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Versal Deformations and Algebraic Stacks

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This paper continues the study of abstract moduli problems begun in our previous paper Algebraization of formal moduli I [3]. Some of the results were announced in [4].

In Section 3, we analyze the properties of Schlessinger's formally versal deformation at a point, giving conditions which imply that is is versal in the algebraic sense (3.3). We then show (4.4) that formal versality is an *open condition*, under some mild hypotheses which are easy to verify in practice. A model for this study was the example of deformations of isolated singularities (4.5). These results are used in Section 5 to give criteria for representability as an algebraic stack, generalizing the results of [3] for algebraic spaces, and in Section 6 it is proved that every flat groupoid is equivalent to an algebraic stack. Some corrections to [3] are made in the appendix.

We have tried to make this paper as independent of [3] as was possible, and the only essential references are to its Sections 1, 2. However, it seemed too clumsy to introduce algebraic stacks without using algebraic spaces, and so, starting with Section 5, the basic notion of algebraic space [12] is assumed.

An improvement over the treatment of [3] is given by the introduction of an explicit obstruction theory. We avoided this before, because we didn't know what was behind such a theory. We still do not completely understand that, but it is certainly clear that a much neater list of axioms is obtained this way (compare (5.4) with [3, Theorem 5.3]).

Throughout the paper, the symbol S denotes a scheme (or algebraic space) of finite type over an excellent Dedekind ring.

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1. Basic Terminology

We have been unable to be consistent about the use of schemes as opposed to rings, but in the first sections we have chosen to work as much as possible with rings, rather than with their spectra. In later sections we will pass informally from one category to the other.

Let C be a subcategory of the category of noetherian rings. By *groupoid* F over C we mean a functor $p: F \to C$ which is cofibred in groupoids, i.e., such that the following conditions hold (cf. [6]). (We will use capital letters to denote objects of C, and small letters for objects of F.)

(a) *(Existence of extension of scalars.) Given a map $A \xrightarrow{\Phi} B$ in C and an element $a \in F$ with $p(a) = A$, there is a lifting $a \to b$* of Φ to F.

(b) *(Uniqueness.) Given a commutative diagram*

$$\begin{array}{ccc} A & \longrightarrow & B \\ & \searrow \quad \swarrow & \\ & C & \end{array}$$

in C, every partial lifting to F of solid arrows

$$\begin{array}{ccc} & & b \\ & \nearrow & \vdots \\ a & & \vdots \\ & \searrow & \downarrow \\ & & c \end{array}$$

can be completed by a unique dotted arrow.

In order to facilitate passage to the language of schemes, we adopt the convention that when we replace rings by their spectra, i.e., pass to the dual category C°, we also pass to the dual category F°. Thus a groupoid over a category of schemes will be a functor *fibred* in groupoids, i.e., a functor such that the dual axioms (a)°, (b)° hold.

We will write $F(A)$ for the fibre of p over $A \in C$. This is a groupoid because of axiom (b). The existence and uniqueness of extension of scalars defines, up to unique isomorphism, a "direct image" functor $\Phi_*: F(A) \to F(B)$ for each map $A \xrightarrow{\Phi} B$ in C, with canonical isomorphisms $(\Phi\Psi)_* \approx \Phi_* \Psi_*$. If $b \in F(B)$ and $A \xrightarrow{\Phi} B$ are given, we will denote by $F_b(A)$ the groupoid of maps $a \to b$ in F lying over Φ, an isomorphism being a commutative diagram

$$\begin{array}{ccc} a_1 & & \\ \vdots & \searrow & \\ \alpha \, \vdots & & b \\ \downarrow & \nearrow & \\ a_2 & & \end{array}$$

with $p(\alpha) = id_A$.

We call a groupoid F *limit preserving* if it is compatible with filtering direct limits in C in the obvious sense, i.e., if $\varinjlim F(A_i) \to F(\varinjlim A_i)$ is an equivalence of categories whenever $\varinjlim A_i$ is in C, the limit on the left being the obvious 2-categorical limit. This property is what we have previously referred to (following Grothendieck) as "local finite presentation". When working with schemes, we require only that F be compatible with the same limits, i.e., with inverse limits of *affine* schemes.

In general, a bar will denote the set of isomorphism classes in a groupoid. Thus $\bar{F}(A)$, $\bar{F}_b(A)$ denote the isomorphism classes of $F(A)$ and $F_b(A)$ respectively. Using extension of scalars, we can view the symbols $\bar{F}, \bar{F}_b$ as *functors*

$$\bar{F}: C \to (\text{sets}),$$

$$\bar{F}_b: C/B \to (\text{sets}).$$

The groupoid F is a *stack* over C [6] if C is closed under tensor products $A^* \otimes_A \cdot$ when $A \to A^*$ is etale, and if the following conditions hold:

(1.1) (i) *For every pair $a_1, a_2 \in F(A)$, the functor*

$$Isom(a_1, a_2) = I: A \backslash C \to (sets)$$

defined as follows: Given $A \xrightarrow{\Phi} B$,

$$I(B) = \{isomorphisms\ \Phi_*(a_1) \xrightarrow{\beta} \Phi_*(a_2)\ satisfying\ p(\beta) = id_B\},$$

is a sheaf on $A \backslash C$ for the etale topology.

(ii) *Let $\{A \xrightarrow{\Phi_i} B_i\}$ be an etale covering family in C. Every descent datum* [6, 10] *for F relative to $\{\Phi_i\}$ is effective.*

2. A Review of Schlessinger's Conditions

Let F be a groupoid over C, as in the previous section. By *infinitesimal extension* of a ring A we mean a surjective map $A' \to A$ having a nilpotent kernel. To generalize Schlessinger's conditions, we consider arbitrary maps of infinitesimal extensions of a given *reduced*[1] ring A_0, and in particular, diagrams

(2.1) $$A' \to A \to A_0, \quad \ker(A' \to A) = M,$$

of infinitesimal extensions of A_0, where $A' \to A$ is surjective and M is a (finite) A_0-module.

[1] The restriction to reduced rings A_0 is not very important, except in condition (4.1)(iii).

(2.2) **Condition** (S 1). (a) *Let*

$$\begin{array}{ccc} & & B \\ & & \downarrow \\ A' & \longrightarrow & A \end{array}$$

be a diagram in C, where $A' \to A$ *is as in* (2.1). *Assume that the composed map* $B \to A_0$ *is surjective. Let* $a \in F(A)$. *Then the canonical map*

$$\bar{F}_a(A' \times_A B) \to \bar{F}_a(A') \times \bar{F}_a(B)$$

is surjective.

(b) *Let* $B \to A_0$ *be a surjection,* A_0 *reduced, and let* M *be a finite* A_0*-module. Let* $b \in F(B)$ *have direct image* $a_0 \in F(A_0)$. *Then the canonical map*

$$\bar{F}_b(B+M) \to \bar{F}_{a_0}(A_0+M)$$

is bijective.

Here $B+M$ denotes the ring $B[M]$, with $M^2=0$. Of course, when we say these conditions hold, it is to be understood that in particular the rings and maps which appear are in C.

A strengthened form of (S 1) which will often hold is

(2.3) **Condition** (S 1′). *With the notation of* (S 1)(a), *the functor*

$$F_a(A' \times_A B) \to F_a(A') \times F_a(B)$$

is an equivalence of groupoids (i.e., *of categories*).

Exactly as in [15, 16], condition (S 1)(b) gives $\bar{F}_{a_0}(A+M)$ a structure of A_0-module, and when (S 1)(a) holds the additive group underlying this module acts transitively on the set $\bar{F}_a(A')$. Conditions (S 1) and (S 1′) are called "semi-homogeneity" and "homogeneity" respectively by Rim [15].

We will use the notation

$$\bar{F}_{a_0}(A_0+M) = D_{a_0}(M), \tag{2.4}$$

so that D is a functor of (a_0, M), depending linearly on (A_0, M). Schlessinger's final condition is

(2.5) **Condition** (S 2). $D_{a_0}(M) = \bar{F}_{a_0}(A_0+M)$ *is a finite* A_0*-module.*

Remark. In the cases we consider, all rings in C will be filtering direct limits in C of rings having some finiteness property, such as being of finite type over S. Then provided that F is a limit preserving groupoid, (S 1′) or (S 1) will hold for all of C if and only if it holds for these special rings. We will need to assume (S 2) only for them.

