BROWNIAN MOTION

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ABSTRACT. This paper introduces Brownian motion and covers several invariances of Brownian motion, some of which follow from the definition and another which follows from the strong Markov property of Brownian motion. We go on to show the nondifferentiability of Brownian motion, describe the set of times linear Brownian motion hits zero, and describe the area of planar Brownian motion.

CONTENTS

1. INTRODUCTION

In 1827, Robert Brown, for whom Brownian motion is named, observed the movements of pollen grains suspended in water. Later, Einstein studied similar movements and used his observations to provide important support for the atomic theory of matter. The mathematical model used to describe this random movement of a small particle suspended in a fluid is known as Brownian motion.

2. DEFINITION AND SIMPLE INVARIANCES

Definition 2.1. A real-valued stochastic process $\{B(t)|t \geq 0\}$ is called a (**linear**) **Brownian motion** with start in $x \in \mathbb{R}$ if

i) $B(0) = x$,

ii) the process has **independent increments**, i.e. for all times $0 \leq t_1 \leq t_2 \leq ... \leq t_n$, the increments $B(t_n) - B(t_{n-1}), B(t_{n-1}) - B(t_{n-2}), ..., B(t_2) - B(t_1)$ are independent random variables,

iii) for all $t \geq 0$ and $h > 0$, $B(t + h) - B(t)$ is a normally distributed random variable with mean 0 and variance h,

iv) almost surely (i.e. with probability 1), the function $t \mapsto B(t)$ is continuous.

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If $x = 0$ in i), then $\{B(t)|t \geq 0\}$ is called a **standard Brownian motion**.

We can generalize this definition to higher dimensions.

Definition 2.2. Suppose $B_1, ..., B_d$ are independent linear Brownian motions started in $x_1, ..., x_d$, respectively. Then the stochastic process $\{B(t)|t \geq 0\}$ defined by

$$B(t) := (B_1(t), ..., B_d(t))^T$$

is called a d-**dimensional Brownian motion** started in $(x_1, ..., x_d)^T$. The d-dimensional Brownian motion started at the origin is called standard Brownian motion and two-dimensional Brownian motion is called **planar Brownian motion**.

It is a nontrivial issue that there exists a random process that satisfies the conditions for Brownian motion without encountering a contradiction. We will omit the proof that Brownian motion exists but note that Paul Lévy's construction of Brownian motion as the uniform limit of continuous functions can be found in Chapter 1 of Morters' *Brownian Motion* [1].

There are several invariances of Brownian motion that will be of use in later proofs. Clearly, Brownian motion is translation invariant, i.e. if $\{B(t)|t \geq 0\}$ is a Brownian motion and $x \in \mathbb{R}$, then $\{B(t)+x|t \geq 0\}$ is also a Brownian motion. The following theorem shows that Brownian motion is also time-shift invariant.

Theorem 2.3. *(Markov Property) Suppose that $\{B(t)|t \geq 0\}$ is a Brownian motion started at $x \in \mathbb{R}^d$. Let $s > 0$, then the process $\{B(t+s) - B(s)|t \geq 0\}$ is a standard Brownian motion independent of the process $\{B(t)|0 \leq t \leq s\}$.*

Proof. Properties ii) and iii) follow from the cancellation of the $B(s)$ terms and the fact that $\{B(t)|t \geq 0\}$ is a Brownian motion. Because the map $t \mapsto B(t+s) - B(s)$ is the composition of (almost surely) continuous functions, the map $t \mapsto B(t+s) - B(s)$ is continuous. Finally, $\{B(t+s) - B(s)|t \geq 0\}$ is a standard Brownian motion since $B(0+s) - B(s) = 0$.

Recall that two stochastic processes $\{X(t)|t \geq 0\}$ and $\{Y(t)|t \geq 0\}$ are said to be independent if for any sets of times $t_1, t_2, ..., t_n \geq 0$ and $s_1, s_2, ..., s_m \geq 0$ the vectors $(X(t_1), ..., X(t_n))$ and $(Y(s_1), ..., Y(s_m))$ are independent. Let $t_1, ..., t_n \geq 0$ and $s \geq s_1, ..., s_m \geq 0$. Because Brownian motion has independent increments, it follows that $(B(t_1 + s) - B(s), ..., B(t_n + s) - B(s))$ and $(B(s_1), ..., B(s_m))$ are independent random vectors. □

Brownian motion is also scaling invariant, that is, if we "zoom in" or "zoom out" on a Brownian motion, it is still a Brownian motion. In this way, the paths of Brownian motion are in some sense random fractals, as they have a nontrivial geometric structure at all scales.

Lemma 2.4. *(Scaling Invariance) Suppose $\{B(t)|t \geq 0\}$ is a standard linear Brownian Motion and let $a > 0$. Then the process $\{X(t)|t \geq 0\}$ where $X(t) := \frac{1}{a}B(a^2t)$ is also a standard Brownian Motion.*

Proof. Clearly, $X(0) = B(0) = 0$.

If $0 \leq t_1 \leq t_2 \leq ... \leq t_n$, then $0 \leq a^2t_1 \leq a^2t_2 \leq ... \leq a^2t_n$. By the definition of Brownian motion,

$$B(a^2t_n) - B(a^2t_{n-1}), B(a^2t_{n-1}) - B(a^2t_{n-2}), ..., B(a^2t_2) - B(a^2t_1)$$

are independent random variables. It then follows that

$$\frac{1}{a}B(a^2t_n) - \frac{1}{a}B(a^2t_{n-1}), \frac{1}{a}B(a^2t_{n-1}) - \frac{1}{a}B(a^2t_{n-2}), ..., \frac{1}{a}B(a^2t_2) - \frac{1}{a}B(a^2t_1)$$

are independent random variables, i.e. the stochastic process $\{X(t)|0 \leq t\}$ has independent increments.

The fact that $X(t+h) - X(t)$ is normally distributed follows immediately from the fact that $B(a^2t + a^2h) - B(a^2t)$ is normally distributed. Furthermore,

$$\begin{aligned}\mathbb{E}[X(t+h) - X(t)] &= \mathbb{E}[\frac{1}{a}B(a^2t + a^2h) - \frac{1}{a}B(a^2t)] \\ &= \frac{1}{a}\mathbb{E}[B(a^2t + a^2h) - B(a^2t)] = 0\end{aligned}$$

where the last equality follows from the definition of Brownian Motion. To show that the variance equals h, observe that

$$\begin{aligned}Var(X(t+h) - X(t)) =& Var(\frac{1}{a}B(a^2t + a^2h) - \frac{1}{a}B(a^2t)) \\ =& \frac{1}{a^2}Var(B(a^2t + a^2h) - B(a^2t) \\ =& \frac{1}{a^2}a^2h = h.\end{aligned}$$

Because the function $t \mapsto B(t)$ is almost surely continuous, the function $t \mapsto X(t) = \frac{1}{a}B(a^2t)$ is the composition of (almost surely) continuous functions and is therefore almost surely continuous. □

Remark 2.5. The above theorem (with $a = -1$) implies that standard Brownian motion is symmetric about 0. In other words, if $\{B(t)|t \geq 0\}$ is a standard Brownian motion and $t \geq 0$, then $B(t)$ has the same distribution as $-B(t)$.

