HOMOTOPY GROUPS OF SPHERES

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ABSTRACT. We consider the elementary theory of higher homotopy groups through its application to the homotopy groups of spheres. We prove the Freudenthal suspension theorem and introduce stable homotopy groups. We then show that the stable homotopy groups of spheres form a graded ring. Turning to the theory of fiber bundles and fibrations, we define the Hopf bundle $S^1 \to S^3 \to S^2$, which allows us to compute $\pi_3(S^2)$. Throughout, our proofs and general presentation follow [1].

CONTENTS

1. INTRODUCTION

We assume the reader has familiarity with elementary algebraic topology, chiefly homotopy groups and CW complexes. We review the following notation and definitions. Recall that the nth homotopy group $\pi_n(X, x_0)$ of a space X with basepoint x_0 is defined as equivalence classes under homotopy of maps $(I^n, \partial I^n) \to (X, x_0)$, or equivalently $(S^n, s_0) \to (X, x_0)$. In the former case the group operation is given by

$$(f \cdot g)(x_1, x_2 \ldots, x_n) = \begin{cases} f(2x_1, x_2, \ldots, x_n) & 0 \leqslant x_1 \leqslant 1/2 \\ g(1 - 2x_1, x_2, \ldots, x_n) & 1/2 \leqslant x_1 \leqslant 1, \end{cases}$$

and in the latter case the group operation is given by $S^n \to S^n \vee S^n \to X$, where the map $S^n \to S^n \vee S^n$ sends the equator S^{n-1} of S^n to a point and the map $S^n \vee S^n \to X$ is f on the first copy of S^n and g on the second.

For a subset A of X and basepoint $x_0 \in A$, we may also define the relative homotopy groups $\pi_n(X, A, x_0)$. Let us agree on the convention that we will identify I^{n-1} as the subset $I^{n-1} \times \{0\}$ of I^n. We then define $J^{n-1} \subset I^n$ as the closure of $I^n \setminus I^{n-1}$. If we consider the boundary of I^n as $2n$ copies of I^{n-1}, then J^{n-1} includes $2n - 1$ copies of I^{n-1}: all of them except for $I^{n-1} \times \{0\}$. We now define $\pi_n(X, A, x_0)$ as equivalence classes under homotopy of maps from $(I^n, \partial I^n, J^{n-1})$ to

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(X, A, x_0), or equivalently from (D^n, S^{n-1}, s_0) to (X, A, x_0). The group operation for relative homotopy groups is the same as for absolute homotopy groups, with trivial modifications. For $n = 1$, there is no natural group structure on $\pi_n(X, A, x_0)$, though we may consider the set of equivalence classes simply as a set. We remind the reader of the following long exact sequence for relative homotopy groups, which we will require later on.

Theorem 1.1. *Given a space X, with subsets $B \subset A \subset X$, and a point $x_0 \in B$, there exists a long exact sequence*

$$\cdots \longrightarrow \pi_i(A, B, x_0) \longrightarrow \pi_i(X, B, x_0) \longrightarrow \pi_i(X, A, x_0) \longrightarrow \pi_{i-1}(A, B, x_0) \longrightarrow$$
$$\cdots \longrightarrow \pi_1(X, A, x_0).$$

Finally, we recall the homotopy groups of the circle.

Theorem 1.2. *$\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_i(S^1) = 0$ for $i > 1$*

The circle is unique among spheres in the simplicity of its homotopy groups. Namely, the circle is the only sphere S^n whose homotopy groups are trivial in dimensions greater than n. For the homology groups $H_k(S^n)$, the property that $H_k(S^n) = 0$ for k greater than n is always true, since S^n may be given a CW structure consisting of a 0-cell and an n-cell, and cellular homology tells us that the kth homology group $H_k(X)$ of a CW complex X has a number of generators at most the number of k-cells of X. The richness and complexity of homotopy groups, and those of spheres in particular, arise from the failure of this property: $\pi_n(X)$ need not be trivial for n greater than the dimension of X.

2. Cellular Approximation

Let $H : X \times I \to Y$ be a homotopy. Throughout this paper, we will use the convention that $h_t(x) = H(x, t)$.

Definition 2.1. Let X be a space containing a subset A. We say that the pair (X, A) has the *homotopy extension property* if, given any space Y and a map $f : X \times \{0\} \cup A \times I \to Y$, there exists a map $\tilde{f} : X \times I \to Y$ extending f. In other words, there exists a map $\tilde{f}$ such that the following diagram commutes:

$$\begin{tikzcd}
X \times I \arrow[dr, dotted, "\tilde{f}"] & \\
X \times \{0\} \cup A \times I \arrow[u, "i"] \arrow[r, "f"] & Y
\end{tikzcd}$$

Proposition 2.2. *Every CW pair (X, A) has the homotopy extension property.*

Proof. We show that $X \times I$ deformation retracts onto $X \times \{0\} \cup A \times I$. The extension homotopy $\tilde{f}$ can then be obtained by precomposing the original map f with the deformation retract.

From the observation that $D^n \times I$ deformation retracts onto $D^n \times \{0\} \cup \partial D^n \times I$ it follows that $X^n \times I$ deformation retracts on $X^n \times \{0\} \cup X^{n-1} \times I \cup A \times I$. This is because $X^n \times I$ is obtained from $X^n \times \{0\} \cup X^{n-1} \times I \cup A \times I$ by attaching copies of $D^n \times I$, where $D^n \times \{0\}$ attaches to $X \times \{0\}$ and $\partial D^n \times I$ attaches to $X^{n-1} \times I$. Therefore, $D^n \times I$ attaches to $X^n \times \{0\} \cup X^{n-1} \times I$ along $D^n \times \{0\} \cup \partial D^n \times I$. The deformation retracts of each $D^n \times I$ taken together form a deformation retract of $X^n \times I$ onto $X^n \times \{0\} \cup X^{n-1} \times I \cup A \times I$.

A second iteration of the process gives us that $X^n \times \{0\} \cup X^{n-1} \times I \cup A \times I$ deformation retracts onto $X^n \times \{0\} \cup X^{n-2} \times I \cup A \times I$. Therefore, by finitely many iterations of the process, we see that $X^n \times I$ deformation retracts onto $X^n \times \{0\} \cup A \times I$. This proves the claim for X a finite dimensional CW complex.

Suppose X is infinite and let r_n be the deformation retract of $X^n \times I$. By composing the countably infinite collection of deformation retracts r_n such that r_n occurs in the time interval $[1/2^n, 1/2^{n-1}]$, we obtain the required deformation retract. The composition is continuous at $t = 0$, since its restriction to any skeleton X^k is continuous and X is given the weak topology. □

Proposition 2.3. *Let X be a CW complex, $\{e_\alpha\}$ be a collection of cells in X, and $\{p_\alpha\}$ be a collection of points such that $p_\alpha \in e_\alpha$ for all α. If $P = \bigcup_\alpha p_\alpha$ and $E = \bigcup_\alpha e_\alpha$, then the map induced by inclusion $i_* : \pi_n(X \setminus E) \to \pi_n(X \setminus P)$ is an isomorphism for all n.*

Proof. We first make the following observation. If e^n is an n-dimensional cell and p is a point in e^n, then $e^n \setminus \{p\}$ deformation retracts onto ∂e^n. Therefore, $X^{n-1} \cup e^n$ deformation retracts onto X^{n-1}. By the homotopy extension property, we may extend the deformation retract $X^{n-1} \cup e^n \to X^{n-1}$ to a homotopy on all of X. The homotopy $H : X \times I \to X$ has the property that $h_1(X)$ does not intersect e^n.