We do not know to what extent a theory of obstructions is determined by the groupoid F, so we will treat such a theory as *extra structure*. By an *obstruction theory* for F we mean the following data:

(2.6) (i) *For each infinitesimal extension* $A \to A_0$ *and element* $a \in F(A)$, *a functor*

$$\mathcal{O}_a\colon \text{(finite } A_0\text{-modules)} \to \text{(finite } A_0\text{-modules)}.$$

(ii) *For each deformation situation* (2.1) *and* $a \in F(A)$, *an element* $o_a(A') \in \mathcal{O}_a(M)$ *which is zero if and only if* $\bar{F}_a(A')$ *is not empty.*

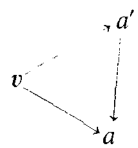
We require that these data be functorial, and linear in (A_0, M) in the obvious sense.

3. Existence of Versal Deformations at a Point

Since the results of this section are local for the etale topology, it is convenient to work with henselian rings. So, let S be the base scheme, and let k be a field which is finitely generated as field over $\mathcal{O}_S$. Let C denote the category of noetherian *henselian* local $\mathcal{O}_S$-algebras with residue field k. Consider a limit preserving groupoid F over C such that $F(k)$ consists of a single object with the identity as its only automorphism. We will call an element $a \in F(A)$ *algebraic* if A is algebraic, i.e., is the henselization of an $\mathcal{O}_S$-algebra of finite type. If A is a complete local ring, we also have the notion of *formal element* $a \in \hat{F}(A)$, meaning a sequence $\cdots \to a_n \to a_{n-1} \to \cdots$ with $a_n \in F(A/\mathfrak{m}^{n+1})$.

An algebraic element $v \in F(R)$ is called *versal* if

(3.1) *every diagram of solid arrows*



such that $p(a' \to a) = A' \to A$ *is surjective, can be completed by a dotted arrow.*

Similarly, an element or formal element v is called *formally versal* if it has the same property (3.1) on the subcategory C_f of finite length algebras, i.e., if (3.1) holds whenever A' has finite length.

According to Schlessinger [16] and Rim [15], there is a complete local ring $\hat{R}$ and a formally versal, formal element $v \in \hat{F}(\hat{R})$ if conditions (S1, 2) of Section 2 hold on the category C_f. Using the results of [3], we obtain

(3.2) **Corollary.** *Let F be as above. Assume that* (S 1, 2) *hold on C_f, and that if $\hat{A}$ is a complete local ring in C, the map*

$$\bar{F}(\hat{A}) \to \varprojlim \bar{F}(\hat{A}/\mathfrak{m}^n)$$

has a dense image.

Then there is an algebraic ring R and an element $v \in F(R)$ which is formally versal.

For, since (S 1, 2) hold for F they also hold for the functor $\bar{F}$, and we may apply Theorem (1.6) of [3].

(3.3) **Theorem.** *Let $v \in F(R)$ be an algebraic element which is formally versal, and consider the conditions*

(i) S 1, 2 *hold for all algebraic elements,*

(ii) (*D is compatible with completions*). *If $a_0 \in F(A_0)$ is algebraic and M is a finite A_0-module,*

$$D_{a_0}(M) \otimes \hat{A}_0 \xrightarrow{\sim} \varprojlim D_{a_0}(M/\mathfrak{m}^n M).$$

(iii) *Let A be an algebraic ring with an ideal I, and let $\bar{A}$ be its I-adic completion. Let $a, b \in F(\bar{A})$. If there exists a compatible sequence of isomorphisms $a_n \xrightarrow{\sim} b_n$ between the truncations in $F(A/I^{n+1})$, then there is an isomorphism $a \xrightarrow{\sim} b$ compatible with the given isomorphism $a_0 \xrightarrow{\sim} b_0$.*

If (i), (ii) *hold, then* (3.1) *holds for all infinitesimal extensions $A' \to A$. If* (i)–(iii) *hold, then v is versal.*

Proof. Consider a lifting problem (3.1) when $A' \to A$ is an infinitesimal extension. Since F is limit preserving, we may assume A' algebraic, and by induction we may further assume the extension is of the type of (2.1). It is permissible to replace R by the henselization R' of some polynomial ring $R[x_1, \ldots, x_r]$ at the origin, and v by its image v' under extension of scalars to R'. Versality for v and v' are equivalent. Doing so suitably results in a situation in which $R \to A$ is surjective, hence A is a finite R-module.

(3.4) **Lemma.** *There is some dotted arrow completing the diagram*

$$\begin{array}{ccc} R & \dashrightarrow & A' \\ & \searrow \quad \swarrow & \\ & A. & \end{array}$$

To see this, note that since v is formally versal, there is a dotted arrow

$$\begin{array}{ccc} R & \dashrightarrow & A'_n \\ & \searrow \quad \swarrow & \\ & A_n & \end{array} \tag{3.5}$$

for each n, where the subscript denotes truncation (modulo $\mathfrak{m}^{n+1}$). For given n, the set of these dotted arrows is a principal homogeneous space under the group of derivations $\mathrm{Der}_{\mathcal{O}_S}(R, M(n))$, $M(n)=\ker(A'_n \to A_n)$. This is a finite length module over A_0, and the map $\mathrm{Der}_{\mathcal{O}}(R, M(n)) \to \mathrm{Der}_{\mathcal{O}}(R, M(m))$ for $n \geqq m$ is A_0-linear. Thus the Mittag-Leffler condition holds for the sets of arrows (3.5), and so there is a map $R \to \hat{A}'$ compatible with $R \to A$. It may be approximated by a map $R \to A'$ [2].

Now let the map ϕ': $R \to A'$ be given, and let $b' = \phi'_*(v)$. We have the following diagram in F:

$$\begin{array}{ccccc} v & \longrightarrow & b' & & a' \\ & \searrow & \downarrow & \swarrow & \\ & & a & & \end{array}.$$

The group $D_{a_0}(M)$ acts transitively on $\bar{F}_a(A')$, and so for some $d \in D_{a_0}(M)$, a' and db' are a-isomorphic. If we show that $\mathrm{Der}_{\mathcal{O}}(R, M)$ maps surjectively to $D_{a_0}(M)$, we can adjust ϕ' by a derivation so that $a' = b'$ in $\bar{F}_a(A')$. This will give the dotted arrow (3.1). Since $\mathrm{Der}_{\mathcal{O}}(R, M) = \mathrm{Hom}_A(R, A+M)$, and $D_{a_0}(M) = \bar{F}_a(A+M)$ by S1(b), this surjectivity is precisely the versality assertion in the case $A' = A+M$. So we are reduced to that case.

If M has length 1, then $\bar{F}_a(A+M) = \bar{F}(k+M)$ by $S1(b)$, and so the assertion follows from the fact that v is formally versal. By induction, the lifting property holds whenever M has finite length. Therefore, setting $M_n = M/\mathfrak{m}^{n+1}M$, we find that $\mathrm{Der}_{\mathcal{O}}(R, M_n) \to D_{a_0}(M_n)$ is surjective for each n. Both sides are finite length R-modules, and so the map between inverse limits $\varprojlim_n$ is also surjective. We have

$$\varprojlim \mathrm{Der}_{\mathcal{O}}(R, M_n) = \varprojlim \mathrm{Hom}_R(\Omega_{R/\mathcal{O}}, M_n) = \mathrm{Hom}_R(\Omega, M) \otimes \hat{R}$$

and

$$\varprojlim D_{a_0}(M_n) = D_{a_0}(M) \underset{A_0}{\otimes} \hat{A}_0 = D_{a_0}(M) \underset{R}{\otimes} \hat{R}$$

by assumption (ii) of the theorem. Thus the map $\mathrm{Der}_{\mathcal{O}}(R, M) \to D_{a_0}(M)$ becomes surjective when tensored with $\hat{R}$, and hence is surjective, as required.