3. Strong Markov Property and the Reflection Principle

Definition 3.1. A **filtration** on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a family $(\mathcal{F}(t)|t \geq 0)$ of σ-algebras such that $\mathcal{F}(s) \subset \mathcal{F}(t) \subset \mathcal{F}$ for all $s < t$. A probability space together with a filtration is called a **filtered probability space**. A stochastic process $\{X(t)|t \geq 0\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called **adapted** if, for any $t \geq 0$, $X(t)$ is measurable with respect to $\mathcal{F}(t)$.

Suppose we have a Brownian motion $\{B(t)|t \geq 0\}$ defined on some probability space. We define the filtration $(\mathcal{F}^0(t)|t \geq 0)$ by letting $\mathcal{F}^0(t)$ be the σ-algebra generated by the random variables $\{B(s)|0 \leq s \leq t\}$. Clearly, the Brownian motion $\{B(t)|t \geq 0\}$ is adapted to this filtration $(\mathcal{F}^0(t)|t \geq 0)$. Intuitively, the σ-algebra $\mathcal{F}^0(t)$ contains all the information available from observing the Brownian motion up to time t.

Furthermore, the Markov property of Brownian motion implies that the process $\{B(t+s) - B(s)|t \geq 0\}$ is independent of $\mathcal{F}(s)$. In the next theorem, we improve this statement of independence by showing that the process is independent of the slightly larger σ-algebra $\mathcal{F}^+(t) := \bigcap_{s>t} \mathcal{F}^0(s)$. Clearly, the family $(\mathcal{F}^+(t)|t \geq 0)$ is also a filtration and $\mathcal{F}^0(t) \subset \mathcal{F}^+(t)$ for any time $t \geq 0$. Intuitively, the σ-algebra $\mathcal{F}^+(t)$ contains all the information available from observing the Brownian motion up to time t plus the information available from an additional infinitesimal glance beyond time t.

Theorem 3.2. *For every $s \geq 0$, the process $\{B(t+s)-B(s)|t \geq 0\}$ is independent of $\mathcal{F}^+(s)$.*

Proof. By continuity, $B(t+s)-B(s) = \lim_{n\to\infty} B(s_n+t)-B(s_n)$ for a strictly decreasing sequence $s_n \searrow s$. For any $t_1,...,t_m \geq 0$, the vector $(B(t_1+s)-B(s),...,B(t_m+s)-B(s)) = \lim_{j\to\infty}(B(t_1+s_j)-B(s_j),...,B(t_m+s_j)-B(s_j))$. By the Markov property of Brownian motion, $(B(t_1+s_j)-B(s_j),...,B(t_m+s_j)-B(s_j))$ is independent of $\mathcal{F}^+(s)$ for all j. Hence, $(B(t_1+s)-B(s),...,B(t_m+s)-B(s))$ is independent of $\mathcal{F}^+(s)$, which then implies that the process $\{B(t+s)-B(s)|t \geq 0\}$ is independent of $\mathcal{F}^+(s)$. □

With a filtration, we can also define stopping times, a certain type of random time that will be essential to describe the strong Markov property of Brownian motion.

Definition 3.3. A random variable T with values in $[0,\infty]$ defined on a probability space with filtration $(\mathcal{F}(t)|t \geq 0)$ is called a **stopping time** if $\{T < t\} \in \mathcal{F}(t)$ for every $t \geq 0$. It is called a **strict stopping time** if $\{T \leq t\} \in \mathcal{F}(t)$ for every $t \geq 0$.

Intuitively, a random time T is a stopping time if, for any time t, we can decide whether the event that $T < t$ occurs by knowing the process up to time t. While it is easy to show that a strict stopping time is always a stopping time, it is not always the case that a stopping time is a strict stopping time. However, strict stopping times will be equivalent to stopping times if we are defining stopping times with respect to the filtration $(\mathcal{F}^+(t)|t \geq 0)$.

Theorem 3.4. *Every stopping time T with respect to the filtration $(\mathcal{F}^+(t)|t \geq 0)$ is automatically a strict stopping time.*

Proof. We first claim that $(\mathcal{F}^+(t)|t \geq 0)$ is right continuous, i.e. $\bigcap_{\epsilon>0} \mathcal{F}^+(t+\epsilon) = \mathcal{F}^+(t)$. To see this note that

$$\bigcap_{\epsilon>0} \mathcal{F}^+(t+\epsilon) = \bigcap_{n=1}^{\infty}\bigcap_{k=1}^{\infty} \mathcal{F}^0(t+\frac{1}{n}+\frac{1}{k}) = \mathcal{F}^+(t).$$

Because T is a stopping time, $\{T < t+\frac{1}{n}\} \in \mathcal{F}^+(t+\frac{1}{n})$. Thus,

$$\{T \leq t\} = \bigcap_{n=1}^{\infty}\{T < t+\frac{1}{n}\} \in \bigcap_{n=1}^{\infty} \mathcal{F}^+(t+\frac{1}{n}) = \mathcal{F}^+(t).$$

□

Before continuing, we provide several examples of stopping times with respect to the filtration $(\mathcal{F}^+(t)|t \geq 0)$:

Examples 3.5.

i) Every deterministic time $t \geq 0$ is also a stopping time.

ii) Suppose $G \subset \mathbb{R}^d$ is open. Then the first time the Brownian motion arrives in G, i.e. $T = \inf\{t \geq 0|B(t) \in G\}$, is a stopping time.

Proof.

$$\{T < t\} = \bigcup_{s\in\mathbb{Q}\cap(0,t)} \{B(s) \in G\} \in \mathcal{F}^+(t).$$

□

iii) If $T_n \nearrow T$ is an increasing sequence of stopping times, then T is also a stopping time.

Proof.

$$\{T \leq t\} = \bigcap_{n=1}^{\infty} \{T_n \leq t\} \in \mathcal{F}^+(t).$$

□

iv) Suppose $H \subset \mathbb{R}^d$ is a closed set. Then $T = \inf\{t \geq 0 | B(t) \in H\}$ is a stopping time.