To show surjectivity of the map $i_* : \pi_n(X \setminus E) \to \pi_n(X \setminus P)$, let $[f]$ be an element of $\pi_n(X \setminus P)$. It suffices to show that f may be homotoped to a map whose image does not intersect any cell e_α in E. Since I^n is compact, $f(I^n) \subset X \setminus P$ is compact. Therefore, $f(I^n)$ intersects only finitely many cells of X, and so it intersects only finitely many cells of E. Let e^n be a cell of highest dimension that intersects the image of f and H be the homotopy constructed above. We define a homotopy $H' : I^n \times I \to X$ by the composition $I^n \times I \to X \times I \to X$, where the second map is given by H and the first map is given by $(x,t) \mapsto (f(x),t)$, where $x \in I^n$ and $t \in I$. We see that $h_0 = f$ and $h_1(I^i)$ does not intersect e^n. Thus we have homotoped f off of the cell e^n. By repeating this process finitely many times, once for each cell that the image of f intersects, we are able to homotope f off of all the cells in E.

To prove injectivity, let $[f]$ be an element of $\pi_n(X \setminus E)$ such that $i_*([f]) = 0$. Let $H : I^n \times I \to X \setminus P$ be a homotopy of if to the constant map. The previous argument allows us to homotope H off of all the cells in E. Thus H is homotopic to map $H' : I^n \times I \to X \setminus E$. It is easy to verify that H' is a homotopy of f to the constant map. Thus $[f] = 0$, as required. □

For the proof of the following technical lemma we refer the reader to [1, p. 350].

Lemma 2.4. *Let $f : I^n \to Y$ be a continuous map and e^k be a cell in Y. Then f is homotopic rel $Y \setminus e^k$ to a map $g : I^n \to Y$ which satisfies the following: there exists a simplex $\Delta^k \subset e^k$ such that $g^{-1}(\Delta^k)$ is a finite, possibly empty, union of convex polyhedra on each of which g is the restriction of a linear surjection $\mathbb{R}^i \to \mathbb{R}^k$.*

Definition 2.5. We say that a map $f : X \to Y$ between CW complexes is *cellular* if $f(X^k) \subset Y^k$ for all k.

Theorem 2.6 (Cellular Approximation)**.** *Every map $f : X \to Y$ between CW complexes is homotopic to a cellular map.*

Proof. We first prove by induction on k that f is homotopic to a map which is cellular on the k-skeleton X^k of X. For the base case, let x be a point in the discrete set X^0. It is clear that the 0-skeleton of any CW complex Y intersects all the path components of Y. Thus let h be a path from $f(x)$ to a point $y \in Y^0$. Since for each $x \in X^0$ we may choose such a path, we see that $f|_{X^0}$ is homotopic to a cellular map. By the homotopy extension property for CW complexes, we may extend the homotopy of $f|_{X^0}$ to a homotopy of f. Thus f is homotopic to a map which is cellular on X^0.

Suppose that f has been homotoped to be cellular on X^{n-1}. Let e^n be an n-cell of X. Since the closure of e^n is compact, its image under f is also compact, and therefore intersects only finitely many cells of Y. Let e^k be a cell of greatest dimension which intersects $f(e^n)$. We may assume that k is greater than n, since otherwise f is already cellular on e^n. Lemma 2.4 allows us to homotope f to be nonsurjective on e^k. To see this, let Δ^k be a simplex in e^k such that $f^{-1}(\Delta^k)$ is a finite union of convex polyhedra on each of which f is the restriction of a linear surjection $\mathbb{R}^n \to \mathbb{R}^k$. Since there are no linear surjections $\mathbb{R}^n \to \mathbb{R}^k$, $f^{-1}(\Delta^k)$ is empty. Therefore, f is nonsurjective on e^k, as claimed.

By Proposition 2.3 we may homotope $f|_{X^{n-1}\cup e^n}$ rel X^{n-1} off of e^k entirely. By finitely many iterations of this process, we may homotope $f|_{X^{n-1}\cup e^n}$ rel X^{n-1} off of all cells in Y of dimension greater than n. We may do the same for all n-dimensional cells of X. This gives us a homotopy rel X^{n-1} of $f|_{X^k}$ to map which is cellular on X^k. By the homotopy extension property, we extend this homotopy to one on all of f.

This completes the proof for the case of a finite CW complex X. If X is infinite dimensional, we require one further step. Let H_0 be a homotopy of f to a map f_0 which is cellular on X^0 and H_1 be a homotopy of f_0 to a map f_1 which is cellular on X^1. Recursively, let H_{n+1} be a homotopy of f_n to a map f_{n+1} which is cellular on X^{n+1}. Note that $H_n \circ H_{n-1} \circ \cdots \circ H_0$ is a homotopy of f to a map which is cellular on X^n. The required homotopy H is obtained by composing the homotopies H_n in time units of length $1/2^{n+1}$. Thus for $0 \leqslant t \leqslant 1/2$ H equals H_0, for $1/2 \leqslant t \leqslant 3/4$ H equals H_1, etc. □

Corollary 2.7. $\pi_i(S^n) = 0$ *for* $i < n$

Proof. Let $[f]$ be an element of $\pi_i(S^n)$. By the previous theorem we may assume that f is cellular. If S^n is given the standard cellular structure with a 0-cell and an n-cell, then the i-skeleton of S^n for $i < n$ is simply the 0-cell. Therefore, $f : S^i \to S^n$ is the trivial map. □

3. Freudenthal Suspension Theorem

The next simplest homotopy groups to compute are $\pi_n(S^n)$. This computation will prove to be a simple consequence of the Freudenthal suspension theorem, and will carry us into the theory of stable homotopy groups. First, however, we require a fundamental result about CW complexes. In a spirit similar to cellular approximation, which allows us to approximate maps between CW complexes by cellular maps, we might ask whether spaces themselves can be approximated by CW complexes. The affirmative answer, known as *CW approximation*, in its simplest form states that for any space Y there exists a CW complex X and a homotopy equivalence $f : X \to Y$. The consequence of CW approximation which we require is given by the following proposition.

Definition 3.1. We say that a space X is *n-connected* if X is path connected and $\pi_i(X) = 0$ for all $i \leqslant n$. We say that a pair (X, A) is n-connected if A intersects all the path components of X and $\pi_i(X, A) = 0$ for all $i \leqslant n$.