It remains to verify versality for v when (3.3) (iii) holds. Let $I \subset A'$ be the kernel of a map $A' \to A$, and let $a'_n \in F(A'/I^{n+1})$ be the element induced by $a' \in F(A')$. Using what has been proved, we can find a compatible sequence of maps $v \to a'_n$, lying over some maps ϕ'_n: $R \to A'/I^{n+1}$. Let ϕ': $R \to \bar{A}'$ be the map to the I-adic completion determined by $\{\phi_n\}$, let $\bar{b}' = \phi_*(v)$ and let $\bar{a}'$ be the direct image of a'. Then we are in a position to apply (3.3) (iii). It gives us an isomorphism $\bar{a}' \xrightarrow{\sim} \bar{b}'$, and hence a dotted arrow

$$\begin{array}{ccc} v & \dashrightarrow & \bar{a}' \\ & \searrow & \downarrow \\ & & a \end{array}.$$

By [2], this arrow can be approximated by the required map $v \dashrightarrow a'$.

(3.6) *Remark.* The assertion of the above theorem can be varied somewhat. In particular, one can weaken the assumptions (i), (ii) at the cost of strengthening condition (iii). This does lead to a better formulation in the absence of automorphisms, i.e., when F is a functor. We obtain the result announced in [4].

(3.7) **Theorem.** *Let F be a limit preserving functor on C, and let $v \in F(R)$ be an algebraic element which is formally versal. Then v is versal if*

(i) $S1(a)$ *holds whenever M is of length* 1.

(ii) *If A is algebraic and $\hat{A}$ is its completion, then the map*

$$F(\hat{A}) \to \varprojlim F(A/\mathfrak{m}^n)$$

is injective.

Proof. We may assume A' algebraic. The case that $A' \to A$ is a length 1 extension is treated in the same way as the infinitesimal case of (3.3). Consider the general case. Let $A_n^* = A_n' \times_{A_n} A$, and let $a_n^* \in F(A_n^*)$ be the element induced by a'. Then $A_n^* \to A_m^*$ is surjective and a finite length extension if $n \geqq m$, and so we can find successively a compatible sequence of maps $v \to a_n^*$. These maps lie over a sequence of homomorphisms $R \to A_n^* \subset \hat{A}_n^*$, which in turn induce a map α: $R \to \hat{A}' = \varprojlim \hat{A}_n^*$. By construction, $\alpha_*(v)$ and $\hat{a}'$ have the same image $\hat{a}_n^*$ in $F(\hat{A}_n^*)$, hence the same image in $F(A_n')$ for every n. By assumption (ii), we have $\alpha_*(v) = \hat{a}'$. We now approximate the formal lifting

$$\begin{array}{ccc} v & \longrightarrow & \hat{a}' \\ & \searrow \ \swarrow & \\ & \hat{a} & \end{array}, \qquad \begin{array}{ccc} R & \longrightarrow & \hat{A}' \\ & \searrow \ \swarrow & \\ & \hat{A} & \end{array}$$

by an algebraic one.

4. Formal Versality in a Neighborhood

In this section C will denote the category of noetherian $\mathcal{O}_S$-algebras. Let F be a limit preserving groupoid over C. An element $a \in F(A)$ is *algebraic* if A is algebraic, i.e., is of finite type over $\mathcal{O}_S$. We assume given an obstruction theory $\mathcal{O}$ for F, and that Schlessinger's conditions $(S1, 2)$ hold. In addition we assume that the following are true for algebraic elements:

(4.1) (i) *The modules D and $\mathcal{O}$ are compatible with étale localization: If $p: A \to B$ is étale, $b = \phi_*(a)$ etc. ..., then*

$$D_{b_0}(M \otimes B_0) \approx D_{a_0}(M) \otimes B_0,$$

and

$$\mathcal{O}_b(M \otimes B_0) \approx \mathcal{O}_a(M) \otimes B_0.$$

(ii) *D is compatible with completions: If $\mathfrak{m}$ is a maximal ideal of A_0, then*

$$D_{a_0}(M) \otimes \hat{A}_0 \approx \varprojlim D_{a_0}(M/\mathfrak{m}^n M).$$

(iii) *Constructibility: There is an open dense set of points of finite type $p \in \operatorname{Spec} A_0$ so that*

$$D_{a_0}(M) \otimes k(p) \approx D_{a_0}(M \otimes k(p)),$$

and

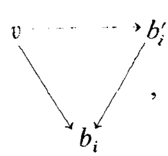
$$\mathcal{O}_a(M) \otimes k(p) \subseteq \mathcal{O}_a(M \otimes k(p)).$$

In these conditions the tensor products are taken over A_0.

We call an algebraic element $v \in F(R)$ *formally smooth* over F if the lifting property (3.1) holds whenever $A' \to A$ is an infinitesimal extension. The element v is *formally versal at a point* $p \in \operatorname{Spec} R$ if (3.1) holds when A' is a finite length extension of the residue field $k(p)$. We have

(4.2) **Proposition.** *An algebraic element v is smooth over F if and only if it is formally versal at every point $p \in \operatorname{Spec} R$ of finite type.*

Proof. We assume v formally versal at every p, the other implication being trivial. Consider a diagram (3.1) with $A' \to A$ an infinitesimal extension, which we may assume to be of the form (2.1). Using condition (4.1)(ii), we can apply Theorem (3.3) to conclude that there is a local lifting (local for the etale topology) at every point p, so that in particular there is an etale covering map $e': A' \to \prod B'_i$ and a family of dotted arrows



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where $b_i' = e_{i*}'(a')$, etc. .... Now the obstruction to the existence of a dotted arrow

$$\begin{array}{ccc} R & \dashrightarrow & A' \\ & \searrow^{\phi} \quad \swarrow & \\ & A & \end{array},$$

lies in $\operatorname{Ext}^1(\phi_* L_{R/\mathcal{O}_S}, M)$ [11, Ch. III,2], which is local for the etale topology. Thus this obstruction vanishes, and there is some arrow $\phi'\colon R \to A'$, determined up to a derivation $d \in \operatorname{Der}_{\mathcal{O}_S}(R, M)$. As in the proof of (3.3), we are led to showing that the map $\operatorname{Der}_{\mathcal{O}}(R, M) \to D_{a_0}(M)$ is surjective. Both terms are compatible with henselization, and so it suffices to check surjectivity when A_0 is henselian. Then it follows from the local assertion (3.3).

(4.3) **Proposition.** *Formal versality is stable under etale localization. In other words, let $x \in F(R)$ be an algebraic element, let $e\colon R \to R^*$ be etale, and $x^* = e_*(x)$. Then x^* is formally versal at a point $p^* \in \operatorname{Spec} R^*$ if and only if x is formally versal at $p = e^{-1}(p^*)$.*

Proof. The formal structure of R^* at p^* is determined by that of R and by the separable field extension $k(p^*)$ of $k(p)$. So to prove the proposition, we go back to the construction [16] of a formal versal deformation, and verify that it is compatible with separable field extensions. There is no problem in first order, i.e., on tangent spaces, by (4.1)(i). Say that we have already constructed a deformation $v_{n-1} \in F(R_{n-1})$ which is formally versal for $(n-1)$st order deformations (i.e., which has the lifting property (3.1) whenever $\mathfrak{m}_{A'}^n = 0$), and is such that $\mathfrak{m}_{R_{n-1}}^n = 0$. Write R_{n-1} as quotient of a smooth $\mathcal{O}_S$-algebra P:

$$0 \to J_{n-1} \to P \to R_{n-1} \to 0,$$

with $J_{n-1} \subset \mathfrak{m}^2$. Then $J_{n-1}/\mathfrak{m} J_{n-1}$ is a finite dimensional vector space. We choose an ideal J_n with $J_{n-1} \supset J_n \supset \mathfrak{m} J_{n-1}$, which is maximal with the property that v_{n-1} extends to $R_n = P/J_n$. The extension to R_n is versal for n-th order deformations. This can be explained using the obstruction theory: Let $M = J_{n-1}/\mathfrak{m} J_{n-1}$, $R' = P/\mathfrak{m} J_{n-1}$, and let $o \in \mathcal{O}_{v_{n-1}}(M)$ be the obstruction to extending v_{n-1} to R'. Then the set of $\phi \in \operatorname{Hom}(M, k)$ sending v to zero in $\mathcal{O}_{v_{n-1}}(k)$ is a linear subspace V, and a maximal quotient M on which the obstruction vanishes is obtained by mapping $M \to k^r$ via a basis $\phi_1, \ldots, \phi_r$ of V. Clearly this is all preserved by an etale extension, because of (4.1)(i). The proposition now follows by induction on n.