Proof. Let $G(n) := \{x \in \mathbb{R}^d |$ there exists $y \in H$ with $|x - y| < \frac{1}{n}\}$ so that $H = \bigcap_{n=1}^{\infty} G(n)$. Then $T_n := \inf\{t \geq 0 | B(t) \in G(n)\}$ are stopping times which are increasing to T. □

v) Let T be a stopping time. Define the stopping times T_n by $T_n := (m+1)2^{-n}$ if $m2^{-n} \leq T < (m+1)2^{-n}$. In other words, we stop at the first time of the form $k2^{-n}$. It is easy to see that T_n is a stopping time which we can use as a discrete approximation to T.

Definition 3.6. Let T be a stopping time. Define the σ-algebra $\mathcal{F}^+(T) := \{A \in \mathcal{A} | A \cap \{T < t\} \in \mathcal{F}^+(t)$ for all $t \geq 0\}$.

Intuitively, $\mathcal{F}^+(T)$ can be viewed as the set of events that happen before the stopping time T. With this definition, we can now state and prove the strong Markov property of Brownian motion, which says that a Brownian motion time-shifted by a stopping time is also a Brownian motion and is independent of all the events that happen before the stopping time.

Theorem 3.7. *(Strong Markov Property) For every almost surely finite stopping time T, the process $\{B(T+t) - B(T) | t \geq 0\}$ is a standard Brownian motion independent of $\mathcal{F}^+(T)$.*

Proof. Recall that $T_n := (m+1)2^{-n}$ if $m2^{-n} \leq T < (m+1)2^{-n}$. We first show our statement holds for the stopping times T_n which discretely approximate T from above. Define the random processes $\{B_k(t) | t \geq 0\}$ and $\{B_*(t) | t \geq 0\}$ where $B_k(t) := B(t + \frac{k}{2^n}) - B(\frac{k}{2^n})$ and $B_*(t) := B(t + T_n) - B(T_n)$. Suppose that $E \in \mathcal{F}^+(T_n)$. Then for every event $\{B_* \in A\}$, we have

$$\begin{aligned}
\mathbb{P}(\{B_* \in A\} \cap E) &= \sum_{k=0}^{\infty} \mathbb{P}(\{B_k \in A\} \cap E \cap \{T_n = k2^{-n}\}) \\
&= \sum_{k=0}^{\infty} \mathbb{P}(\{B_k \in A\}) \mathbb{P}(E \cap \{T_n = k2^{-n}\}),
\end{aligned}$$

using the fact that $\{B_k \in A\}$ is independent of $E \cap \{T_n = k2^{-n}\} \in \mathcal{F}^+(k2^{-n})$ by Theorem 3.2. Now, by the Markov property, $\mathbb{P}(\{B_k \in A\}) = \mathbb{P}(\{B \in A\})$, where B is a standard Brownian motion. Thus, we get

$$\begin{aligned}
\sum_{k=0}^{\infty} \mathbb{P}(\{B_k \in A\}) \mathbb{P}(E \cap \{T_n = k2^{-n}\}) &= \mathbb{P}(\{B \in A\}) \sum_{k=0}^{\infty} \mathbb{P}(E \cap \{T_n = k2^{-n}\}) \\
&= \mathbb{P}(\{B \in A\}) \mathbb{P}(E).
\end{aligned}$$

Taking E to be the entire probability space, we see that B_* has the same distributions as B. Hence, B_* is a standard Brownian motion. It then follows that $\mathbb{P}(\{B_* \in A\} \cap E) = \mathbb{P}(\{B \in A\})\mathbb{P}(E) = \mathbb{P}(\{B_* \in A\})\mathbb{P}(E)$, which shows that B_* is a Brownian motion and independent of E, hence of $\mathcal{F}^+(T_n)$, as claimed.

It remains to generalize this to general stopping times T. As $T_n \searrow T$, we have that $\{B(s+T_n) - B(T_n) | s \geq 0\}$ is a Brownian motion independent of $\mathcal{F}^+(T_n) \supset \mathcal{F}^+(T)$. Hence, the increments

$$B(s+t+T) - B(t+T) = \lim_{n\to\infty} B(s+t+T_n) - B(t+T_n)$$

of the process $\{B(r+T) - B(T) | r \geq 0\}$ are independent and normally distributed with mean zero and variance s. As the process is clearly almost surely continuous, it is a Brownian motion. Moreover, all increments $B(s+t+T_n) - B(t+T_n)$ are independent of $\mathcal{F}^+(T)$. Therefore, all increments $B(s+t+T) - B(t+T) = \lim_{n\to\infty} B(s+t+T_n) - B(t+T_n)$ are independent of $\mathcal{F}^+(T)$. Thus, the process $\{B(t+T) - B(T) | t \geq 0\}$ is independent of $\mathcal{F}^+(T)$. □

The first application of the strong Markov property comes in proving the reflection principle.

Theorem 3.8. *(Reflection Principle) If T is a stopping time and $\{B(t)|t \geq 0\}$ is a standard Brownian motion, then the process $\{B^*(t)|t \geq 0\}$ (called Brownian motion reflected at T) defined by*

$$B^*(t) = B(t)1_{\{t \leq T\}} + (2B(T) - B(t))1_{\{t > T\}}$$

is also a standard Brownian motion.

Proof. If T is finite, the strong Markov property implies that both

$$\{B(t+T) - B(T) | t \geq 0\} \text{ and } \{-(B(t+T) - B(T)) | t \geq 0\}$$

are Brownian motions independent of the beginning $\{B(t) | t \in [0, T]\}$. Hence, the concatenations of the process up to time T with each of the processes after time T have the same distributions, i.e. the processes defined by

$$B(t)1_{t \leq T} + (B(t+T) - B(T) + B(T))1_{t > T} \text{ and}$$
$$B(t)1_{t \leq T} + (-B(t+T) + B(T) + B(T))1_{t > T}$$

have the same distributions. Because the first process is $\{B(t)|t \geq 0\}$ and the second is $\{B^*(t)|t \geq 0\}$, it follows that $B^*(t)$ is also a standard Brownian motion. □

Let $M(t) = \sup_{0 \leq s \leq t} B(s)$ for a standard linear Brownian motion $\{B(t)|t \geq 0\}$. The reflection principle allows us to describe the distributions of $M(t)$. These distributions will later be used to prove the law of large numbers for Brownian motion.

