POINTERS TO REU PAPER TOPICS (IN AND AROUND MODEL THEORY)

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Below are a few possible suggestions for REU paper topics in and around model theory. Any interested people are encouraged to discuss with me before or after class. Some topics involve substantial logic, some involve applying logic to other areas, and some are ideal for those with a background in other areas interested in what logic might have to say about their field. Others require only "mathematical maturity" and an inquiring mind. Books referenced are all ones our library should have, and any paper mentioned can be found either on the author's webpage or from mathscinet (www.ams.org/mathscinet, accessible from campus).

(1) The classic model theory text is Chang and Keisler, available in the library. Topics covered in the book which would make a reasonably self-contained paper include: (a) ultraproducts, (b) the Ax-Kochen-Ersov theorem for those with a strong algebra background, (c) Morley's theorem for those with a strong model theory background, (d) the Keisler-Shelah theorem that elementarily equivalent structures have isomorphic ultrapowers (requires some model theory and set theory).

(2) Model theory and topology: Type spaces. Investigate the topology of the Stone spaces $S_n(T)$ and its connection to the compactness theorem. (This would be a nice way to organize a first paper on model theory. Or, for those who already know some model theory, the connections go deeper: e.g. what does it mean when the isolated types are dense?)

(3) For those interested in ultraproducts, and familiar with infinite cardinals, there is a deep theory of *regular* ultrapowers. See e.g. expository intro to "Hypergraph sequences..." article on my webpage. A paper could involve explaining the basics of how regular ultrapowers work and the proof that some theories always have saturated regular ultrapowers, while others do not.

(4) Model theory and algebraic geometry: How does the model-theoretic language approach algebraically closed fields? What are types, etc in this context? You might look at Bouscaren's book "Model theory and algebraic geometry," which was written to explain Hrushovski's 1996 proof of the Mordell-Lang conjecture using model theory, or at Marker's model theory book. For this topic ideally one should know some algebra or some model theory, but not necessarily both.

(5) For those interested in higher orders of infinity: set theory (e.g. ordinals, cardinals and their arithmetic), another topic which could be taken in many directions. Classic sources are Jech and Kunen. There is a wonderful book of Komjath and Totik, Problems and Theorems in Classical Set Theory, which teaches you the subject through a series of progressively harder problems. Possible paper topics here would include cardinal arithmetic, the continuum hypothesis, the power of transfinite induction as a tool, the Banach-Tarski paradox, etc.

(6) In model theory, Shelah's proof that the order property implies the independence property or the strict order property. This is a beautiful and quite short result, maybe three pages, but putting the theorem in context and understanding what it does requires a nontrivial amount of understanding.

(7) What are Fraissé constructions and how do they work?

(8) Why study first-order logic? Lindström's theorem gives a characterization (this is a section of the more recent edition of Chang and Keisler).

(9) For those who already know some model theory, there is a rapidly growing field of *abstract elementary classes* which you can read about in John Baldwin's monograph *Categoricity.* A reasonable paper topic would be to explain abstract elementary classes and prove Shelah's (mysterious!) Presentation Theorem. Alternately, one could investigate Zilber's program to prove Schanuel's conjecture in transcendental number theory (also discussed in Baldwin's book).

(10) Fagin, "Probabilities on finite models." The phenomenon of zero-one laws.

(11) Henson proved in "A family of countable homogeneous graphs" that the countable "universal K_n-free" graphs are homogeneous (K_n is the complete graph on n vertices). The paper is 16 pages (large type), readable and involves various interesting digressions and some open problems.
(12) Szemerédi's regularity lemma. This is a very useful tool in graph theory (with interesting connections to model theory, as the course will discuss) and there are many expositions of it.
(13) Of particular interest to algebraic geometers is Scanlon's paper on "automatic uniformity" as a consequence of definability of types in stable theories (on his Berkeley webpage). 9 pages.
(14) A theorem due to Ax and Grothendieck (independently) says that every injective polynomial function from a complex vector space to itself is surjective. Ax's proof is model-theoretic: "by compactness"! This is written up in Marker's model theory book, and was the subject of my REU course last year. There are various other proofs, of independent interest.
(15) For those interested in probability, you might look at Laskowski's influential paper "Vapnik-Chervonenkis classes of definable sets" which makes a surprising connection between logic and probability. This theory has wide ranging applications to e.g. statistical learning theory. 9 pages.
(16) The Kirby-Paris theorem, and the Paris-Harrington theorem. See Kirby and Paris, "Accessible independence results for Peano arithmetic," and Paris, "Some independence results for Peano arithmetic." This requires knowing some model theory and a little about the ordinals (we discussed it at the end of 278) but has its rewards: a rigorous proof that Hercules will always win against the Hydra is a topic you can enjoy explaining at math teas for years to come.

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