(4.4) **Theorem.** *With the assumptions of the beginning of this section, let $v \in F(R)$ be an algebraic element. If v is formally versal at $p \in \operatorname{Spec} R$, then it is formally smooth in an open neighborhood of p. In particular, formal versality is an open condition on* $\operatorname{Spec} R$.

(4.5) *Example.* Let F be the groupoid whose objects are families of equidimensional isolated singularities: An element $a \in F(A)$ is given by an *affine* scheme X_A flat over $\operatorname{Spec} A$, such that the locus $\Delta_A \subset X_A$ on which X_A is not smooth is finite over $\operatorname{Spec} A$. A map $a \to b$ is a commutative square

$$\begin{array}{ccc} X_A & \longrightarrow & X_B \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \longrightarrow & \operatorname{Spec} B \end{array}$$

such that $X_B \approx X_A \otimes_A B$. Schlessinger [17] has defined an obstruction theory for this groupoid, which is easily seen to satisfy (4.1). So, Theorem (4.4) applies to F. Moreover, Elkik [8, 9] has shown that the hypothesis of Corollary (3.2) holds, and that therefore every affine scheme X_0 with isolated singularities admits a formally versal deformation $v \in F(R)$. It is formally versal in a neighborhood by (4.4). In order for this formally versal deformation to be versal at the given point, i.e., to satisfy (3.1) locally for extensions which are not infinitesimal, it is necessary to change the groupoid slightly by working in etale neighborhoods of the singular set. We can define another groupoid F' as follows: Its objects are those of F, but a map $a \to b$ is given by a diagram

$$\begin{array}{ccccc} & & X'_A & & \\ & \swarrow^{e} & & \searrow & \\ X_A & & & & X_B \\ \downarrow & & & & \downarrow \\ \operatorname{Spec} A & & \longrightarrow & & \operatorname{Spec} B \end{array},$$

where X'_A is an etale neighborhood of Δ_A in X (i.e., e is etale and $e^{-1}(\Delta_A) \approx \Delta_A$), and $X'_A \otimes_A B$ is an etale neighborhood of Δ_B in X_B. Then condition (iii) of Theorem (3.3) holds (cf. Elkik [9]), and so v is versal.

Proof of the Theorem. This proof has arguments similar to that of [3, 5.3]. We first show that if x is a generalization of p, then v is also formally versal at x. Consider a "test map" $a' \to a$ lying over a surjection $A' \to A$ of infinitesimal extensions of $k(x)$. We have to verify the lifting property (3.1) for such a map. If v were not formally versal at x, there would be a test map for which the lifting property failed. Moreover, it is easy to see that a test map would exist so that the lifting property failed also after any etale localization of $A' \to A$. (Take $A = R/\mathfrak{m}^n$ and $A' = R'/\mathfrak{m}'^n$, where $v' \in F(R')$ is versal, and use (4.3).) Therefore it suffices to check that a lifting exists for the given test map, after some etale localization at x.

Let $\mathfrak{p} \subset R$ be the prime ideal of x, and $B_0 = R/\mathfrak{p}$. To prove that (3.1) holds, it suffices to show

(4.6) **Lemma.** *The diagram* (3.1) *extends in some way to a diagram of solid arrows*

$$\begin{array}{ccc} v & \dashrightarrow & b' \\ & \searrow \quad \swarrow_{\beta} & \\ & b & \end{array} ,$$

where $p(\beta) = (B' \to B)$ is a surjective map of infinitesimal extensions of B_0 whose localization at x is $A' \to A$, and such that the localization of this diagram is (3.1).

For, condition (3.3)(ii) holds at p because of (4.1)(i), (ii). So we may apply Theorem (3.3) to complete the above diagram with a dotted arrow, locally at p for the étale topology. This gives a local lifting of (3.1), as required.

Proof of the Lemma. It is a simple exercise to find a diagram

$$\begin{array}{ccccc} & & R & & \\ & & \downarrow & & \\ B' & \longrightarrow & B & \longrightarrow & B_0 \end{array}$$

which localizes to the given one. Next, we have to extend $a' \in F(A')$ to $b' \in F(B')$. Working step by step using induction on the nilradical of B', we are reduced to finding b' when $b \in F(B)$ is already given. Let $M = \ker(B' \to B)$. We may assume M torsion-free. The obstruction $o_b(B')$ to lifting b in some way to $F(B')$ lies in $\mathcal{O}_b(M)$, and it vanishes at x. Therefore it is zero in some neighborhood of x since F is limit preserving, and so by (4.1)(i) it is a torsion element of $\mathcal{O}_b(M)$, say killed by some $t \neq 0$ in B_0. Let

$$B'' = B'[t^{-1}M] = \{b' + t^{-1}m \mid b' \in B', m \in M\}.$$

Then B'' is an infinitesimal extension of B with kernel $t^{-1}M$. There is a diagram

$$\begin{array}{ccccc} M & \longrightarrow & B' & \longrightarrow & B \\ \cap & & \cap & & \| \\ t^{-1}M & \longrightarrow & B'' & \longrightarrow & B \end{array} ,$$

and we may identify the inclusion $M \subset t^{-1}M$ with the map $M \xrightarrow{t} M$. By linearity of the obstruction theory, we have $o_b(B'') = t\, o_b(B') = 0$. Thus if we replace B' by B'', we can lift b in some way, say to $b^* \in F(B')$. Let $a^* \in F(A')$ be its image. Then there is some $\delta \in D_{b_0}(M) \otimes k(x) = D_{a_0}(M \otimes k(x))$ such that $\delta a^* = a'$ in $\bar{F}_a(A')$. Write $\delta = t^{-1} d$ for some $d \in D_{b_0}(M)$ and $t \in B_0$. Then if we replace B' by B'' and identify $\ker(B'' \to B)$ with M as above, we have $da^* = a'$, and so $b' = da^*$ is the required lifting. Finally, we want b to be isomorphic to the image of v under the map $R \to B$. Since this is true after localization, an argument similar to the above shows it is true when B is chosen suitably.

We return to the proof of Theorem (4.4). If v is formally versal at p but not smooth in any neighborhood of p, there is some irreducible closed set $Y \subset \operatorname{Spec} R$, containing p, and a dense set $\mathscr{S}$ of points $y \in Y$ at which v is not formally versal. Since we have shown that formal versality is preserved under generalization, it is clear that we may suppose the points $y \in \mathscr{S}$ of finite type. Making a localization of the base if necessary, we may assume they are closed points. We also know that v is formally versal at the generic point η of Y. This will lead to a contradiction.

For each $y \in \mathscr{S}$, the failure of versality yields a surjective map $A'_y \to A_y$ of infinitesimal extensions of $k(y)$ with kernel of length 1, and a diagram of solid arrows

(4.7)
$$\begin{array}{ccc} v & \dashrightarrow & a'_y \\ & \searrow \quad \swarrow & \\ & a_y & \end{array}$$

which can not be completed by a dotted arrow. By Schlessinger's axiom $S\,1(a)$, there is a $v'_y \in F(R'_y)$, where $R'_y = A'_y \times_{A_y} R$, and a diagram

$$\begin{array}{ccc} v'_y & \longrightarrow & a'_y \\ \downarrow & & \downarrow \\ v_y & \longrightarrow & a_y \end{array}$$

completing (4.7).

Let $B_0 = R/\mathfrak{p}$, where $\mathfrak{p}$ is the prime ideal of Y. By Lemma (5.9) of [3], there is an infinitesimal extension B of B_0 (in fact of the form $B = R/I$, with $\sqrt{I} = \mathfrak{p}$), and a further infinitesimal extension $B' \to B$, with the following property: Every R'_y is of the form $B'_y \times_B R$ for some length 1 extension $B'_y \to B$ which is a quotient of B'. Let $M = \ker(B' \to B)$. We may assume M is a B_0-module, and since our problem is local in a neighborhood of the generic point η of Y, we may in fact suppose that M is free, say of rank r, over B_0.