Proposition 3.2. *Let (X, A) be an n-connected CW pair. Then there exists a CW pair (Z, A) such that all cells of $Z - A$ have dimension greater than n and a homotopy equivalence $h : (Z, A) \to (X, A)$ relative to A.*

Proof. See [1, p. 352]. □

Theorem 3.3 (Homotopy Excision)**.** *Let X be a CW complex and A, B subcomplexes such that $X = A \cup B$. Suppose that the intersection $C = A \cap B$ is connected, (A, C) is m-connected, and (B, C) is n-connected. Then the inclusion map $A \hookrightarrow B$ induces isomorphisms $\pi_i(A, C) \to \pi_i(X, B)$ for $i < m + n$ and a surjection for $i = m + n$.*

Proof. We begin with the following two simplifications of the problem.

(1) By proposition 3.2 we may assume without loss of generality that $A \setminus C$ and $B \setminus C$ consist of cells of dimensions $\geqslant m + 1$ and $n + 1$, respectively.
(2) By cellular approximation, since the statement of the theorem only concerns homotopy groups of dimension $\leqslant m + n$, it suffices to consider the problem for $A \setminus C$ consisting of cells of dimension $\leqslant m + n + 1$.

We first consider a preliminary case, where $A \setminus C$ consists only of cells of dimension $m + 1$. This will require the following proposition.

Proposition 3.4. *Let X, A, B, C, satisfy the hypotheses of theorem 3.3. Let $A \setminus C$ consist of cells e_j^{m+1} and $B \setminus C$ consist of a single cell e^{n+1}. Then any map $(I^i, \partial I^i, J^{i-1}) \to (X, B, b_0)$, for $i \leqslant m + n$, is homotopic to a map f for which there exists a point $b \in e^{n+1}$ such that $b \notin f(I^i)$ and points $a_j \in e_j^{m+1}$ such that $a_j \notin f(\partial I^i)$.*

Let us see how this proposition implies the result for the preliminary case. Suppose, first, that $B \setminus C$ consists of a single cell e^{n+1}. To show surjectivity of the map induced by inclusion $\pi_i(A, C) \to \pi_i(X, B)$ for $i \leqslant m + n$, let $[f]$ be an element of $\pi_i(X, B)$. We may assume that f satisfies the conclusions of proposition 3.4, and let A' be the union of the points a_j. Consider the following commutative diagram, where all the arrows are induced by the inclusion map.

$$\begin{tikzcd}
\pi_i(A, C) \arrow[r] \arrow[d, "l_*"] & \pi_i(X, B) \arrow[d, "k_*"] \\
\pi_i(X \setminus \{b\}, X \setminus (A' \cup \{b\}) \arrow[r] & \pi_i(X, X \setminus A')
\end{tikzcd}$$

Proposition 2.3 tells us that l_* and k_* are isomorphisms. We may therefore consider $[f]$ to be an element of $\pi_i(X, X \setminus A')$. Proposition 3.4 tells us precisely that $f(I^i) \subset X \setminus \{b\}$ and $f(\partial I^i) \subset X \setminus (A' \cup \{b\})$. Therefore, we may consider $[f]$ to be an element of $\pi_i(X \setminus \{b\}, X \setminus (A' \cup \{b\})$. Finally, since l_* is an isomorphism, we may consider $[f]$ to be an element of $\pi_i(A, C)$. Surjectivity now follows from the commutativity of the diagram.

The proof of injectivity is similar. Let $[f_0]$ and $[f_1]$ be elements of $\pi_i(A, C)$ such that $[j_*(f_0)] = [j_*(f_1)]$ in $\pi_i(X, B)$. Equality still holds if we consider $[j_*(f_0)]$ and $[j_*(f_1)]$ as elements of $\pi_i(X, X \setminus A')$. Let h be a homotopy between $j_*(f_0)$ and $j_*(f_1)$ in $\pi_i(X, X \setminus A')$. Since h is a map $(I^i \times I, \partial I^{i-1} \times I, J^{i-1} \times I) \to (X, X \setminus A', b_0)$, $[h]$ is an element of $\pi_{i+1}(X, X \setminus A')$. Proposition 3.4 allows us to consider $[h]$ as an element of $\pi_{i+1}(X \setminus \{b\}, X \setminus (A' \cup \{b\})$. Therefore, we may consider $[h]$ as an element of $\pi_{i+1}(A, C)$. Any representative of the equivalence class $[h]$ is a homotopy between f_0 and f_1 in $\pi_i(A, C)$. Therefore, $[f_0] = [f_1]$. However, since h is a map $I^i \times I \to X$, $i + 1$ replaces i in the case of surjectivity. Therefore, injectivity holds only for $i < m + n$.

Now let $B \setminus C$ consist of an arbitrary collection of cells, as in the original statement of the preliminary case. Let E denote the cells of $B \setminus C$, and recall that the cells of E may be assumed to have dimension $\geqslant n+1$. Let e^k be a cell of highest dimension in E and let $f : (I^i, \partial I^i, J^{i-1}) \to (X, B, b_0)$. The hypotheses of proposition 3.4 are satisfied if $A \cup E \setminus e^k$ replaces A, $C \cup \setminus e^k$ replaces C, and e^k replaces e^{n+1}. We may therefore homotope f in the manner described. Then let A' and b denote the same as previously. Note that b, in this case, is an element of e^k, but the elements A' are still points of the $(m+1)$-dimensional cells of $A \setminus C$, as before. Consider the following commutative diagram, which resembles the previous one.

$$\begin{array}{ccc}
\pi_i(A \cup E \setminus \{e^k\}, C \cup E \setminus \{e^k\}) & \xrightarrow{j_*} & \pi_i(X, B) \\
\downarrow{\scriptstyle l_*} & & \downarrow{\scriptstyle k_*} \\
\pi_i(X - \{b\}, X - A' - \{b\}) & \longrightarrow & \pi_i(X, X - A')
\end{array}$$

An identical argument to the one given above shows that j_* is injective for $i \leqslant m+n$ and surjective for $i < m + n$. If $e^{k'}$ is a cell of highest dimension of $E \setminus \{e^k\}$, then similar reasoning shows that

$$j_* : \pi_i(A \cup E \setminus \{e^k, e^{k'}\}, C \cup E \setminus \{e^k, e^{k'}\}) \to \pi_i(A \cup E \setminus \{e^k\}, C \cup E \setminus \{e^k\})$$

is surjective and injective for the specified values of i. Therefore, by composition, surjectivity and injectivity also hold for

$$j_* : \pi_i(A \cup E \setminus \{e^k, e^{k'}\}, C \cup E \setminus \{e^k, e^{k'}\}) \to \pi_i(X, B).$$

Since E consists of only finitely many cells, the result for the map $j_* : \pi_i(A, C) \to (X, B)$ is given by finitely many iterations of this process. This conclude the preliminary case, so we now prove proposition 3.4.