Corollary 3.9. *If $a > 0$, then $\mathbb{P}(M(t) > a) = 2\mathbb{P}(B(t) > a)$.*

Proof. Let $T = \inf\{t \geq 0 | B(t) = a\}$ and let $\{B^*(t)|t \geq 0\}$ be a Brownian motion reflected at T. Then, conditioning on the value of $B(t)$, the event $\{M(t) > a\}$ is the disjoint union of the events $\{B(t) > a, M(t) > a\} = \{B(t) > a\}$ and $\{M(t) > a, B(t) \leq a\}$. The latter event is equivalent to the event $\{B^*(t) \geq a\}$. Since the reflection principle implies that $B^*(t)$ has the same distribution as $B(t)$, it follows that $\mathbb{P}(M(t) > a) = 2\mathbb{P}(B(t) > a)$. □

4. Time Inversion

The reflection principle allows us to prove another invariance of Brownian motion. In this section we will show that, if $\{B(t)|t \geq 0\}$ is a standard Brownian motion, then the process $\{X(t)|t \geq 0\}$ defined by

$$X(t) = \begin{cases} tB(\frac{1}{t}) & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

is also a standard Brownian motion.

Definition 4.1. A random vector $(X_1, ..., X_n)$ is called a **Gaussian random vector** if there exists an $n \times m$ matrix A and an n-dimensional vector b such that $X^T = AY + b$, where Y is an $m-$dimesional vector with independent standard normal entries.

Definition 4.2. A stochastic process $\{Y(t)|t \geq 0\}$ is called a **Gaussian process**, if for all $t_1 < t_2 < ... < t_n$ the vector $(Y(t_1), Y(t_2), ..., Y(t_n))$ is a Gaussian random vector.

Proposition 4.3. *A Brownian motion $\{B(t)|t \geq 0\}$ is a Gaussian process.*

Proof. Let $0 < t_1 < t_2 < ... < t_n$. It follows that

$$\begin{pmatrix} B(t_1) \\ B(t_2) \\ \vdots \\ B(t_n) \end{pmatrix} = \begin{pmatrix} B(t_1) - B(0) \\ B(t_2) - B(t_1) + B(t_1) - B(0) \\ \vdots \\ (\sum_{i=2}^{n} B(t_i) - B(t_{i-1}) + B(t_1) - B(0)) + B(0) \end{pmatrix} + \begin{pmatrix} B(0) \\ B(0) \\ \vdots \\ B(0) \end{pmatrix}$$

$$= XY \begin{pmatrix} \frac{B(t_1)-B(0)}{\sqrt{t_1 - 0}} \\ \frac{B(t_2)-B(t_1)}{\sqrt{t_2 - t_1}} \\ \vdots \\ \frac{B(t_n)-B(t_{n-1})}{\sqrt{t_n - t_{n-1}}} \end{pmatrix} + \begin{pmatrix} B(0) \\ B(0) \\ \vdots \\ B(0) \end{pmatrix}$$

where X is the lower left triangular matrix of all 1's and Y is the diagonal matrix with entries $y_{ii} = \sqrt{t_i - t_{i-1}}$. Therefore, $(B(t_1), ..., B(t_n))$ is a Gaussian random vector, and it follows that $\{B(t)|t \geq 0\}$ is a Gaussian process. □

If $\{B(t)|t \geq 0\}$ is a standard Brownian motion and $0 < t_1 < t_2 < ... < t_n$, it is easy to see that $\mathbb{E}(B(t_i)) = 0$ and $\text{Cov}(B(t_i), B(t_j)) = \min(t_i, t_j)$. Because Gaussian random vectors are characterized by their expectation and covariance matrices, if two Gaussian processes have the same expectation and covariance matrices for all sets of times $0 < t_1 < t_2 < ... < t_n$, then those two Gaussian processes have the same distributions. We will use this fact to prove that Brownian motion is invariant under time inversion.

We now state without proof an elementary result in probability theory that will be used now and in later proofs.

Lemma 4.4. *(Borel-Cantelli Lemma) Let (E_n) be a sequence of events on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If $\sum_{n=1}^{\infty} \mathbb{P}(E_n) < \infty$ then $\mathbb{P}(\limsup_{n\to\infty} E_n) = 0$.*

Before we prove that Brownian motion is invariant under time inversion, we will need the following lemma to show continuity at time $t = 0$.

Lemma 4.5. *(Law of Large Numbers) Almost surely,* $\lim_{t\to\infty} \frac{B(t)}{t} = 0$.

Proof. First, we will show that $\lim_{n\to\infty} \frac{B(n)}{n} = 0$. Let $\epsilon > 0$ and consider $\sum_{n=1}^{\infty} \mathbb{P}\left(\frac{B(n)}{n} > \epsilon\right)$. It follows from the fact that $\frac{B(n)}{\sqrt{n}}$ has a standard normal distribution that

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\frac{B(n)}{n} > \epsilon\right) = \sum_{n=1}^{\infty} \mathbb{P}\left(\frac{B(n)}{\sqrt{n}} > \epsilon\sqrt{n}\right) \leq \sum_{n=1}^{\infty} \frac{1}{\epsilon\sqrt{n}} \frac{1}{\sqrt{2\pi}} e^{-(n\epsilon^2)/2}.$$

Because the last sum is finite, $\sum_{n=1}^{\infty} \mathbb{P}\left(\frac{B(n)}{n} > \epsilon\right) < \infty$. The Borel-Cantelli Lemma then implies that there exists N such that, for all $n \geq N$, $\frac{B(n)}{n} < \epsilon$ almost surely. Since $\mathbb{P}\left(\frac{B(n)}{n} < -\epsilon\right) = \mathbb{P}\left(-\frac{B(n)}{n} > \epsilon\right) = \mathbb{P}\left(\frac{B(n)}{n} > \epsilon\right)$, it follows that, for sufficiently large n, $\frac{B(n)}{n} > -\epsilon$ almost surely. Therefore, $\lim_{n\to\infty} \frac{B(n)}{n} = 0$.

To extend the result to all nonnegative times t, it suffices to show that

$$\limsup_{n\to\infty} \frac{1}{n}\left(\max_{n\leq t\leq n+1} B(t) - B(n)\right) = 0.$$

First, note that, for $\epsilon > 0$, the reflection principle and the fact the $B(1)$ has a standard normal distribution imply that

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{0\leq t\leq 1} B(t) > \epsilon n\right) = \sum_{n=1}^{\infty} 2\mathbb{P}(B(1) \geq \epsilon n) \leq 2\sum_{n=1}^{\infty} \frac{1}{\epsilon n} \frac{1}{\sqrt{2\pi}} e^{-\frac{\epsilon^2 n^2}{2}}.$$

Again, the last sum is finite so the Borel-Cantelli Lemma implies that $\mathbb{P}(\limsup_{n\to\infty} (\max_{0\leq t\leq 1} B(t)) > \epsilon n) = 0$. Because the distributions of $\max_{0\leq t\leq 1} B(t)$ and $\max_{n\leq t\leq n+1} B(t) - B(n)$ are the same by time-shift invariance, it follows that for sufficiently large n, $\frac{1}{n}\max_{n\leq t\leq n+1} B(t) - B(n) < \epsilon$ almost surely. Thus, $\limsup_{n\to\infty} \frac{1}{n}(\max_{n\leq t\leq n+1} B(t) - B(n)) = 0$. Since the distribution of $B(t)$ is symmetric about 0, it follows that $\lim_{n\to\infty} \frac{1}{n}(\max_{n\leq t\leq n+1} |B(t) - B(n)|) = 0$. Because $\frac{B(t)}{t}$ is bounded by $\frac{1}{n}(B(n) + \max_{n\leq s\leq n+1} |B(s) - B(n)|)$ for $n \leq t \leq n+1$, the squeeze theorem implies that $\lim_{t\to\infty} \frac{B(t)}{t} = 0$. ☐

Theorem 4.6. *(Time Inversion) Suppose* $\{B(t)|t \geq 0\}$ *is a standard Brownian Motion. Then the process* $\{X(t)|t \geq 0\}$ *defined by*

$$X(t) = \begin{cases} tB(\frac{1}{t}) & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

is also a standard Brownian Motion.