Let $b \in F(B)$ be the image of v. The obstruction to lifting this element to $F(B')$ lies in $\mathcal{O}_b(M) = \mathcal{O}_b(B_0)^r$; say it is $o = (\xi_1, \ldots, \xi_r)$ with $\xi_i \in \mathcal{O}_b(B_0)$.

The surjective map $B' = B'_y$ induces a map $M \to k(y)$, and since b does lift to $F(B'_y)$, the obstruction o maps to zero in $\mathcal{O}_b(k(y))$ under the induced map. Since $\mathcal{O}_b(M) \otimes k(y) \subset \mathcal{O}_b(k(y))^r$ by (4.1)(iii) (after localization if necessary), it follows that the residues of the ξ_i in $\mathcal{O}_b(k(y))$ are linearly dependent for each $y \in \mathcal{S}$. Therefore the ξ_i are linearly dependent in $\mathcal{O}_b(B_0)$, and so there is a non-zero map $M \xrightarrow{\phi} B_0$ sending the obstruction to zero. After a further localization, we may assume ϕ surjective.

Let $B^* = B/\ker \phi$. Then b lifts in some way to $b^* \in F(B^*)$. Since v is formally versal at η, we can complete the diagram

$$\begin{array}{ccc} v & \dashrightarrow & b^* \\ & \searrow \quad \swarrow & \\ & b & \end{array} \tag{4.8}$$

with a dotted arrow in a neighborhood of η. This gives us a map $R \to B^*$, after localization. Now there are two cases:

Case 1. For a dense subset of $\mathcal{S}$, B'_y is not a quotient of B^*. In this case we replace $\mathcal{S}$ by this subset, and consider the products $B^*_y = B'_y \times_B B^*$. Then B^*_y is a quotient of B' which is a length 1 extension of B^*, and clearly $R'_y = B^*_y \times_{B*} R$. So we can replace B by B^* in the above discussion. By induction, this reduces us to

Case 2. B'_y is a quotient of B^* for a dense subset of $\mathcal{S}$. Now we replace $\mathcal{S}$ by this subset and B' by B^*, which reduces us to the case $M = B_0$ free of rank 1, and that b lifts in some way, say to $b^* \in F(B')$.

Again using versality of v at η, the diagram (4.8) can be completed by a dotted arrow after localization. Therefore $R' = B' \times_B R$ splits: $R' = R + M$. We now apply Condition (4.1)(iii) for D and Schlessinger's axiom $S\,1(b)$:

$$D_v(M) = \bar{F}_v(R + M) \approx D_{b_0}(M),$$

and

$$D_{b_0}(M) \otimes k(y) \approx D_{b_0}(M \otimes k(y))$$

for all y in a dense open set. Localize so this holds for all $y \in \mathcal{S}$. Let $a_{y0} \in F(k(y))$ be the element induced by v. By $S\,1(b)$,

$$D_{b_0}(M \otimes k(y)) = F_{b_0}(B_0 + k(y)) \approx D_{a_{y0}}(k(y)).$$

Thus

$$D_v(M) \otimes k(y) \approx D_{a_{y0}}(k(y)).$$

Denote by v^* the image of v via the map $R \to R + M$, and by a^*_y its image in A'_y. There is an element $d \in D_v(M)$ whose residue $\bar{d} \in D_{a_{y0}}(k(y))$ carries a^*_y to a'_y: $\bar{d} a^*_y \approx a'_y$.

Now the set $\mathrm{Der}_{\mathcal{O}}(R, M)$ of splittings of $R' \to R$ identifies with the set of liftings of $R \to B$ to B', and we have the canonical map

$$\mathrm{Der}_{\mathcal{O}}(R, M) \to D_b(M) \approx D_v(M).$$

Both terms are compatible with localization. Since v is formally versal at η, this map is surjective at η. A final localization allows us to suppose this map surjective. Let $\delta \in \mathrm{Der}_{\mathcal{O}}(R, M)$ represent d in $D_v(M)$. Then if we map R to R' via δ, the image of v becomes dv^*. Hence under the composed map $R \to R' \to A'_y$, the image of v is isomorphic to $\bar{d}a^*_y = a'_y$. We obtain a diagram

$$\begin{array}{ccc} v & \dashrightarrow & a'_y \\ & \searrow \quad \swarrow & \\ & a_y & \end{array},$$

contradicting the choice of $\mathcal{S}$. This completes the proof of the theorem.

(4.9) *Remark.* In constructing an obstruction theory for a given groupoid F, it may be convenient to rigidify F by some auxiliary structure, such as a choice of basis, etc. .... For applications to the above theorem, there is no need to check that the theory thus obtained is independent of the choices made, since the assertion of (4.4) is local. More precisely, returning to the notation of Section 2, let us call a *local* obstruction theory *at* $a_{00} \in F(A_{00})$ (A_{00} reduced) a collection of data of the following type:

(4.10) (i) For every diagram

$$\begin{array}{ccc} a & \longrightarrow & a_0 \\ & & \uparrow \\ & & a_{00} \end{array}, \qquad \begin{array}{ccc} A & \xrightarrow{\alpha} & A_0 \\ & & \uparrow \beta \\ & & A_{00} \end{array},$$

where α is an infinitesimal extension and β is étale, a functor

$$\mathcal{O}_a\colon \text{(finite } A_0\text{-modules)} \circlearrowleft .$$

(ii) For each deformation situation (2.1) with $a \in F(A)$ as in (i), an element $o_a(A') \in \mathcal{O}_a(M)$ which is zero if and only if $F_a(A')$ is non-empty.

Again, this data must be functorial and linear in (A_0, M). Then the proof of (4.4) shows

(4.11) **Corollary.** *The conclusion of Theorem* (4.4) *remains valid if local obstruction theories exist for a base for the étale topology on algebraic elements of C, satisfying conditions* (4.1).

5. Representation of Algebraic Stacks

We adopt a slightly more general definition of algebraic stack than is given by Deligne and Mumford [6].

(5.1) *Definition. Let C be the category of schemes over S. A limit preserving stack F over C is an algebraic stack if*

(i) *(F is relatively representable). For every pair of algebraic elements $x \in F(X)$, $y \in F(Y)$, the fibred product $X \times_F Y$ is represented by an algebraic space locally of finite type over S.*

(ii) *There is an $X \in C$ locally of finite type and an $x \in F(X)$, which is smooth and surjective, i.e., such that for every $y \in F(Y)$ the map $X \times_F Y \to Y$ is smooth (hence locally of finite type) and surjective.*

This is more general in two respects. First, we do not assume that the fibred products in (i) are schemes, which would be unnatural (cf. [6, 4.6]), but only that they are algebraic spaces. To fix ideas, let us assume the algebraic spaces at least locally quasi-separated. This means for an algebraic space Z, that the diagonal map $Z \to Z \times Z$ is of finite type[2]. Secondly, we require only that the map $x: X \to F$ be *smooth*, and not necessarily etale. It is clear that the notion of smoothness coincides with that of formal smoothness (Section 4) when F is relatively representable.

Let F be an algebraic stack, and $X^0 = X \to F$ the smooth, surjective map of (5.1)(ii). Let $X^1 = X \times_F X$. Then a map $Z \to X^1$ is given by a pair (f, g) of maps $Z \to X^0$ and an isomorphism $\phi: f^*(x) \xrightarrow{\sim} g^*(x)$. Thus there is a law of composition making $X^1 \rightrightarrows X^0$ into an *algebraic groupoid*, i.e., a groupoid-object in the category of algebraic spaces. Conversely, an algebraic groupoid $X^1 \rightrightarrows X^0$ determines a stack, namely the stack associated to the groupoid functor represented by $X^1 \rightrightarrows X^0$. (Passage to associated stack means that the functor is closed under descent etc. ... [10], [6, 4.1].) It is a routine exercise to check that if the algebraic groupoid is obtained from an algebraic stack F as above, then the stack so determined is *equivalent* to F. Note that in this case the structure maps $X^1 \rightrightarrows X^0$ are always *smooth*.