Proof of Proposition 3.4. Let $g : (I^i, \partial I^i, J^{i-1}) \to (X, B, b_0)$, for $i \leqslant m + n$, be the map referred to in the hypotheses of the proposition. Since $g(I^i)$ is compact, it intersects only finitely many of the cells e_j^{m+1}. By a finite number of applications of Lemma 2.4, we may homotope g so that there exist simplices $\Delta^{n+1} \subset e^{n+1}$ and $\Delta_j^{m+1} \subset e_j^{m+1}$ whose preimages under g are finite unions of convex polyhedra, on each of which g is the restriction of a linear surjection $\mathbb{R}^i \to \mathbb{R}^{n+1}$ or $\mathbb{R}^i \to \mathbb{R}^{m+1}$. Note that if the image of f does not intersect a given e_j^{m+1}, then any simplex will suffice, and the union of convex polyhedra is empty.

The next step in the proof is to construct a function $\phi : I^{i-1} \to [0, 1)$ such that there exist points $b \in \Delta^{n+1}$, $a_j \in \Delta_j^{m+1}$ that satisfy

(1) $g^{-1}(b)$ lies below the graph of ϕ,

(2) $g^{-1}(a_j)$ lies above the graph of ϕ, if this preimage is nonempty, and
(3) $\phi(\partial I^{i-1}) = 0$.

We now carry out this construction. Let b be any point in Δ^{n+1}. Let $g^{-1}(\Delta^{n+1})$ be the finite union of polyhedra $P_1, \ldots, P_\alpha$ and let $T_1, \ldots, T_\alpha$ be linear surjections $\mathbb{R}^i \to \mathbb{R}^{n+1}$ such that $T_j|_{P_j} = g|_{P_j}$ Each P_j is of dimension $\leqslant i$ since it is contained in I^i. Since T_j is surjective, $\mathrm{Ker}(T_j)$ is of dimension $\leqslant i-(n+1)$. Therefore, $g^{-1}(b)$ is a finite union of polyhedra of dimension $\leqslant i-(n+1)$. Let W be the union of all line segments $\{x\} \times I$ intersecting $f^{-1}(b)$. Thus $W = \pi^{-1}(\pi(f^{-1}(b)))$, where π is the projection of I^i onto I^{i-1}. Since $\pi(g^{-1}(b))$ is a finite union of convex polyhedra of dimension $\leqslant i-n-1$, W is a finite union of convex polyhedra of dimension $\leqslant i-n$.

Now let $g^{-1}(\Delta_j^{m+1})$ be a finite union of convex polyhedra $Q_{j_1}, \ldots, Q_{j_\alpha}$ and let $S_{j_1}, \ldots, S_{j_\alpha}$ be linear surjections $\mathbb{R}^i \to \mathbb{R}^{m+1}$ such that $S_{j_i}|_{Q_{j_i}} = g|_{Q_{j_i}}$. Then

$$g(W) \cap \Delta_j^{m+1} = S_{j_1}(W \cap Q_{j_1}) \cup \cdots \cup S_{j_\alpha}(W \cap Q_{j_\alpha}).$$

From the fact that each $W \cap Q$ is a convex polyhedron of dimension $\leqslant i-n$ and each S is linear, it follows that $g(W) \cap \Delta_j^{m+1}$ is the finite union of convex polyhedra of dimension $\leqslant i-n$. The assumption that $i \leqslant m+n$ gives us that $i-n < m+1$ and therefore that none of these polyhedra is all of Δ_j^{m+1}. In particular, $g(W) \cap \Delta_j^{m+1}$ does not equal Δ_j^{m+1}. For each j, choose a point $a_j \in g(W) \setminus \Delta_j^{m+1}$. The collection of points $\{a_j\}$ satisfies that $g^{-1}(a_j) \cap W$ is empty for all j.

Now choose a neighborhood U of $\pi(W) = \pi(g^{-1}(b))$ to be disjoint from $\pi(g^{-1}(a_j))$. We claim that Urysohn's lemma allows us to construct a map $\phi : I^{i-1} \to [0,1)$ with support contained in U such that $g^{-1}(b)$ lies under the graph of ϕ. It is easy to verify that the map ϕ would then satisfy the three properties specified above. There is, however, a small detail we must address in order to justify this construction. We claim that the projection of $g^{-1}(b)$ onto the I summand of $I^{n-1} \times I$ has an upper bound which is strictly less than 1. If p is such an upper bound, then Urysohn's lemma allows us to construct a map $\phi : I^{i-1} \to [0,p]$ which equals p on $\pi(W)$ and 0 outside of U.

To prove the claim, first note that the basepoint b_0 of X does not equal b, since former lies in C and the latter does not. We may therefore choose an open neighborhood V of b_0 which is disjoint from b. It follows that $g^{-1}(V)$ is an open set in I^n containing $I^{n-1} \times \{0\}$ and disjoint from $g^{-1}(b)$. Each point of $I^{n-1} \times \{0\}$ has an open neighborhood contained in $g^{-1}(V)$. By compactness, finitely many of these neighborhoods cover $I^{n-1} \times \{0\}$. We now project the open neighborhoods of this cover onto the I summand of $I^{n-1} \times I$, giving us a finite collection of sets of real numbers. The infimum of each of these sets is strictly less than 1. The upper bound p can be taken as the minimum of the infimums of these sets. This completes the construction of ϕ.

We now use the map ϕ to homotope g to satisfy the conclusions of proposition 3.4. Note that, given any $t \in [0,1]$, the region R_t above the graph of $t\phi$ is homeomorphic to all of I^i, where $t\phi$ denotes simply the multiplication of ϕ by the constant t. We may therefore consider a map $R_t \to X$ as a map $I^i \to X$, so we define g_t to equal g restricted to R_t, but considered as a map from I^i. Let $f = g_1$. If we define $G : I^i \times I \to X$ by $G(x,t) = g_t(x)$, then G is a homotopy from g to f. The continuity of G follows from the continuity of ϕ. Since $g^{-1}(b)$ lies below the graph of ϕ, $b \notin f(I^i)$. Since $g^{-1}(a_j)$ lies above the graph of ϕ, it does not lie anywhere on the boundary of R_t. The homeomorphism from R_t to I^i maps the boundary of R_t to the boundary of I^i. Therefore, $a_j \notin f(\partial I^i)$ for all j. We see that f satisfies the conclusions of proposition 3.4. $\square$

We now prove the general case of theorem 3.3, where $A \setminus C$ consists of cells of dimension $\geqslant m+1$ and $\leqslant n+m+1$, and $B \setminus C$ consists of cells of dimension $\geqslant n+1$. Let A_k be the union of C and the cells of $A \setminus C$ of dimension $\leqslant k$. Let X_k be the union of B and A_k. We use induction on k to show that $\pi_i(A_k, C) \to \pi_i(X_k, B)$ is an isomorphism for $i < m+n$ and surjective for $i = m+n$. This will prove the result, since it will prove the inductive statement for $k = m+n+1$.

