Singularity properties of convolutions of algebraic morphisms and applications

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2. and if $f \in C^k(\mathbb{R})$ and $g \in C^l(\mathbb{R})$ then $f * g \in C^{k+l}(\mathbb{R})$.

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Question

Is there a geometric analogue to this phenomenon?

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2. Let G be any algebraic group and let $[\ ,\ ] : G \times G \to G$ be the commutator map $[x, y] = xyx^{-1}y^{-1}$. Then $[\ ,\ ] * [\ ,\ ](x_1, y_1, x_2, y_2) = [x_1, y_1] \cdot [x_2, y_2] = x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1}$.

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Let A be a finite ring, and consider the maps $(\varphi_i)_A : X(A) \to G(A)$.

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- *By $(\dagger)$ we have,*

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- Properties preserved under convolutions: dominance, flatness, flatness with reduced or normal fibers, smoothness.
- If S is a property of morphisms which is preserved under base change and compositions and $X \to \mathrm{Spec}(K)$ satisfies S, then it is preserved under convolutions.

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1. *To see (4), take* $\varphi(x_1, \ldots, x_m) = (x_1^2, (x_1x_2)^2, (x_1x_3)^2, \ldots, (x_1x_m)^2)$.

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2. *In general, we should not expect to get a smooth morphism if we start from a non-smooth morphism (e.g.* $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ *by* $\varphi(x) = x^2$, *then* $d\varphi^{*n}_{(0,\ldots,0)} = 0$ *for all* $n \in \mathbb{N}$*).*

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An affine variety X has rational singularities if it is Cohen-Macaulay, normal and if for every strong resolution of singularities $p : \widetilde{X} \to X$ and regular top differential form $\omega \in \Omega^{\mathrm{top}}_{X^{\mathrm{sm}}}(X^{\mathrm{sm}})$ there exists a regular top differential form $\widetilde{\omega} \in \Omega^{\mathrm{top}}_{\widetilde{X}}(\widetilde{X})$ such that $\omega = \widetilde{\omega}|_{X^{\mathrm{sm}}}$.

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Example

Consider the variety $X = \{\sum_{i=1}^{k} x_i^{n_i} = 0\} \subseteq \mathbb{A}^k$ $(k > 1)$.
X has rational singularities if $\sum_{i=1}^{k} \frac{1}{n_i} > 1$ (and $(0, \ldots, 0)$ is not a rational singularity if $\sum_{i=1}^{k} \frac{1}{n_i} < 1$).

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Theorem (Glazer-H. 2018)

Let X be a smooth, absolutely irreducible variety, G be an algebraic group and let $\varphi : X \to G$ be a dominant morphism. Then there exists $N \in \mathbb{N}$ such that for any $n > N$ the n-th convolution power φ^{*n} is (FRS).

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φ^{*4}	$\{x^3 + y^3 + z^3 + w^3 = 0\}$	✓	✓	✓

$\Rightarrow$ φ^{*4} **is (FRS).**

Rational singularities and counting points over finite rings

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If X is smooth, we have for almost all primes,

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Example

Let $X = \mathrm{Spec}\left(\mathbb{Z}[x]/(x^2)\right)$, then $|X(\mathbb{Z}/p^{2k}\mathbb{Z})| = p^k$ but $\dim X_{\mathbb{Q}} = 0$.

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1. *$X_{\mathbb{Q}}$ has rational singularities.*

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Let $\varphi : X \to Y$ be a $\mathbb{Z}$-morphism with absolutely irreducible fibers. Then $\varphi_{\mathbb{Q}}$ is (FRS) $\Rightarrow$ for every k we have $\lim_{p\to\infty} \sup_{y\in Y(\mathbb{Z}/p^k\mathbb{Z})} \frac{|\varphi^{-1}_{\mathbb{Z}/p^k\mathbb{Z}}(y)|}{p^{k(\dim X - \dim Y)}} = 1$.

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Theorem (Aizenbud-Avni)

Let G be a semi-simple group and let $[\ ,\ ] : G \times G \to G$ be the commutator map $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$. Then $[\ ,\ ]^{*21} : (G \times G)^{21} \to G$ is (FRS).

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Conjecture

Let G be a semi-simple group, then $[\ ,\ ] * [\ ,\ ] : (G \times G)^2 \to G$ is (FRS).

Proof of main theorem

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Let K be a field of characteristic 0. Recall we want to show the following:

Theorem (Glazer-H. 2018)

Let X be a smooth, absolutely irreducible K-variety, G be an algebraic K-group and let $\varphi : X \to G$ be a dominant morphism. Then there exists $N \in \mathbb{N}$ such that for any $n > N$ the n-th convolution power φ^{*n} is (FRS).

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Proposition

Assume a K-morphism $\psi : X^{*N} \to G$ is (FRS) at $(x,\dots,x)$ for every $x \in X(\overline{K})$, then $\psi^{*2N} : X^{2N} \to G$ is (FRS).

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5. Restrict scalars to get a $\mathbb{Q}$-morphism $\mathrm{Res}_{\mathbb{Q}}^{K}(\varphi)$. Now, if the morphism $\mathrm{Res}_{\mathbb{Q}}^{K}(\varphi)^N = \mathrm{Res}_{\mathbb{Q}}^{K}(\varphi^N)$ is (FRS) then so is φ^N by noting the structure of $\mathrm{Res}_{\mathbb{Q}}^{K}(\varphi^N) \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(K)$.