Proof. By construction, $X(0) = 0$.

By Proposition 4.3, for all times $t_1, t_2, ..., t_n$, the finite dimensional marginals $\big(B(t_1), B(t_2), ..., B(t_n)\big)$ of Brownian motion are Gaussian random vectors and are therefore characterized by the mean and covariance matrix. Thus, it suffices to prove that for all times $t_1, t_2, ..., t_n$ the finite dimensional marginals $\big(X(t_1), X(t_2), ..., X(t_n)\big)$ are Gaussian random vectors with $\mathbb{E}[X(t_i)] = \mathbb{E}[B(t_i)] = 0$ for all $1 \leq i \leq n$ and $\mathrm{Cov}\big(X(t_i), X(t_j)\big) = \mathrm{Cov}\big(B(t_i), B(t_j)\big) = \min(t_i, t_j)$.

Let $t_1, t_2, ..., t_n > 0$. Then

$$\begin{pmatrix} X(t_1) \\ X(t_2) \\ \vdots \\ X(t_n) \end{pmatrix} = \begin{pmatrix} t_1 B(\frac{1}{t_1}) \\ t_2 B(\frac{1}{t_2}) \\ \vdots \\ t_n B(\frac{1}{t_n}) \end{pmatrix} = \begin{pmatrix} t_1 & 0 & ... & 0 \\ 0 & t_2 & ... & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & ... & t_n \end{pmatrix} \begin{pmatrix} B(\frac{1}{t_1}) \\ B(\frac{1}{t_2}) \\ \vdots \\ B(\frac{1}{t_n}) \end{pmatrix}$$

which is a Gaussian random vector by similar logic as in the proof of theorem 4.3.

The expectations $\mathbb{E}\big(X(t_i)\big) = t_i \mathbb{E}\big(B(\frac{1}{t_i})\big)$ equal zero. The covariances, for $t > 0$ and $h \geq 0$, are given by

$$\mathrm{Cov}\big(X(t+h), X(t)\big) = (t+h)t\mathrm{Cov}\left(B\left(\frac{1}{t+h}\right), B\left(\frac{1}{t}\right)\right) = t(t+h)\frac{1}{t+h} = t.$$

Thus, for all times $t_1, t_2, ..., t_n$, the distributions of the finite dimensional marginals $\big(X(t_1), X(t_2), ..., X(t_n)\big)$ are the same as for $\big(B(t_1), B(t_2), ..., B(t_n)\big)$. This implies that the process $\{X(t) | t \geq 0\}$ has independent increments and that, for all $t \geq 0$ and $h > 0$, the increments $X(t+h) - X(t)$ are normally distributed with expectation zero and variance h.

Clearly, the map $t \mapsto X(t)$ is continuous away from 0. For continuity at $t = 0$, we use the law of large numbers and see that, almost surely,

$$0 = \lim_{t \to \infty} \frac{B(t)}{t} = \lim_{t \to 0} tB\left(\frac{1}{t}\right) = \lim_{t \to 0} X(t).$$

Hence, the map $t \mapsto X(t)$ is continuous for all $t \geq 0$ almost surely. □

5. Nondifferentiability of Brownian Motion

Though Brownian motion is almost surely continuous, this section will show that Brownian motion is almost surely differentiable nowhere. To give an indication as to how Brownian motion is quite erratic, we present the following proposition:

Proposition 5.1. *Almost surely, for all* $0 < a < b < \infty$*, Brownian motion is not monotone on the interval* $[a, b]$*.*

Proof. Fix an interval $[a, b]$. Suppose that the Brownian motion $\{B(t) | t \geq 0\}$ is monotone on $[a, b]$. Partition the interval $[a, b]$ into n sub-intervals by picking times $a = a_1 < a_2 < ... < a_n < a_{n+1} = b$. If B is monotone on $[a, b]$, then each increment $B(a_i) - B(a_{i-1})$ must have the same sign. Because the increments are independent and normally distributed, the event that all the increments $B(a_i) - B(a_{i-1})$ have the same sign occurs with probability $2 \cdot 2^{-n}$. Taking $n \to \infty$ shows that $[a, b]$ is an interval of monotonicity with probability 0.

It follows that if we take the countable union over all the intervals $[a, b]$ where a and b are rational endpoints, we see that almost surely Brownian motion is not monotone on any interval with rational endpoints. For any other interval $[r, s]$, we can find a sub-interval $[a, b] \subset [r, s]$ where a and b are rational. Because monotonicity on $[r, s]$ implies monotonicity on $[a, b]$, it follows that almost surely Brownian motion is not monotone on any interval $[a, b]$. □

Now that we have an intuition as to how Brownian motion can be erratic, we can begin showing that Brownian motion is almost surely nowhere differentiable.

The proof of the next lemma uses the definition of exchangeable events and the Hewitt-Savage 0-1 law which we now recall. A proof of the Hewitt-Savage 0-1 law may be found in Chapter 3 of Durrett's *Probability: Theory and Examples* [2].

Definition 5.2. Let (X_n) be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and consider a set A of sequences such that $\{\omega \in \Omega | X_n(\omega) \in A\} \in \mathcal{F}$. The event $\{X_1, X_2, ... \in A\}$ is called **exchangeable** if

$$\{X_1, X_2, ... \in A\} \subset \{X_{\sigma_1}, X_{\sigma_2}, ... \in A\}$$

for all finite permutations $\sigma : \mathbb{N} \to \mathbb{N}$. Here, finite permutations means that σ is a bijection with $\sigma_n = n$ for all sufficiently large n.