In the following commutative diagram, the rows represent the long exact sequences for the triples (A_k, A_{k-1}, C) and (X_k, X_{k-1}, B) and the vertical arrows are induced by the inclusion maps.

$$\begin{tikzcd}
\pi_{i+1}(A_k, A_{k-1}) \arrow[r] \arrow[d] & \pi_i(A_{k-1}, C) \arrow[r] \arrow[d] & \pi_i(A_k, C) \arrow[r] \arrow[d] & \pi_i(A_k, A_{k-1}) \arrow[r] \arrow[d] & \pi_{i-1}(A_{k-1}, C) \arrow[d] \\
\pi_{i+1}(X_k, X_{k-1}) \arrow[r] & \pi_i(X_{k-1}, B) \arrow[r] & \pi_i(X_k, B) \arrow[r] & \pi_i(X_k, X_{k-1}) \arrow[r] & \pi_{i-1}(X_{k-1}, B)
\end{tikzcd}$$

The base case, $k = m+1$, is addressed by the preliminary case. This also gives that the first and fourth vertical maps are isomorphisms for all k. For the induction step, we suppose that $\pi_i(A_{k-1}, C) \to \pi_i(X_{k-1}, B)$ is an isomorphism for $i < m+n$. Thus the second and fifth vertical maps are isomorphisms. Therefore, by the five lemma, for $i \geqslant 2$ the middle map is an isomorphism. For $i = m+n$ the second and fourth maps are surjective and the fifth map is injective. A simple excercise in diagram chasing shows that the middle map is surjective. This completes the induction.

It still remains to prove the theorem for $j_* : \pi_1(A, C) \to \pi_1(X, B)$. We may assume that m and n do not both equal 0, since in this case there is nothing to prove. Suppose $m+n \geqslant 1$. We claim $\pi_i(A, C)$ and $\pi_i(X, B)$ are both trivial. If $m = 0$, then $n \geqslant 1$. Then $\pi_1(X, B)$ is trivial by the definition of connectivity and $\pi_i(A, C)$ is trivial by cellular approximation, since A is obtained by from C by adding 0-cells. Similarly, for $n = 0$ and $m \geqslant 1$, connectivity and cellular approximation again imply the claim. $\square$

3.1. **Freudenthal Suspension Theorem.**

Definition 3.5. Given a space X with basepoint x_0, we define the *suspension* SX of X by $SX = X \times I/\sim$, where each of $X \times \{1\}$, $X \times \{0\}$, and $x_0 \times I$ are identified to a point.

Example 3.6. The suspension of a sphere is a sphere of one dimension higher. We may think of $S^i \times I$ as a cylinder. By "pinching" the ends we obtain a sphere.

Given a map $f : X \to Y$ between two spaces, there is a natural map $Sf : SX \to SY$ between the suspensions of the two spaces. Namely, we set $Sf(x,t) = (f(s), t)$. In particular, given a space X, the preceding example shows that the suspension of a map in $\pi_i(X)$ is a map in $\pi_{i+1}(X)$. The reader may verify that the map $S : \pi_i(X) \to \pi_{i+1}(SX)$ is a well-defined homomorphism.

Theorem 3.7 (Freudenthal Suspension Theorem)**.** *Let X be an n-connected CW complex. Then the suspension map $S : \pi_i(X) \to \pi_{i+1}(SX)$ is an isomorphism for $i \leqslant 2n$ and is a surjection for $i = 2n+1$.*

Proof. Let C_1X and C_2X be two cones which constitute SX. In other words, $C_1X = [1/2, 1] \times X$ with $\{1\} \times X$ identified to a point, and $C_2X = [0, 1/2] \times X$ with $\{0\} \times X$ identified to a point.

The suspension map $\pi_i(X) \to \pi_{i+1}(SX)$ may be decomposed as follows, where the middle arrow is the inclusion induced map and the first and third arrows are isomorphisms taken from the long exact sequences of the pairs (C_1X, X) and (SX, C_2X), respectively.

$$\pi_i(X) \longrightarrow \pi_{i+1}(C_1X, X) \longrightarrow \pi_{i+1}(SX, C_2X) \longrightarrow \pi_{i+1}(SX)$$

We can see that the decomposition is correct, since the first arrow is described by extending a map $I^i \to X$ to a map $I^{i+1} \to C_1X$ in the obvious way and the third arrow is described by taking a map $(I^{i+1}, \partial I^{i+1}) \to (SX, C_2X)$ and "pinching" the bottom to produce a map $(I^{i+1}, \partial I^{i+1}) \to (SX, s_0)$.

We claim that the pair (C_iX, X) is $(n+1)$-connected. This is true because the long exact sequence gives us

$$\ldots \longrightarrow \pi_{i+1}(C_iX) \longrightarrow \pi_{i+1}(C_iX, X) \longrightarrow \pi_i(X) \longrightarrow \pi_i(C_iX) \longrightarrow \ldots,$$

where $\pi_k(C_iX) = 0$ for all k, and $\pi_i(X) = 0$ for $i \leqslant n$. By theorem 3.3, the middle map is an isomorphism for $i + 1 < 2n + 2$ and surjective for $i + 1 = 2n + 2$, as required. □

Corollary 3.8. *For all n, $\pi_n(S^n)$ is isomorphic to $\mathbb{Z}$.*

Proof. Since S^n is $(n-1)$-connected, the suspension map $S : \pi_i(S^n) \to \pi_{i+1}(S^{n+1})$ is an isomorphism for $i < 2n - 1$. In particular, $S : \pi_n(S^n) \to \pi_{n+1}(S^{n+1})$ is an isomorphism for $n \geqslant 2$, since, in this case $n + 1 < 2n$. The Freudenthal suspension theorem, however, tells us only that $S : \pi_1(S^1) \to \pi_2(S^2)$ is surjective. The computation of $\pi_1(S^1)$, therefore, cannot be used to prove the result. Instead, problem reduces to the computation of $\pi_2(S^2)$. In Proposition 5.11 we show that $\pi_2(S^2)$ is isomorphic to $\mathbb{Z}$, which then implies the same of $\pi_n(S^n)$ for all n. □

4. Stable Homotopy Groups

If X is an n-connected CW complex, then the suspension map $S : \pi_i(X) \to \pi_{i+1}(SX)$ is an isomorphism for $i < 2n+1$. In particular, S is an isomorphism for $i \leqslant n$. Since $\pi_i(X)$ is trivial for $i \leqslant n$, it follows that the same is true for $\pi_{i+1}(SX)$, again for $i \leqslant n$. This imples that SX is an $(n+1)$-connected CW complex. It follows that S^kX is $(n+k)$-connected, where S^kX denotes the kth suspension of X.