A *separation property* of an algebraic stack F is a property of the map $X^1 = X^0 \times_F X^0 \to X^0 \times X^0$, (or, equivalently, of the diagonal map $F \to F \times F$, cf. [6, 4.4]). For instance, F is locally quasi-separated if this map is of finite type. For a locally quasi-separated F, the condition of being an algebraic stack in the sense of Deligne and Mumford [6], i.e., of admitting an *etale* covering $X \to F$, is also a separation condition. Namely, it is that the map $X^1 \to X^0 \times X^0$ be unramified (an immersion). This can be shown easily using a slice (quasi-section) argument of the type made in [7], and is roughly the assertion of [6, 4.21]. Another way

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[2] Products are to be interpreted as fibred products over S.

of expressing this condition is by saying that the objects of F do not admit infinitesimal automorphisms. Let us say that such a stack admits an *etale structure*. The same slice argument shows that F is an (separated) *algebraic space* if $X^1 \to X^0 \times X^0$ is a (closed immersion) monomorphism.

Let F be a limit preserving stack. Then to show F is an algebraic stack, we can apply the results of the previous sections to verify axiom (ii) of (5.1). This gives

(5.2) **Corollary.** *Let F be a limit preserving stack on C, and assume given an obstruction theory $\mathcal{O}$ for F. Then F is an algebraic stack locally of finite type over S if*

(1) *F is relatively representable.*

(2) *Schlessinger's conditions* (S 1, 2) *hold.*

(3) *If $\hat{A}$ is a complete local $\mathcal{O}_S$-algebra with residue field of finite type over S, then $\bar{F}(\hat{A}) \to \varprojlim \bar{F}(\hat{A}/\mathfrak{m}^n)$ has a dense image.*

(4) *The modules D and $\mathcal{O}$ satisfy conditions* (4.1).

On the other hand, we may use this Corollary again to verify relative representability, and can collect the various conditions together. Now relative representability of F is equivalent with the following property: For every pair $x_1, x_2 \in F(X)$, the functor

$Isom(x_1, x_2) = I$: (Schemes/X) $\to$ (Sets),

$$I(Z) = \text{set of isoms. between the images of } x_i \text{ in } F(Z)$$

is represented by an algebraic space. We can also consider the related functor for $x \in F(X)$:

Aut_x: (Schemes/X) $\to$ (Sets)

$Aut_x(Z)$ = set of autos. of the image of x in $F(Z)$.

In the assertions below, it is convenient to pass back to the category of $\mathcal{O}_S$-algebras, and so we will write Aut_a for $a \in F(A)$. We revert to the notation of the previous sections. Collecting together the various conditions gives

(5.3) **Theorem.** *Let F be a limit preserving stack with obstruction theory $\mathcal{O}$. Then F is an algebraic stack, locally of finite type over S, if*

(1) (S 1, 2) *hold for F, and if $a_0 \in F(A_0)$ is an algebraic element then $Aut_{a_0}(A_0 + M)$ is a finite A_0-module.*

(2) *For any complete local $\mathcal{O}_S$-algebra $\hat{A}$ with residue field of finite type over S, the canonical map*

$$F(\hat{A}) \to \varprojlim F(\hat{A}/\mathfrak{m}^n)$$

is faithful, and has a dense image, i.e., the projection to $F(\hat{A}/\mathfrak{m}^n)$ is essentially surjective for every n.

(3) *D and $\mathcal{O}$ satisfy conditions* (4.1), *and also* $\mathrm{Aut}_{a_0}(A_0+M)$ *satisfies the conditions analogous to* (4.1) *for D.*

(4) *If $a_0 \in F(A_0)$ is algebraic and ϕ is an automorphism of a_0 which induces the identity in $F(k)$ for a dense set of points $A_0 \to k$ of finite type, then $\phi = \mathrm{id}$ on a non-empty open subset of* $\mathrm{Spec}\, A_0$.

Here the limit in Condition (2) must be taken in the obvious 2-categorical sense. That is, an object of $\varprojlim F(A/\mathfrak{m}^n)$ consists of objects $a_n \in F(A/\mathfrak{m}^n)$ for each n, together with isomorphisms of a_n with the image of a_{n+1} for each n. Condition (4) is just to insure that the products $X \times_F Y$ are at least locally quasi-separated, as we assumed above.

We remark that the converse of Theorem (5.3) is also true. That is, given an algebraic stack F locally of finite type over S, there exists an obstruction theory so that conditions (1)–(4) of (5.3) hold. Local obstruction theories can be found using [11] and the auxiliary schemes V constructed in the next section. These can be shown to be independent of the choices made in constructing V, and hance give a global theory. We omit the proofs of these assertions [3].

Copying Theorem (5.3) over for functors yields

(5.4) **Corollary.** *Let F: ($\mathcal{O}_S$-algebras)$\to$(Sets) be a limit preserving functor which is a sheaf for the etale topology, and let $\mathcal{O}$ be an obstruction theory for F. Then F is represented by an algebraic space locally of finite type over S if*

(1) (S 1′, 2) *hold for F.*

(2) *If $\hat{A}$ is a complete local $\mathcal{O}_S$-algebra with residue field of finite type, then $F(\hat{A}) \to \varprojlim F(\hat{A}/\mathfrak{m}^n)$ is injective and has a dense image.*

(3) *$D \in \mathcal{O}$ satisfy conditions* (4.1).

(4) *(local quasi-separation). If $a_0 \in F(A_0)$ is algebraic and $x, y \in F(A_0)$ are equal at a dense set of points of finite type, they are equal on a non-empty open set.*

Stronger separation conditions such as those of [3, 3.4] may be added to these conditions. Note that of course remark (4.9) applies, so that (5.3) and (5.4) remain valid when only local obstruction theories are given.

(5.5) *Example.* (Moduli stack for surfaces of general type.) Here F is the stack over $S = \mathrm{Spec}\, \mathbb{Z}$ such that $F(A)$ is the groupoid of smooth proper algebraic spaces X_A over $\mathrm{Spec}\, A$, all of whose geometric fibres are non-ruled surfaces with $(K^2) > 0$. It is a standard fact that (S 1′, 2) hold, and that the modules $Aut_{a_0}(A_0+M)$, $D_{a_0}(M)$, $\mathcal{O}_a(M)$ are $H^q(X_{A_0}, \Theta)$ for $q = 0, 1, 2$ respectively, Θ being the relative tangent bundle. These

[3] This point was clarified in some correspondence with Illusie.

certainly satisfy (5.3) (1, 2), and (5.3) (4) presents no problem. The only serious point is the effectivity of formal elements (5.3) (2), i.e., that if $\{X_n\}$ is a compatible sequence in $F(\hat{A}/\mathfrak{m}^{n+1})$, then it can be approximated by an $X \in F(\hat{A})$.

Now since X_0 is of general type, a sufficiently high multiple of the canonical bundle defines a morphism $X_0 \to \mathbb{P}^N$, whose image is a normal surface $\bar{X}_0$ birational to X_0 and having only rational double points as singularities [13, 14]. This map extends to the infinitesimal deformations X_n, and hence $\{X_n\}$ gives rise to a family $\{\bar{X}_n\}$ of subschemes of projective space. By Grothendieck's existence theorem, this family is induced by a scheme $\bar{X}$ proper and flat over Spec $\hat{A}$, and its fibres have only rational singularities (cf. proof of Proposition (3.4) of [5]). The family $\{X_n\}$ is a formal resolution of $\bar{X}$, which is algebraic by [5, 2.2]. Thus F satisfies the hypotheses of (5.3), and is an algebraic stack.

It is known that in characteristic zero a surface of general type has no infinitesimal automorphisms. So F admits an etale structure in characteristic zero. I don't know any examples in characteristic p either. In any case a surface of general type can not have a positive dimensional group of automorphisms. Thus the map $X^1 \to X^0 \times X^0$ is quasi-finite in all characteristics.

Proof of Theorem (5.3). This is routine. We just have to check that the conditions of (5.2) hold for the functors $Isom(x_1, x_2) = I$. Condition (S 1') for I is

$$I(A' \times_A B) \xrightarrow{\sim} I(A') \times_{I(A)} I(B),$$

which is the full and faithful property of the corresponding map for F, and so it is included in (S 1') for F. Similarly, the map $I(\hat{A}) \to \varprojlim I(\hat{A}/\mathfrak{m}^n)$ is controlled by condition (2) of the theorem.

