is the string simplex with one empty string.

Now, given a string simplex S_m of dimension $2^m - 1$, we construct a string simplex S_{m+1} of dimension $2^{m+1} - 1$.

First, let the operation $A(S)$ append a 0 to the end of each row of S, and let $B(S)$ replace every 0 with a 1 in S, and every 1 with a 0, and append a 1 to the end of each row.

Then let

$$S_{m+1} = \begin{pmatrix} A(S_m) & S_m \\ B(S_m) & S_m \end{pmatrix}.$$

Now assume, for any $a, b \in S_m$, $H(a, b) = d_m$ (for $m \geq 0$). Then

$$H(A(a), A(b)) = H(B(a), B(b)) = H(A(a), B(b)) = H(B(a), A(b)) = d_m$$

and

$$H(A(a), B(a)) = H(A(b), B(b)) = 2d_m$$

and we can let $d_{m+1} = 2d_m$.

□

The first few examples of this "reflecting and connecting" construction are:
Dimension 0:

$$S_0 = ()$$

Dimension 1

$$S_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Dimension 3:

$$S_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Dimension 7:

$$S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

5. Corrections

The proof of the "only if" part of the statement of the last theorem may have seemed a bit unclear to the reader, and that is because it is false. To see why, let us consider a construction suggested by the abstracts of [1] and [4].

Definition 5.1. A **Hadamard matrix** of order n is an n-by-n square matrix such that all entries are ± 1 and all rows are mutually orthogonal (i. e. the rows differ pairwise in half their entries).

The striking similarity between this definition and that of a string simplex as above lets us again consider the problem of regular simplices inscribed in hypercubes.

Lemma 5.2. *If a Hadamard matrix of order n exists, then there exists a Hadamard matrix of order n such that there is a column of exclusively 1 or exclusively -1.*

Proof. Consider the set V of all vectors that are acceptable as the rows of a Hadamard matrix (these vectors' endpoints form a hypercube of side length 2 centered at the origin). Let W_i be the set of ordered pairs of vectors $(v, -v), v \in V$ such that the ith coordinate of v is 1.

Let H be a Hadamard matrix considered as a set of row vectors. Note that replacing v in H with $-v$ produces another Hadamard matrix. Then, let H_i^+ be defined as the matrix created by taking all $v \in H$ to the element of each v's vector pair in W which has a 1 in the ith position, and let H_i^- be defined analogously.

Thus if all vectors in H were mutually orthogonal, then the vectors in H_i^+ and H_i^- are also. □

Theorem 5.3. *It is possible to inscribe a regular simplex of dimension n in a hypercube of dimension n if and only if a Hadamard matrix of order $n+1$ exists.*

Proof. Whenever a string simplex of dimension n exists, we can simply replace all 0's with -1's and add a constant column of 1's or -1's to get a Hadamard matrix of order $n+1$. Whenever a Hadamard matrix of order $n+1$ exists, we can use the previous lemma to get a constant column, and then reverse this process to get a string simplex of dimension n. □

As a final note, this recursion that takes a Hadamard matrix of order n and creates a Hadamard matrix of order $2n$ is a bit cleaner than the equivalent recursion presented for string simplices:

$$H_{2n} = \begin{pmatrix} H_n & H_n \\ -H_n & H_n \end{pmatrix}.$$

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