Consider the sequence of suspensions

$$\pi_i(X) \longrightarrow \pi_{i+1}(SK) \longrightarrow \ldots \longrightarrow \pi_{i+k}(S^kX) \longrightarrow \ldots$$

Since S^kX is $(n+k)$-connected, the map $\pi_{i+k}(S^kX) \to \pi_{i+k+1}(S^{k+1}X)$ is an isomorphism for $i + k < 2(n+k) + 1$, thus for $k \geqslant i - 2n$. Therefore, for k sufficiently large, the sequence of maps in the above diagram are all isomorphisms. The resulting group is denoted the stable homotopy group $\pi_i^s(X)$.

We now turn to consider the stable homotopy groups $\pi_i^s(S^n)$ of spheres. Note that, in this case, since $SS^n = S^{n+1}$, the diagram becomes

$$\pi_i(S^n) \longrightarrow \pi_{i+1}(S^{n+1}) \longrightarrow \pi_{i+2}(S^{n+2}) \longrightarrow \ldots$$

Since we may add any finite number of terms to the beginning of the diagram without affecting the resulting stable homotopy group, $\pi_i^s(S^n) = \pi_{i-n}^s(S^0)$. Thus the only stable homotopy groups of spheres are the ones $\pi_i^s(S^0)$ for some value of i, which we may write simply as π_i^s. These stable homotopy groups classify the mappings of $(i+k)$-dimensional spheres onto k-dimensional spheres, for sufficiently large values of k.

There is a natural multiplication operation $\pi_i^s \times \pi_j^s \to \pi_{i+j}^s$. Namely, the composition of maps $S^{i+j+k} \to S^{j+k}$ and $S^{j+k} \to S^k$ gives a map $S^{i+j+k} \to S^k$. The following theorem characterizes this multiplication in terms of the algebraic concept of a graded ring.

Definition 4.1. We say that a ring R is a *graded ring* if R can be decomposed as the direct sum $\bigoplus_i A_i$, where $\{A_i\}$ is a countable collection of abelian groups such that the multiplication operation in R satisfies that $a_i a_j \in A_{i+j}$ whenever $a_i \in A_i$ and $a_j \in A_j$.

Example 4.2. The ring of polynomials $\mathbb{R}[X]$ in one variable over $\mathbb{R}$ is a graded ring. The abelian groups A_i may be taken to be the group of polynomials of degree i.

Theorem 4.3. *Let $\pi_*^s = \bigoplus_i \pi_i^s$. Then π_*^s is a graded ring which satisfies the following commutativity relation: for $f \in \pi_i^s$ and $g \in \pi_j^s$, $fg = (-1)^{ij} gf$.*

Proof. We need only verify distributivity and the commutativity relation. Let $f, g : S^{i+k} \to S^k$ and $h : S^{j+k} \to S^k$. The following consideration shows that $h(f+g) = hf + hg$. In both cases, the map is $S^{i+j+k} \to S^k$. Given a point x in the upper hemisphere of S^{i+j+k}, $h(f+g)(x) = hf(x)$, since $(f+g)(x) = f(x)$, by the definition of addition in homotopy groups. Likewise, if x lies on the lower hemisphere of S^{i+j+k} then $h(f+g)(x) = hg(x)$. Thus $h(f+g)$ and $hf + hg$ agree on the two hemispheres of S^{i+j+k}.

The same argument, however, does not suffice for distributivity on the right. For instance, if x lies on the upper hemisphere of S^{i+j+k}, we cannot conclude that $(f+g)h = fh$, because we cannot assume that h maps x to the upper hemisphere of S^{i+k}. Right hand distributivity will follow from left hand distributivity and the commutativity relation. We now prove the latter.

We note that the suspension SS^n of a sphere may be regarded as the smash product $S^n \wedge S^1$. Given a map $f : S^n \to S^k$, we may also regard the suspension map $Sf : S^{n+1} \to S^{k+1}$ as the smash product map $f \wedge \mathbf{1}$. This equality is also valid in higher iterations. Thus $S^j f = f \wedge \mathbf{1}$, where $S^j f$ is a map $S^{n+j} \to S^{k+j}$ and $\mathbf{1}$ is the identity on S^j.

Let $f : S^{i+k} \to S^k$ and $g : S^{j+k} \to S^k$. Then $f \wedge g : S^{i+j+2k} \to S^{2k}$ is map in π_{i+j}^s. Likewise, gf is a map in π_{i+j}^s. We claim that $f \wedge g \simeq gf$. By symmetry, this would give us that $fg \simeq g \wedge f$, and the problem would reduce to showing that $f \wedge g = (-1)^{ij}(g \wedge f)$. To prove the claim, we first note the following:

(1) $f \wedge g = (\mathbf{1} \wedge g)(f \wedge \mathbf{1})$, where $\mathbf{1} \wedge g : S^{j+2k} = S^k \wedge S^{j+k} \to S^{2k} = S^k \wedge S^k$ and $f \wedge \mathbf{1} : S^{i+j+2k} = S^{i+k} \wedge S^{j+k} \to S^{j+2k} = S^k \wedge S^{k+j}$. This is easily verified.

(2) $gf = (g \wedge \mathbf{1})(f \wedge \mathbf{1})$, in sufficiently high dimensions. To see this, note that f and Sf are considered identical elements of π_i^s. But since $Sf = f \wedge \mathbf{1}$, this result follows.

Therefore, to prove that $f \wedge g \simeq fg$, it suffices to show that $\mathbf{1} \wedge g \simeq g \wedge \mathbf{1}$. To see this consider the following commutative diagram, where σ and τ permute the coordinates of $S^k \wedge S^{j+k}$ and $S^k \wedge S^k$, respectively, and where the smash products may be considered as product spaces.

$$\begin{tikzcd} S^k \wedge S^{j+k} \arrow[r, "\mathbf{1}\times g"] \arrow[d, "\sigma"] & S^k \wedge S^k \arrow[d, "\tau"] \\ S^{j+k} \wedge S^k \arrow[r, "g\times\mathbf{1}"] & S^k \wedge S^k \end{tikzcd}$$

Let us consider $S^k \wedge S^{j+k}$ as I^{j+2k} with the proper identifications. Thus

$$\sigma(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{j+2k}) = (x_{k+1}, \ldots, x_{j+2k}, x_1, \ldots, x_k).$$

We see that σ is the composition of $k(j+k)$ permutations of adjacent coordinates, each such permutation having degree -1. Therefore, σ has degree $(-1)^{k(j+k)}$. We may assume without loss of generality that k is even, so σ has degree 1 and is homotopic to the identity. A similar argument shows that τ is homotopic to the identity. By the commutativity of the above diagram it follows that $\mathbf{1} \times g$ is homotopic to $g \times \mathbf{1}$, and so $\mathbf{1} \wedge g$ is homotopic to $g \wedge \mathbf{1}$, as claimed.