Next, let $\alpha \in I(A)$. Then the tangent space $I_{\alpha_0}(A_0 + M)$ can be identified with $\mathrm{Aut}_{a_0}(A_0 + M)$, where a denotes the image of x_1 in $F(A)$. Our hypotheses include (S 2) and (4.1) for these modules.

Finally, the obstruction to lifting α to $I(A')$ can be expressed this way: Let $a'_1, a'_2 \in F(A)$ be the images of x_1, x_2. Then via the map α^{-1} we can view a'_2 as element of $F_a(A')$. Clearly α lifts if and only if $a'_1 = a'_2$ in $\bar{F}_a(A')$, i.e., $(a'_1, a'_2) = (a'_1, a'_1)$ in

$$\bar{F}_a(A') \times \bar{F}_a(A') \approx \bar{F}_a(A' \times_A A') = \bar{F}_a(A' + M) \approx \bar{F}_a(A') \times D_{a_0}(M).$$

Hence we can identify (a'_1, a'_2) with the pair (a'_1, d) for some $d \in D_{a_0}(M)$. This shows that α lifts if and only if $d = 0$. Therefore the element $d \in D_{a_0}(M)$ gives an obstruction theory for I. It satisfies the necessary axioms (4.1) by assumption.

It remains only to check the relative representability of I. This is done by resubstituting into the same Corollary (5.2), remembering that the deformation theory for the corresponding sygyzy functor is trivial. Condition (4) of the theorem is merely the assertion that I is locally quasi-separated (cf. [3, p. 59, paragraph 2]).

6. Reduction of a Flat Groupoid to an Algebraic Stack

Consider an algebraic groupoid $\mathscr{X} = X^1 \rightrightarrows X^0$ locally of finite type over S, such that the structure maps $X^1 \rightrightarrows X^0$ are flat. Let $\mathscr{X}$ denote also the functor of points. In this situation, it is natural to pass to the stack F associated to $\mathscr{X}$ with respect to the *flat* (*fppf*) topology. This means [10, Ch. II] that an element $Z \in F(Z)$ is given by a flat covering $Z' \to Z$, a map $Z' \to X^0$, and descent data $Z' \times_Z Z' \to X^1$.

(6.1) **Theorem.** *Assume $\mathscr{X}$ locally quasi-separated, i.e., that $X^1 \to X^0 \times X^0$ is of finite type. Then F is an algebraic stack locally of finite type over S, i.e., there is an algebraic groupoid $Y^1 \rightrightarrows Y^0$ with smooth structure maps, whose associated stack is equivalent to F.*

Here we mean of course this: If the stack F is viewed "by restriction" as a stack for the etale topology, then it satisfies the axioms of Definition (5.1). In this connection, we should remark that taking the associated stack to a groupoid $Y^1 \rightrightarrows Y^0$ with smooth structure maps, for the flat or the etale topologies, leads to equivalent groupoids over C. So nothing is gained from the flat topology in that case.

(6.2) *Example.* Let G be a finite, flat, commutative group scheme over S, and consider the classifying stack BG associated to the algebraic groupoid $G \rightrightarrows S$. It is known that G embeds into a smooth group scheme H' over S, and that $H = G/H'$ is again smooth. Thus H' is a principal G-bundle over H, which is determined by a map $H \to BG$. This is the smooth surjective map required by (5.1), and so $H \times_{BG} H \rightrightarrows H$ is an algebraic stack with smooth structure maps equivalent to BG. For, let $Y' \to Y$ be any principal G-bundle, given by a map $Y \to BG$. Then $H \times_{BG} Y$ represents the functor $Isom(p_1^* H', p_2^* Y')$ over $H \times Y$. To show this smooth and surjective over Y is a local problem and Y for the flat topology. So we may assume Y' trivial: $Y' = Y \times G$. Hence we may in fact assume that $Y = S$ and $Y' = G$, i.e., we have to show $H \times_{BG} S$ is smooth over S. But this scheme is easily seen to be H', which is smooth by construction.

If we write out Theorem (6.1) in the case that X^1 is an equivalence relation on X^0, i.e., $X^1 \to X^0 \times X^0$ is a monomorphism, we obtain the assertion of [1, § 6]:

(6.3) **Corollary.** *Let G be an algebraic space locally of finite type over S, and let $R \to X \times X$ be a finite type equivalence relation on X such that the maps $R \rightrightarrows X$ are flat. Then the quotient X/R as sheaf for the flat topology is represented by an algebraic space.*

Turning to the proof of Theorem (6.1), we note first the following

(6.4) **Lemma.** *It suffices to prove the theorem in the case that F is already known to be relatively representable.*

Proof. First, $X^0 \times_F X^0 = X^1$ [10], and hence this product is representable. Next, if a map $Y \to F$ factors through X^0, then $X^0 \times_F Y = X^1 \times_{X^0} Y$ is representable. By definition of F, any map lifts, locally for the flat topology. Thus $X^0 \times_F Y$ is representable, locally on Y for the flat topology. This reduces the problem of constructing $X^0 \times_F Y$ to one of flat descent, i.e., to a passage to quotient by a flat equivalence relation. Similarly, if $Y \to F$, $Z \to F$ are two maps, and if the first, say, lifts to X^0, then $Y \times_F Z = Y \times_{X^0} X^0 \times_F Z$. So, the construction of $Y \times_F Z$ also reduces to a quotient problem. Assuming the theorem proved under the relative representability hypothesis, we can apply it (via Corollary (6.3)) to these quotient problems. So, the proof of the lemma is reduced to that case. In other words, we have to prove relative representability when $X^1 \rightrightarrows X^0$ is an equivalence relation. Then a product $Y \times_F Z$ is a certain subfunctor of $Y \times Z$. Running through the above argument once more, we are reduced finally to the case that $X^1 \rightrightarrows X^0$ is an equivalence relation, and that moreover the quotient F is a subfunctor of a representable functor: $F \subset W$ for some algebraic space W. Then $Y \times_F Z = Y \times_W Z$ is indeed representable.

It remains now to find the smooth surjective map $Y^0 \to F$ required by Definition (5.1). We will construct it directly. Let U^i be affine $\mathcal{O}_S$-schemes etale over the $(i+1)$st fibre power X^i of X^0 over F, and let n be an integer. Consider the problem of giving, for a variable affine S-scheme Z, the following data:

(1) A finite Z-scheme Z^0 such that $\mathcal{O}_{Z^0}$ is a free, rank n, $\mathcal{O}_Z$-module with chosen basis.

(2) Maps $Z^i \to U^i$ (Z^i the $(i+1)$st fibre power of Z^0 over Z) such that the induced diagram

$$\begin{array}{ccccc} Z^2 & \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} & Z^1 & \rightrightarrows & Z^0 \\ \downarrow & & \downarrow & & \downarrow \\ X^2 & \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} & X^1 & \rightrightarrows & X^0 \end{array}$$

defines descent data for a map $Z \to F$ relative to the cover $Z^0 \to Z$. Clearly this problem is represented by an affine S-scheme W of finite type. Given such data we obtain a diagram

$$\begin{array}{cccc} & \overset{\phi}{\nearrow} Z \times_F X^0 & \longrightarrow & X^0 \\ Z^0 & \downarrow \pi & & \downarrow \\ & \searrow Z & \longrightarrow & F \end{array},$$

where π is of course flat. We impose the further conditions

(3) The map ϕ is a closed immersion, and Z^0 is a local relative complete intersection in $Z \times_F X^0$ over Z.

It is known and easy to see that these are *open* conditions.

Let $V \subset W$ be the open subset representing this problem. To complete the proof, it suffices to show that the universal map $V \to F$ is smooth, and that the various V we get from choice of U^i and n cover F. It suffices to check these things for schemes Z of finite length. Given a map $\zeta: Z \to F$, we choose an etale affine Y over $Z \times_F X^0$, and let Z^0 be the closed subscheme of Y defined by an $\mathcal{O}_Y$-sequence $f_1, \ldots, f_r$ ($r = \dim Y$). Since Z is of finite length, Z^0 will be free over Z, and we chose a basis. Next, the descent data $Z^i \to X^i$ is given canonically by our choice of Z^0. It will lift to some affines U^i over X^i [12, II, 6.4]. These choices determine a lifting of ζ to one of the above schemes V. Now given any infinitesimal extension $Z \subset Z_1$, the etale affine Y over $Z \times_F X^0$ extends canonically to Y_1 over $Z_1 \times_F X^0$ [12, III, 3.4], and so on. Thus the lifting to V is unobstructed, which completes the proof.