Theorem 5.3. *(Hewitt-Savage 0-1 law) If A is an exchangeable event for an independent, identically distributed sequence, then $\mathbb{P}(A)$ is 0 or 1.*

Lemma 5.4. *Almost surely,*

$$\limsup_{n\to\infty} \frac{B(n)}{\sqrt{n}} = +\infty, \text{ and } \liminf_{n\to\infty} \frac{B(n)}{\sqrt{n}} = -\infty,$$

Proof. By Fatou's lemma,

$$\mathbb{P}\{B(n) > c\sqrt{n} \text{ infinitely often}\} \geq \limsup_{n\to\infty} \mathbb{P}\{B(n) > c\sqrt{n}\}.$$

Using scaling invariance with $a = \frac{1}{\sqrt{n}}$, we have $\limsup_{n\to\infty} \mathbb{P}\{B(n) > c\sqrt{n}\} = \mathbb{P}\{B(1) > c\}$ which is greater than 0 since $B(1)$ has a standard normal distribution. Let $X_n = B(n) - B(n-1)$ and note that the event

$$\{B(n) > c\sqrt{n} \text{ infinitely often}\} = \left\{\sum_{j=1}^{n} X_j > c\sqrt{n} \text{ infinitely often}\right\}$$

is an exchangeable event. It then follows from the Hewitt-Savage 0-1 law that, almost surely, $B(n) > c\sqrt{n}$ infinitely often. Taking the intersection over all the positive integers c gives that $\limsup_{n\to\infty} \frac{B(n)}{\sqrt{n}} = +\infty$ and the second part of the theorem follows by similar logic. □

Definition 5.5. For a function f, we define the **upper** and **lower right derivatives**

$$D^*f(t) := \limsup_{h\searrow 0} \frac{f(t+h) - f(t)}{h} \text{ and } D_*f(t) := \liminf_{h\searrow 0} \frac{f(t+h) - f(t)}{h}.$$

Theorem 5.6. *Fix $t \geq 0$. Then, almost surely, Brownian motion is not differentiable at t. Furthermore, $D^*B(t) = +\infty$ and $D_*B(t) = -\infty$.*

Proof. Given a standard Brownian motion $\{B(t)|t \geq 0\}$ we construct another standard Brownian motion $\{X(t)|t \geq 0\}$ by time inversion as in Theorem 4.6. It then follows that

$$D^*X(0) \geq \limsup_{n\to\infty} \frac{X(\frac{1}{n}) - X(0)}{\frac{1}{n}} \geq \limsup_{n\to\infty} \sqrt{n}X(\frac{1}{n}) = \limsup_{n\to\infty} \frac{B(n)}{\sqrt{n}}$$

which is infinite by lemma 5.4. By similar logic, $D_*X(0) = -\infty$. Hence, standard Brownian motion is not differentiable at 0.

Now let $t > 0$ be arbitrary and let $\{B(t)|t \geq 0\}$ be Brownian motion. It follows from time-shift invariance that $Y(s) := B(t+s) - B(t)$ defines a standard Brownian motion. We just showed that standard Brownian motion is not differentiable at $s = 0$, which implies that B is not differentiable at t. □

Corollary 5.7. *Let $\{B(t)|t \geq 0\}$ be a standard linear Brownian motion. Then* $\inf\{t > 0|B(t) = 0\} = 0$.

Proof. By theorem 5.6, almost surely

$$D^*B(0) = \limsup_{t \searrow 0} \frac{B(t)}{t} = +\infty$$

which implies that there exist arbitrarily small $t > 0$ such that $B(t) > 0$. Similarly, $D_*B(0) = -\infty$ implies that there exist arbitrarily small $t > 0$ such that $B(t) < 0$. Let $\epsilon > 0$ and choose $0 < t^*, t_* < \epsilon$ such that $B(t^*) > 0$ and $B(t_*) < 0$. It then follows from almost sure continuity of B that there exists some time $t > 0$ between t^* and t_* such that $B(t) = 0$. Thus, there exists $0 < t < \epsilon$ such that $B(t) = 0$ for any $\epsilon > 0$ and so $\inf\{t > 0 | B(t) = 0\} = 0$. □

Corollary 5.7, along with a simple application of translation invariance, shows that almost surely for any $\epsilon > 0$ there exists $0 < t < \epsilon$ such that the Brownian motion hits its starting point at time t.

We have shown that for any time $t \geq 0$, Brownian motion is not differentiable at t almost surely. This statement is quite different from the claim that, almost surely, Brownian motion is not differentiable for any time $t \geq 0$. This claim, however, is in fact true.

Theorem 5.8. *Almost surely, Brownian motion is nowhere differentiable. Furthermore, almost surely, for all times* $t \geq 0$,

$$D^*B(t) = +\infty \text{ or } D_*B(t) = -\infty.$$

Proof. Suppose there is a time $t_0 \in [0,1]$ such that $-\infty < D_*B(t) \leq D^*B(t) < +\infty$. Then

$$\limsup_{h \searrow 0} \frac{|B(t_0 + h) - B(t_0)|}{h} < \infty.$$

It follows that we can find constants $\delta > 0$ and M' such that for all $0 < h < \delta$

$$\frac{|B(t_0 + h) - B(t_0)|}{h} < M'.$$

Because Brownian motion is almost surely continuous, it is almost surely bounded on $[0,2]$. Hence, there exists a constant M'' such that for all $\delta \leq h \leq 1$,

$$\frac{|B(t_0 + h) - B(t_0)|}{h} \leq \frac{|B(t_0 + h) - B(t_0)|}{\delta} \leq M''.$$

Taking $M = \max(M', M'')$, we have that

$$\sup_{h \in (0,1]} \frac{|B(t_0 + h) - B(t_0)|}{h} \leq M.$$

It suffices to show that this event has probability zero for any M.

Fix M and suppose that $\sup_{h \in [0,1]} \frac{|B(t_0+h) - B(t_0)|}{h} \leq M$. If t_0 is contained in the binary interval $\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$ for $n > 2$, then for all $1 \leq j \leq 2^n - k$ the triangle inequality implies that

$$\begin{aligned} \left| B\left(\frac{k+j}{2^n}\right) - B\left(\frac{k+j-1}{2^n}\right) \right| &\leq \left| B\left(\frac{k+j}{2^n}\right) - B(t_0) \right| + \left| B\left(\frac{k+j-1}{2^n}\right) - B(t_0) \right| \\ &\leq M \frac{2j+1}{2^n}. \end{aligned}$$

Define the events

$$\Omega_{n,k} := \left\{ \left| B\left(\frac{k+j}{2^n}\right) - B\left(\frac{k+j-1}{2^n}\right) \right| \le M\frac{2j+1}{2^n} \text{ for } j = 1,2,3 \right\}.$$