We now prove that $f \wedge g = (-1)^{ij}(g \wedge f)$.

$$\begin{tikzcd} S^{i+k} \wedge S^{j+k} \arrow[r, "f\wedge g"] \arrow[d, "\sigma"] & S^k \wedge S^k \arrow[d, "\tau"] \\ S^{j+k} \wedge S^{i+k} \arrow[r, "g\wedge f"] & S^k \wedge S^k \end{tikzcd}$$

In the above diagram, let σ and τ be permutations, as before. The degree of τ is still 1. In this case, however, the degree of σ is $(-1)^{(i+k)(j+k)} = (-1)^{ij}$, since k is even. We claim that $(g \wedge f)\sigma \simeq (-1)^{ij}(g \wedge f)$. This is true, because the additive inverse of an element of a homotopy group is homotopic to the precomposition of that element with a reflection. The result now follows from the commutativity of the diagram. $\square$

5. Fibrations and Fiber Bundles

Definition 5.1. Let $p : E \to B$ be a surjection. We say that p has the *covering homotopy property* with respect to a space X if, given any homotopy $G : X \times I \to B$ and a lift $\tilde{g}_0 : X \to E$ of g_0, there exists a lift $\tilde{G} : X \times I \to E$ of G which extends $\tilde{g}_0$. Diagrammatically, this property may be expressed as follows.

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\tilde{g}_0"] \arrow[d, "i"] & E \arrow[d, "p"] \\ X \times I \arrow[r, "G"] \arrow[ur, dotted, "\tilde{G}"] & B \end{tikzcd}$$

Given any G and $\tilde{g}_0$, the square diagram above commutes. The question is whether there exists a map $\tilde{G}$ after the addition of which the diagram remains commutative. If such is the case for all pairs $(G, \tilde{g}_0)$, then p has the homotopy lifting property with respect to X.

Definition 5.2. We say that a surjective map $p : E \to B$ is a *fibration* if p has the homotopy lifting property with respect to all spaces X. For a point $b \in B$, we say that $F_b = p^{-1}(b)$ is the *fiber* of b.

Example 5.3. A projection map $p : A \times B \to B$ is a fibration. Let X, G, and $\tilde{g}_0$ be given as in the diagram above. By commutativity, the projection of $\tilde{g}_0$ onto B is the identity. Thus all the information which $\tilde{g}_0$ carries is given by its projection $p_A\tilde{g}_0$ onto A. This suggests a way of extending the lift $\tilde{g}_0$ to a lift of all G. Namely, for $x \in X$ and $t \in I$, we define $\tilde{g}_t(x) = (g_t(x), p_A\tilde{g}_0(x))$.

Definition 5.4. We say that a map $p : E \to B$ has the *homotopy lifting property for a pair* (X, A) if, given a homotopy $G : X \times I \to B$ and a lift $\tilde{g} : X \times \{0\} \cup A \times I \to E$ of $G|_{X \times \{0\} \cup A \times I}$, there exists an extension $\tilde{G} : X \times I \to E$ of $\tilde{g}$ which is a lift of G. In other words, in the diagram above, we replace the upper left hand corner with $X \times \{0\} \cup A \times I$ and ask, as before, whether the hypothetical arrow $\tilde{G}$ exists such that the diagram remains commutative.

In the case of the disk D^k, the homotopy lifting property for D^k is equivalent to the homotopy lifting property for the pair $(D^k, \partial D^k)$. This follows from the fact that $D^k \times \{0\}$ is homeomorphic to $D^k \times \{0\} \cup \partial D^k \times I$.

Definition 5.5. We say that $p : E \to B$ is a *Serre fibration* if p has the homotopy lifting property with respect to all disks D^k.

Theorem 5.6. *Let $p : E \to B$ be a Serre fibration. Then given basepoints $b_0 \in B$ and $x_0 \in F = p^{-1}(b_0)$, $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isomorphism for $n \geq 2$ and bijective for $n = 1$.*

Proof. We first show surjectivity of the map $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$. Let $[g]$ be an element of $\pi_n(B, b_0)$. Consider the following commutative diagram, where the map $\partial I^n \to E$ is the constant map to x_0.

$$\begin{tikzcd} \partial I^n \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ I^n \arrow[r, "g"] \arrow[ur, dotted, "f"] & B \end{tikzcd}$$

The homotopy lifting property for the pair $(I^n, \partial I^n)$ allows us to posit the existence of the map f. By commutativity, the map f satisfies that $f(\partial I^n) \subset F$ and $f(J^{n-1}) = x_0$. Therefore, $[f]$ is an element of $\pi_n(E, F, x_0)$ such that $p_*[f] = [g]$.

We now consider injectivity. Let f be a map $(I^n, \partial I^n, J^{n-1}) \to (E, F, x_0)$ such that pf is nullhomotopic. That is, $[pf] = 0$ as an element of $\pi_n(B, b_0)$. To show injectivity, we show that f is nullhomotopic. Consider the following commutative diagram, where G is a nullhomotopy of pf.

$$\begin{tikzcd} \Gamma \arrow[r, "\varphi"] \arrow[d] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "G"] \arrow[ur, dotted, "\tilde{G}"] & B \end{tikzcd}$$

We define Γ as $I^n \times \{0\} \cup I^n \times \{1\} \cup J^{n-1} \times I$. The map φ is defined as follows.

$$\varphi = \begin{cases} f & \text{on } I^n \times \{0\} \\ x_0 & \text{on } J^{n-1} \times I \cup I \times \{1\}. \end{cases}$$

It is simple to verify that φ is well defined and that the diagram is commutative. We note that Γ is homeomorphic to I^n. Therefore, the homotopy lifting property for I^{n+1} allows us to posit the existence of the map $\tilde{G}$. We observe that $\tilde{G}$ is a nullhomotopy of f, as required. □

Corollary 5.7. *Assume the hypotheses of theorem 5.6 and additionally assume that B is path connected. Then there exists a long exact sequence*

$$\cdots \longrightarrow \pi_n(F,x_0) \xrightarrow{i_*} \pi_n(E,x_0) \xrightarrow{p_*} \pi_n(B,b_0) \xrightarrow{\partial} \pi_{n-1}(F,x_0) \longrightarrow$$
$$\cdots \longrightarrow \pi_0(E,x_0) \longrightarrow 0$$

Proof. This follows from the long exact sequence of homotopy groups for the triple (E,F,x_0) and from the fact that $\pi_n(E,F,x_0)$ is isomorphic to $\pi_n(B,b_0)$ for $n \geq 1$. It is clear that the map $p_* : \pi_n(E,x_0) \to \pi_n(B,b_0)$ is given by the composition

$$\pi_n(E,x_0) \xrightarrow{j_*} \pi_n(E,F,x_0) \xrightarrow{p_*} \pi_n(B,b_0).$$

The assumption that B is path connected is neccesary to address the end of the sequence, where $n = 0$. We claim that the map $\pi_0(F,x_0) \to \pi_0(E,x_0)$ is surjective. This is equivalent to the statement that F intersects all of the path components of E. Given a point $x \in E$, there exists a path q in B from $p(x)$ to b_0. By the homotopy lifting property, we may lift q to a path $\tilde{q}$ starting at x. If y is the endpoint of the path, then $y \in F$, since $p(y) = b_0$. □

Definition 5.8. A *fiber bundle* is a projection map $p : E \to B$ together with the fiber $F \subset E$ such that there exists an open covering $\{U_\alpha\}$ of B which satisfies that $p^{-1}(U_\alpha)$ is homeomorphic to $F \times U_\alpha$ for all α and the composition $F \times U_\alpha \to p^{-1}(U) \to U$ is the projection map $F \times U_\alpha \to U_\alpha$. We say that B is the *base space* and E is the *total space*. We often denote a fiber bundle by $F \to E \to B$, where the map p is understood.