Appendix

Corrections to "Algebraization of Formal Moduli *I*" [3].

I am endebted to P. Deligne and M. Raynaud for calling my attention to the following three errors:

1. Theorem (6.1), p. 61 of [3] requires the hypothesis that the map $f: X \to S$ be separated. If one wants to extend the notion of Hilbert scheme to non-separated maps f, the thing to do is to replace closed subschemes by quasi-finite maps, i.e., to consider the stack

$F(S')$ = groupoid of algebraic spaces Y', quasifinite over $X \times_S S'$ and and proper over S'.

It is represented by a algebraic stack for any map $f: X \to S$ locally of finite type. Unfortunately, the proof of this fact requires some foundational work on non-separated schemes which makes it too long to give here.

2. The preliminary reduction in the proof of representability of $\operatorname{Pic} X$ on p. 68 of [3] is incorrect: It is not necessarily true that X is flat over $\bar{S}$. Thus the proof of (7.3) is complete only with this flatness as an extra hypothesis. The general case can be handled with minor changes in the argument.

One can also proceed by applying Theorem (5.3) of this paper. Let $f: X \to S$ be any flat and proper map, and let F be the *relative Picard stack* of X/S, i.e.,

$F(S') =$ groupoid of invertible sheaves on $X \times_S S'$.

It follows easily from (5.3) that F is represented by an algebraic stack $P^1 \rightrightarrows P^0$ over S. The relative Picard functor Pic X/S is obtained from F by passing to isomorphism classes $\bar{F}$, and taking the associated sheaf on S for the etale topology.

Suppose now that f is also cohomologically flat in dimension zero. Then the standard argument by the exponential shows that, for any object $z \in F(Z)$, the algebraic group Aut_z of its automorphisms is smooth over Z. Consider the structure map $P^1 \to P^0 \times P^0$, and let R denote the fibred product of P^1 with itself over $P^0 \times P^0$. Purely formally, the fact that Aut_p is smooth, when $p \in F(\mathrm{P}^0)$ is the given element, implies that the projections $R \to P^1$ are smooth. Thus R is a smooth equivalence relation on P^1. Let $\bar{P}^1$ be the quotient P^1/R as algebraic space. Then clearly $\bar{P}^1 \to P^0 \times P^0$ is the equivalence relation defining $Pic\ X/Y$. The separation properties are verified as in [3], p. 69.

3. A substantial step in the proof of Theorem (1.6) of [3] was omitted. This is the justification of the assertion of line 10, p. 31, which is as follows:

The map ψ (2.5) is not $\hat{R}[[d]]$-linear. However, it is surjective, and hence the images of $d_1, \ldots, d_n$ in $\tilde{\mathscr{A}}$ can be lifted to elements $d_1^{(2)}, \ldots, d_n^{(2)} \in \bar{A}$. Denoting by $d_i^{(1)}$ the image of d_i in $\bar{A}$ via the structure map, we have $d_i^{(1)} \equiv d_i^{(2)} \pmod{(p, d)^{n+1}}$. The elements $d_i^{(2)}$ define a new structure of finite $\hat{R}[[d]]$-algebra on $\bar{A}$, which we denote by a superscript (2), to distinguish it from the fixed structure, denoted by (1). The map ψ is linear with respect to (2).

One sees immediately that $I^{n-\nu}(\bar{A})$ is defined intrinsically in $\bar{A}$, and hence it is independent of the algebra structure. Now since $I^{n-\nu}$ is a finite module over $\hat{B} = \hat{R}[[d_{\nu+1}, \ldots, d_n]]$ with respect to structure (1), the same is true with respect to (2), if N is sufficiently large; and moreover we can choose N so that the two structures of $\hat{B}$-module are congruent modulo $(p, d_{\nu+1}, \ldots, d_n)^c$ where c is any integer given in advance. (Two modules M, M' over a ring B are called congruent modulo an ideal $\mathfrak{a}$ if $M/\mathfrak{a}M \approx M'/\mathfrak{a}M'$.)

By construction, the sequence of line 9, p. 31 is exact with respect to the second structure on $I^{n-\nu}(\bar{A})$. Thus, to justify the assertion of line 10, it suffices to find some measure f of $\hat{B}$-modules, depending on the local structure at the prime ideal p, such that

$$f(I^{n-\nu}(\hat{\mathscr{A}})) < f(I^{n-\nu}(\bar{A})^{(2)}) \leqq f(I^{n-\nu}(\bar{A})^{(1)}).$$

Such a measure is provided by Fitting's ideals:

Let B be a local ring, with maximal ideal $\mathfrak{m}$, M a finite B-module, Λ a localization of B which is a discrete valuation ring, and p the corresponding prime ideal of B. We denote by $F_i(M)$ the i-th Fitting ideal. Recall that if $m_1, \ldots, m_n$ generate M and if

$$\begin{array}{ccc} a_{11}m_1+\cdots+a_{1n}m_n=0 \\ \vdots \quad\quad \vdots \quad \vdots \\ a_{m1}m_1+\cdots+a_{mn}m_n=0 \end{array}$$

is a complete set of relations among the m_j, then $F_i(M)$ is the ideal generated by the $(n-i)$-rowed subdeterminants of the relation matrix (a_{jk}) if $i<n$, and is the unit ideal if $i \geqq n$. Of course, $F_i(M)$ is independent of the presentation.

Denote by $f_i(M)$ the order of zero of $F_i(M)$ at the prime ideal p (so that $0 \leqq f_i(M) \leqq \infty$), and by $f(M)$ the sequence $(f_0, f_1, \ldots)$. Thus, if M is locally isomorphic to $\Lambda^r \oplus \Lambda/p^{e_1} \oplus \cdots \oplus \Lambda/p^{e_s}$, where $e_1 \geqq e_2 \geqq \cdots \geqq e_s$, then $f(M)$ is the sequence

$$\infty, \cdots, \infty, \quad (e_1+\cdots+e_s), (e_2+\cdots+e_s), \ldots, (e_s), \quad 0, 0 \ldots.$$

Lemma. (i) *If M', M'' are finite R-modules and if M'' is a quotient of M', then $f(M'') \leqq f(M')$, i.e., $f_i(M'') \leqq f_i(M')$ for all i. Equality holds if and only if $M' \approx M''$ locally at p.*

(ii) *Given M, there is an integer c so that any finite B-module M' with $M' \equiv M$* (mod $\mathfrak{m}^c$) *satisfies* $f(M') \leqq f(M)$.

This lemma applies in our case, if we set $B=\hat{B}$, $M=I^{n-\nu}(\bar{A})^{(1)}$, $M'=I^{n-\nu}(\bar{A})^{(2)}$, and $M''=I^{n-\nu}(\hat{\mathscr{A}})$.

Proof of the Lemma. The first assertion of (i) is trivial, since the relation matrix for M' will be a submatrix of that for M'', if we take corresponding generators. We leave the last assertion as an exercise.

To prove (ii), choose generators m_i for M, and elements $m_i' \in M'$ which are congruent to the m_i (modulo $\mathfrak{m}^c$). Then the m_i' generate M' by the Nakayama lemma. If $a_1 m_1 + \cdots + a_n m_n = 0$ is a relation among the m_i, then $a_1 m_1' + \cdots + a_n m_n' \equiv 0$ (modulo $\mathfrak{m}^c M'$). Hence there is a relation $a_1' m_1' + \cdots + a_n' m_n' = 0$ with $a_i' \equiv a_i$ (mod $\mathfrak{m}^c$). Thus if (a_{jk}) is a relation matrix for M, we can obtain a relation matrix for M' whose upper entries are $a_{jk}' \equiv a_{jk}$ (mod $\mathfrak{m}^c$), but which may have some more rows. The assertion now follows easily.

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