It then follows from the independence of the increments and scaling invariance $\left(\text{with } a = \frac{1}{\sqrt{2^n}}\right)$ that, for $1 \le k \le 2^n - 3$,

$$\begin{aligned}\mathbb{P}(\Omega_{n,k}) &\le \prod_{j=1}^{3} \mathbb{P}\left(\left| B\left(\frac{k+j}{2^n}\right) - B\left(\frac{k+j-1}{2^n}\right) \right| \le M\frac{2j+1}{2^n} \right\} \\ &\le \mathbb{P}\left(|B(1)| \le M\frac{7}{\sqrt{2^n}} \right)^3,\end{aligned}$$

which is bounded above by $(7M2^{-\frac{n}{2}})^3$ since the normal density is bounded by $\frac{1}{2}$. Thus,

$$\mathbb{P}\left(\bigcup_{k=1}^{2^n-3} \Omega_{n,k} \right) \le 2^n(7M2^{-\frac{n}{2}})^3 = (7M)^3 2^{-\frac{n}{2}}$$

which is summable over all $n \in \mathbb{N}$. It then follows from the Borel-Cantelli lemma that

$$\begin{aligned}&\mathbb{P}\left(\text{there is a } t_0 \in [0,1] \text{ with } \sup_{h\in[0,1]} \frac{|B(t_0+h) - B(t_0)|}{h} \le M \right) \\ &\le \mathbb{P}\left(\bigcup_{k=1}^{2^n-3} \Omega_{n,k} \text{ for infinitely many } n \in \mathbb{N} \right) = 0,\end{aligned}$$

where the inequality follows from the fact that the first event implies the second. Taking the countable union over all $M \in \mathbb{N}$ and intervals $[i, i+1]$ with $i \in \mathbb{N}$ completes the proof. □

6. The Zero Set of Brownian Motion

In the previous section, we saw that standard linear Brownian motion hits 0 for arbitrary small times $t > 0$. In this section, we will show that the set of times that any linear Brownian motion hits 0 is almost surely a closed set with no isolated points and, hence, uncountable.

First, we show that, if we fix a point M, then standard Brownian motion almost surely hits that point.

Lemma 6.1. *Let $\{B(t)|t \ge 0\}$ be a standard Brownian motion and fix a constant M. Define $\tau := \inf\{t \ge 0 | B(t) = M\}$. Then $\tau < \infty$ almost surely.*

Proof. If $M = 0$ then $\tau = 0$ for a standard Brownian motion. Now take the case $M < 0$ and consider the event

$$\{B(t) \ge M \text{ for all } t \ge 0\} = \bigcap_{n=1}^{\infty} \left\{ \min_{0\le t\le n} B(t) \ge M \right\}.$$

It follows from scaling invariance (with $a = -1$) and corollary 3.9 that

$$\begin{aligned}\mathbb{P}\left(\bigcap_{n=1}^{\infty}\left\{\min_{0\le t\le n} B(t) \ge M\right\}\right) &= \lim_{n\to\infty}\mathbb{P}\left(\min_{0\le t\le n} B(t) \ge M\right)\\ &= \lim_{n\to\infty}\mathbb{P}\left(\max_{0\le t\le n} B(t) \le -M\right)\\ &= 1-\lim_{n\to\infty}\mathbb{P}\left(\max_{0\le t\le n} B(t) > -M\right)\\ &= 1-2\lim_{n\to\infty}\mathbb{P}(B(n) > -M)\\ &= 1-2\cdot\frac{1}{2} = 0.\end{aligned}$$

Hence, almost surely τ is finite in the case where $M < 0$. The case where $M > 0$ follows similarly. □

Remark 6.2. By taking the countable intersection over all $M \in \mathbb{N}$ and applying lemma 6.1, we see that Brownian motion gets arbitrarily far away from its starting point.

Theorem 6.3. *Let* $\{B(t)|t \ge 0\}$ *be a linear Brownian motion and define*

$$\mathbf{Zero} := \{t \ge 0|B(t) = 0\}$$

to be the **zero set**. *Then, almost surely,* **Zero** *is a closed set with no isolated points.*

Proof. Clearly, with probability one, **Zero** is closed because Brownian motion is continuous almost surely. To prove that no point of **Zero** is isolated we consider the following construction: For each rational $q \in [0,\infty)$ consider the first zero after q, i.e.,

$$\tau_q = \inf\{t \ge q|B(t) = 0\}.$$

Lemma 6.1 applied to the time-shifted Brownian motion $\{B(t+q) - B(q)|t \ge 0\}$ with $M = -B(q)$ proves that τ_q is an almost surely finite stopping time. Since **Zero** is closed, the inf is almost surely a minimum, i.e. $\tau_q \in$ **Zero**. By the strong Markov property for stopping time τ_q and corollary 5.7, we have that, for each q, almost surely τ_q is not an isolated zero from the right. But, since there are only countably many rationals, we conclude that almost surely, for all rationals q, τ_q is not an isolated zero from the right.

Our next task is to prove that the remaining points of **Zero** are not isolated from the left. So we claim that any $0 < t \in$ **Zero** which is different from τ_q for all rationals q is not an isolated point from the left. To see this take a sequence $q_n \nearrow t$, $q_n \in \mathbb{Q}$. Clearly, $q_n \le \tau_{q_n} < t$ and so $\tau_{q_n} \nearrow t$. Since $\tau_{q_n} \in$ **Zero** for all n almost surely, t is not isolated from the left. □

Since **Zero** is a closed set with no isolated points, the Baire Category theorem will imply that **Zero** is uncountable.

Corollary 6.4. **Zero** *is uncountable.*

Proof. Consider **Zero** as a metric space with the metric inherited from $\mathbb{R}$. Since **Zero** is closed in $\mathbb{R}$, **Zero** is a complete metric space. For any $z \in$ **Zero**, z is not isolated which implies that the set $\{z\}$ is nowhere dense. The Baire Category theorem states that a non-empty complete metric space is not the countable union of nowhere dense sets. Because **Zero** is a complete metric space and **Zero** $= \bigcup_{z\in \mathbf{Zero}}\{z\}$, it follows from that **Zero** must be uncountable. □

7. The Area of Planar Brownian Motion

In this the final section, we turn our attention to Brownian motion in two dimensions and show that the area of the image of Brownian motion in two dimensions is almost surely 0. This fact is nontrivial as there are continuous space-filling curves such as Peano's curve. Throughout the section we denote two dimensional Lebesgue measure by $\mathcal{L}_2$.