In other words, if $F \to E \to B$ is a fiber bundle, then the following diagram must commute for all α, where h_α is the homeomorphism described above and the diagonal arrow is the projection onto the second coordinate.

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[d, "p"] & F \times U_\alpha \arrow[l, "h_\alpha"'] \arrow[dl] \\ U_\alpha & \end{tikzcd}$$

Example 5.9. We construct the Hopf bundle $S^1 \to S^3 \to S^2$. We may consider S^3 as the unit sphere in $\mathbb{C}^2$. We define a map $p : S^3 \to \mathbb{C} \cup \{\infty\} \cong S^2$ by $p(z_1, z_2) = z_1/z_2$. Equivalently, $p(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) = r_1/r_2 e^{i(\theta_1 - \theta_2)}$, where $r_1^2 + r_2^2 = 1$. Given a point $re^{i\theta} \in \mathbb{C} \cup \{\infty\}$, the equations $r_1/r_2 = r$ and $r_1^2 + r_2^2 = 1$ uniquely determine r_1 and r_2. The values of θ_1 and θ_2, however, are not uniquely determined, since if $(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \in p^{-1}(re^{i\theta})$, then so is $(r_1 e^{i\theta_1 + \lambda}, r_2 e^{i\theta_2 + \lambda})$ for all $\lambda \in [0, 2\pi]$. It follows that the fiber of each point of S^2 is homeomorphic to S^1.

It remains to show the local trivialization condition. It is not the case that $p^{-1}(S^2) = S^3$ is homeomorphic to $S^1 \times S^2$. However, if we remove a point x from S^2, then $p^{-1}(S^2 - \{x\}) = S^3 - p^{-1}(x) = S^3 - S^1$. We see that $S^3 - S^1$ is homeomorphic to $S^2 - \{\text{point}\} \times S^1$, as required. For two distinct points x_1 and x_2, the sets $S^2 - \{x_1\}$ and $S^2 - \{s_2\}$ form an open cover of S^2.

The Hopf bundle will allow us to compute $\pi_2(S^2)$ and $\pi_3(S^2)$. For this, however, we require the following theorem.

Theorem 5.10. *A fiber bundle $F \to E \to P$ is a Serre fibration.*

Proof. See [3, p. 364]. $\square$

Proposition 5.11. $\pi_2(S^2) \cong \pi_3(S^2) \cong \mathbb{Z}$

Proof. The previous theorem tells us that the Hopf bundle $S^1 \to S^3 \to S^2$ is a Serre fibration. Since S^2 is path connected, corollary 5.7 allows us to construct a long exact sequence

$$\cdots \longrightarrow \pi_n(S^1) \longrightarrow \pi_n(S^3) \longrightarrow \pi_n(S^2) \longrightarrow \pi_{n-1}(S^1) \longrightarrow \cdots.$$

Since $\pi_0(S^1) = \pi_0(S^3) = \pi_1(S^2) = \pi_1(S^3) = 0$, the end of the sequence can be simplified to

$$\cdots \longrightarrow \pi_2(S^3) \longrightarrow \pi_2(S^2) \longrightarrow \pi_1(S^1) \longrightarrow 0.$$

Since $\pi_2(S^3) = 0$, this tells us that $\pi_2(S^2) \cong \pi_1(S^1) \cong \mathbb{Z}$.

The long exact sequence also gives us that

$$0 = \pi_3(S^1) \longrightarrow \pi_3(S^3) \longrightarrow \pi_3(S^2) \longrightarrow \pi_2(S^1) = 0,$$

which implies that $\pi_3(S^2) \cong \pi_3(S^3)$. By corollary 3.8, $\pi_3(S^3) \cong \mathbb{Z}$. This proves the proposition. $\square$

The elements of $\pi_3(S^2)$ may be given a geometric interpretation by assigning to each element an integer value known as the *Hopf Invariant*. In [2], Hopf proves the following theorem:

Theorem 5.12 (Hopf). *For each map $f : S^3 \to S^2$ there exists a integer $\gamma(f)$ which satisfies*

(1) $\gamma(f) = \gamma(f')$ when f and f' are homotopic.
(2) If $\deg(g) = c$, where g is map $S^3 \to S^3$, then $\gamma(fg) = c\gamma(f)$.
(3) There exists a map $f : S^3 \to S^2$ with $\gamma(f) = 1$.

We note that the existence of the Hopf invariant γ with the above properties only implies the map $\pi_3(S^2) \to \mathbb{Z}$ given by $[f] \mapsto \gamma(f)$ is surjective. We outline the main steps of the proof, for the reader who is familiar with simplicial complexes and simplicial homology.

Proof. We assume that S^3 and S^2 have been given triangulations. The value $\gamma(f)$ is defined for a simplicial map $f : S^3 \to S^2$. The construction is then extended to an arbitrary map $g : S^3 \to S^2$, since by simplicial approximation, g is homotopic to a simplicial map f. We then define $\gamma(g) = \gamma(f)$.

Let $f : S^3 \to S^2$ be a simplicial map. The construction of $\gamma(f)$ be may reduced to four steps.

(1) Let ϵ be a point in the interior of some triangle in the triangulation of S^2. The properties of a simplicial map imply that $f^{-1}(\epsilon)$ is a finite union of polygons in S^3. We may therefore consider $f^{-1}(\epsilon)$ as a 1-cycle in S^3.
(2) Since the first homology group of S^3 is trivial, we may choose a 2-chain K_2 in S^3 such that $\partial K_2 = f^{-1}(\epsilon)$. (The details of this are tedious and involve re-triangulating S^3 such that the map f remains simplicial.)

(3) We now consider $f(K_2)$, which is a 2-chain in S^2. Since f is simplicial, $\partial f(A) = f(\partial A)$ for any subset A of S^3. In particular, $\partial f(K_2) = f(\partial K_2) = f(f^{-1}(\epsilon)) = \epsilon$. Since the boundary of $f(K_2)$ is a point, it follows that $f(K_2)$ is a cycle in S^2. Therefore, it may be associated with a degree γ.

(4) We show that the value γ is independent of ϵ, independent of the triangulations of S^3 and S^2, and independent of the simplicial map f, up to homotopy.

□

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References

[1] A. Hatcher. *Algebraic Topology.* Cambridge University Press. 2001.
[2] H. Hopf. *Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche.* 1931. Mathematische Annalen 104: 637-665.
[3] J. Rotman. *Introduction to Algebraic Topology.* Springer-Verlag. 1988.