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Let $\varphi : X \to Y$ be a map between smooth varieties defined over a finitely generated field K of characteristic 0, and let $x \in X(K)$. Then TFAE:

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2. *For any finite extension K'/K, there exists a non-Archimedean local field $F \supseteq K'$ and a non-negative Schwartz measure μ on $X(F)$ that does not vanish at x such that $(\varphi|_{X(F)})_*(\mu)$ has continuous density.*

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Let $h : G(\mathbb{Z}_p) \to \mathbb{C}$ be a function. If the Fourier transform $\mathcal{F}(h)$ of h is absolutely integrable, then h is continuous.

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Question

Can we find a collection of measures $\{\mu_{\mathbb{Q}_p}\}_{p>M}$ as in the theorem and an integer N such that $\mathcal{F}(\varphi_*(\mu_{\mathbb{Q}_p})^{*N}) = \mathcal{F}(\varphi_*(\mu_{\mathbb{Q}_p}))^N$ is absolutely integrable for every p?

Digression: motivic functions

Languages, formulas and theories

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7. $\forall x, y[x, y \neq 0 \rightarrow x \cdot y \neq 0]$
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The ring of motivic functions is preserved under integration.

Theorem (Cluckers-Loeser, Cluckers-Gordon-Halupczok)

Let X and Y be $\mathcal{L}_{\mathrm{DP}}$-definable sets and let $f \in \mathcal{C}(X \times Y)$ be a motivic function.

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Let X and Y be $\mathcal{L}_{\mathrm{DP}}$-definable sets and let $f \in \mathcal{C}(X \times Y)$ be a motivic function. Then there exists a function $g \in \mathcal{C}(Y)$ and $M \in \mathbb{N}$ such that for every $F \in \mathrm{Loc}_M$ we have

$$g_F(y) = \int_{X_F} f_F(x,y)dx$$

for every $y \in Y_F$ such that $f_F(x,y) \in L^1(X_F)$.

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Theorem (Denef-Pas)

Let η be an $\mathcal{L}_{\mathrm{DP}}$-formula. Then there exists an $\mathcal{L}_{\mathrm{DP}}$-formula η' without quantifiers of the valued field sort and an integer M such that η and η' are equivalent for every non-Archimedean local field of residue characteristic larger than M.

proof of the main theorem: Step 3.5 - defining a motivic measure

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Can we find a collection of smooth measures $\{\mu_F\}_{F\in \mathrm{Loc}_M}$ *such that* $\mathrm{supp}(\mu_F) = X(\mathcal{O}_F)$ *for every* $F \in \mathrm{Loc}_M$ *and* $\mathcal{F}(\varphi_*(\mu_F)^{*N}) = \mathcal{F}(\varphi_*(\mu_F))^N$ *is absolutely integrable for some N (which does not depend on F)?*

- Set $\mu_F := 1_{X(\mathcal{O}_F)}$ and consider the collection $\mu = \{\mu_F\}_{F\in \mathrm{Loc}_M}$.

proof of the main theorem: Step 3.5 - defining a motivic measure

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- Since the ring of motivic functions is preserved under integration, the collection $\sigma = \{\varphi_*(\mu_F)\}_{F\in\mathrm{Loc}_M}$ is motivic as well.

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Claim

Let $h \in \mathcal{C}(G)$ be an absolutely integrable, compactly supported motivic function. Then there exists $N \in \mathbb{N}$ such that h_F^{*N} has continuous density for every $F \in \mathrm{Loc}_M$.

Proof of the main theorem: final step - prove a theorem on motivic functions

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$$|\mathcal{F}(h_F)(y)| < d(F)\min\{|y|^{\alpha}, 1\}$$

for every $F \in \mathrm{Loc}_M$, *where* $d(F)$ *depends only on* F.

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Theorem ($L^1 \Rightarrow L^{1+\epsilon}$, Glazer-H. 2018)

Let X be a smooth algebraic variety, let μ be a motivic measure on X, and let h be a compactly supported motivic function on X such that $h_F \in L^1(X(F), \mu_F)$ for every $F \in \mathrm{Loc}_M$.

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- Write $I_h(s, F) = \sum_{k \in \mathbb{Z}} a_k q_F^{-ks}$ where
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- Each a_k can be simplified, and $I_h(s, F)$ can be written as

$$q_F^{-n} \sum_{\eta \in k_F^r} \sum_{\substack{l_1, \ldots, l_n, k \in \mathbb{Z} \\ \sigma(\eta, l_1, \ldots, l_n, k)}} q_F^{-ks - l_1 - \ldots - l_n}$$

where σ is an $\mathcal{L}_{\mathrm{DP}}$-formula.

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  where σ is an $\mathcal{L}_{\mathrm{DP}}$-formula.
- Using elimination of quantifiers and certain uniformization theorems, we can write the above expression as finitely many sums of the form

$$\sum_{(e_1, \ldots, e_l) \in \mathbb{N}^l} p^{b_1(s) e_1 + \ldots + b_l(s) e_l}$$

  where $b_i(s)$ are simple functions.

Questions?