Lemma 7.1. *If $A_1, A_2 \subset \mathbb{R}^2$ are Borel sets with positive area, then*

$$\mathcal{L}_2\Big(\big\{x \in \mathbb{R}^2 | \mathcal{L}_2\big(A_1 \cap (A_2 + x)\big) > 0\big\}\Big) > 0.$$

Proof. It suffices to prove the lemma in the case where A_1 and A_2 are bounded since otherwise we could take bounded subsets with positive area. Using $*$ to denote convolution, Fubini's theorem implies that

$$\begin{aligned}\int_{\mathbb{R}^2} 1_{A_1} * 1_{-A_2}(x)dx &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 1_{A_1}(w)1_{A_2}(w-x)dw\ dx \\ &= \int_{\mathbb{R}^2} 1_{A_1}(w)\left(\int_{\mathbb{R}^2} 1_{A_2}(w-x)dx\right)\ dw \\ &= \mathcal{L}_2(A_1)\mathcal{L}_2(A_2) > 0.\end{aligned}$$

Thus, $1_{A_1} * 1_{-A_2}(x) > 0$ on a set of positive area. But

$$\begin{aligned}1_{A_1} * 1_{-A_2}(x) &= \int 1_{A_1}(y)1_{-A_2}(x-y)dy \\ &= \int 1_{A_1}(y)1_{A_2+x}(y)dy \\ &= \mathcal{L}_2\big(A_1 \cap (A_2+x)\big),\end{aligned}$$

proving the lemma. □

Theorem 7.2. *Almost surely, $\mathcal{L}_2(B[0,1]) = 0$.*

Proof. Let $X = \mathcal{L}_2(B[0,1])$ denote the area of $B[0,1]$. First, we check that $\mathbb{E}(X) < \infty$. Note that $X > a$ only if the Brownian motion leaves the square centered at the origin with sidelength $\frac{\sqrt{a}}{2}$. Hence, using corollary 3.9

$$\mathbb{P}(X > a) \le 2\mathbb{P}\left(\max_{t\in[0,1]}|W(t)| > \frac{\sqrt{a}}{2}\right) = 4\mathbb{P}\left(W(1) > \frac{\sqrt{a}}{2}\right) \le 4e^{-a/8},$$

for $a > 1$, where $\{W(t)|t \ge 0\}$ is a standard linear Brownian motion. Hence,

$$\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X > a)da \le 4\int_1^\infty e^{-a/8}da + 1 < \infty.$$

Note that $B(3t)$ and $\sqrt{3}B(t)$ have the same distribution by scaling invariance, and hence

$$\mathbb{E}\big(\mathcal{L}_2(B[0,3])\big) = 3\mathbb{E}\big(\mathcal{L}_2(B[0,1])\big) = 3\mathbb{E}(X).$$

Note that we have $\mathcal{L}_2(B[0,3]) \le \sum_{j=0}^{2} \mathcal{L}_2(B[j,j+1])$ with equality if and only if for $0 \le i < j \le 2$ we have $\mathcal{L}_2(B[i,i+1] \cap B[j,j+1]) = 0$. On the other hand, for $j = 0,1,2$ the Markov property implies that $\mathbb{E}\big(\mathcal{L}_2(B[j,j+1])\big) = \mathbb{E}(X)$ and

$$3\mathbb{E}(X) = \mathbb{E}\big(\mathcal{L}_2(B[0,3])\big) \le \sum_{j=0}^{2} \mathbb{E}\big(\mathcal{L}_2(B[j,j+1])\big) = 3\mathbb{E}(X),$$

whence, almost surely, the intersection of any two of the $B[j,j+1]$ has measure zero. In particular, $\mathcal{L}_2(B[0,1] \cap B[2,3]) = 0$ almost surely.

Now we can use the Markov property to define two Brownian motions, $\{B_1(t) | t \in [0,1]\}$ by $B_1(t) := B(t)$ and $\{B_2(t) | t \in [0,1]\}$ by $B_2(t) = B(t+2) - B(2) + B(1)$. Both Brownian motions are independent of the random variable $Y := B(2) - B(1)$ by the Markov property. For $x \in \mathbb{R}^2$, let $R(x)$ denote the area of the set $B_1[0,1] \cap (x + B_2[0,1])$, and note that $\{R(x) | x \in \mathbb{R}^2\}$ is independent of Y because B_1 and B_2 are independent of Y. Then

$$0 = \mathbb{E}\big(\mathcal{L}_2(B[0,1] \cap B[2,3])\big) = \mathbb{E}\big(R(Y)\big) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{\frac{|x|^2}{2}} \mathbb{E}\big(R(x)\big) dx,$$

where we are averaging with respect to the Gaussian distribution of $B(2) - B(1)$. Thus, $R(x) = 0$ almost surely for $\mathcal{L}_2$-almost all x, and hence

$$\mathcal{L}_2(\{x \in \mathbb{R}^2 | R(x) > 0\}) = 0, \text{ almost surely.}$$

From lemma 7.1 we get that, almost surely, $\mathcal{L}_2(B[0,1]) = 0$ or $\mathcal{L}_2(B[2,3]) = 0$. The observation that $\mathcal{L}_2(B[0,1])$ and $\mathcal{L}_2(B[2,3])$ are identically distributed and independent completes the proof that $\mathcal{L}_2(B[0,1]) = 0$ almost surely. □

A consequence of this theorem is that planar Brownian motion almost surely does not hit singletons. Hence, unlike linear Brownian motion which hits 0 uncountably many times, planar Brownian motion with starting point $x \neq 0$ almost surely never hits 0.

Corollary 7.3. *For any points $x, y \in \mathbb{R}^d$, $d \ge 2$, we have $\mathbb{P}(y \in B(0,1]) = 0$.*

Proof. Observe that, by projection onto the first two coordinates, it suffices to prove this result for $d = 2$. Note that theorem 7.2 holds for Brownian motion with arbitrary starting point $y \in \mathbb{R}^2$. By Fubini's theorem, for any fixed $y \in \mathbb{R}^2$,

$$\int_{\mathbb{R}^2} \mathbb{P}_y(x \in B[0,1]) dx = \mathbb{E}_y\big(\mathcal{L}_2(B[0,1])\big) = 0.$$

Hence, for $\mathcal{L}_2$-almost every point x, we have $\mathbb{P}_y(x \in B[0,1]) = 0$. By symmetry of Brownian motion,

$$\mathbb{P}_y(x \in B[0,1]) = \mathbb{P}_0(x - y \in B[0,1]) = \mathbb{P}_0(y - x \in B[0,1]) = \mathbb{P}_x(y \in B[0,1]).$$

We infer that $\mathbb{P}_x(y \in B[0,1]) = 0$, for $\mathcal{L}_2$-almost every point x. For any $\epsilon > 0$, we thus have, almost surely, $\mathbb{P}_{B(\epsilon)}(y \in B[0,1]) = 0$. Hence,

$$\mathbb{P}(y \in B(0,1]) = \lim_{\epsilon \searrow 0} \mathbb{P}(y \in B[\epsilon,1]) = \lim_{\epsilon \searrow 0} \mathbb{P}_{B(\epsilon)}(y \in B[0,1-\epsilon]) = 0,$$

where we have used the Markov property in the second step. □

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