ADOLESCENT WHITTAKER CATEGORIES, REGULARITY, AND HEISENBERG GROUPS

0.1. Let $n \geqslant 2$ and let G_n denote the group scheme of maps from $\mathrm{Spec}(k[[t]]/t^n)$ to G, and let e.g. N_n be defined similarly. Let $\{e_i \in \mathfrak{n}\}_{i \in \mathcal{I}_G}$ be Chevalley generators indexed by $\mathcal{I}_G$ the set of positive simple roots.

Let $\psi : N_n \to \mathbb{G}_a$ be defined as the composition:

$$N_n \to N_n/[N,N]_n = \prod_{i \in \mathcal{I}_G} (\mathbb{G}_a)_n \cdot e_i \xrightarrow{\text{sum}} (\mathbb{G}_a)_n = (\mathbb{G}_a) \underset{k}{\otimes} k[[t]]/t^n \to \mathbb{G}_a$$

where the last map is induced by the functional:

$$k[[t]]/t^n \to k$$
$$\sum a_i t^i \mapsto a_{n-1}.$$

The main result of this section answers the question: for $\mathcal{C} \in D(G_n)\text{–}\mathsf{mod}$, how much information do the invariants $\mathcal{C}^{N_n,\psi}$ remember about $\mathcal{C}$?

0.2. Recall that n is assumed to be at least 2. Therefore, we have a homomorphism:

$$\mathfrak{g} \otimes \mathbb{G}_a \to G_n$$
$$(\xi \in \mathfrak{g}) \mapsto \exp(t^{n-1}\xi).$$

This map realizes $\mathfrak{g} \otimes \mathbb{G}_a$ as a normal subgroup of G_n. Note that the adjoint action of G_n on this normal subgroup is given by:

$$G_n \xrightarrow{\text{ev}} G \overset{\text{adjoint}}{\curvearrowright} \mathfrak{g}.$$

If $\mathcal{C}$ is acted on by G_n, it is thus acted on by $\mathfrak{g} \otimes \mathbb{G}_a$ by restriction, or using Fourier transform, by $D(\mathfrak{g}^\vee)$ equipped with the $\overset{!}{\otimes}$-tensor product. (We omit the tensoring with $\mathbb{G}_a$ because we are not concerned with the additive structure on $\mathfrak{g}^\vee$ here.)

Fix a symmetric, linear G-equivariant identification $\kappa : \mathfrak{g} \simeq \mathfrak{g}^\vee$ once and for all. Therefore, $\mathcal{C}$ is acted on by $D(\mathfrak{g})$ with its $\overset{!}{\otimes}$-monoidal structure. In particular, for S a scheme mapping to $\mathfrak{g}$, we may form $\mathcal{C}|_S := \mathcal{C} \otimes_{D(\mathfrak{g})} D(S)$.

Define $\mathcal{C}_{reg}$ as $\mathcal{C}|_{\mathfrak{g}_{reg}}$ where $\mathfrak{g}_{reg} \subseteq \mathfrak{g}$ is the subset of regular elements. We have adjoint functors:

$$j^* : \mathcal{C} \rightleftarrows \mathcal{C}_{reg} : j_{*,dR}$$

with the right adjoint $j_{*,dR}$ being fully-faithful: indeed, these properties are inherited from the corresponding situation $j^* : D(\mathfrak{g}) \rightleftarrows D(\mathfrak{g}_{reg}) : j_{*,dR}$ for $j : \mathfrak{g}_{reg} \hookrightarrow \mathfrak{g}$ the embedding.

Because $\mathfrak{g}_{reg} \subseteq \mathfrak{g}$ is closed under the adjoint action of G, and since G_n acts on $\mathfrak{g}^\vee \simeq \mathfrak{g}$ through the adjoint action of G, it follows that $\mathcal{C}_{reg}$ is acted on by G_n so that the comparison functors with $\mathcal{C}$ are G_n-equivariant.

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0.3. **Main theorem.** We have $\mathfrak{g} \otimes \mathbb{G}_a \cap N_n = \mathfrak{n} \otimes \mathbb{G}_a$, and under the Fourier transform picture above, we have:

$$\mathcal{C}^{\mathfrak{n}\otimes\mathbb{G}_a,\psi|_{\mathfrak{n}\otimes\mathbb{G}_a}} \simeq \mathcal{C}|_{f+\mathfrak{b}}.$$

Here f a principal nilpotent whose image in $\mathfrak{g}/\mathfrak{b} \simeq \mathfrak{n}^\vee$ is $\psi|_{\mathfrak{n}\otimes\mathbb{G}_a}$.

In particular, because $f + \mathfrak{b} \subseteq \mathfrak{g}_{reg}$, it follows that $\mathcal{C}^{N_n,\psi} \simeq \mathcal{C}_{reg}^{N_n,\psi}$. The following result states that this is the only loss in (N_n,ψ)-invariants.

Theorem 0.3.1. *The functor:*

$$D(G_n)\text{–}\mathsf{mod}_{reg} \xrightarrow{\mathcal{C}\mapsto\mathcal{C}^{N_n,\psi}} \mathsf{DGCat}_{cont}$$

is conservative, where $D(G_n)\text{–}\mathsf{mod}_{reg} \subseteq D(G)\text{–}\mathsf{mod}$ *is the full subcategory consisting of* $\mathcal{C}$ *with* $\mathcal{C}_{reg} = \mathcal{C}$.

Here are some consequences.

Corollary 0.3.2. *For every* $\mathcal{C} \in D(G_n)\text{–}\mathsf{mod}$*, the convolution functor:*

$$D(G_n)^{N_n,-\psi} \underset{\mathcal{H}_n}{\otimes} \mathcal{C}^{N_n,\psi} \to \mathcal{C}$$

is fully-faithful with essential image $\mathcal{C}_{reg}$. *Here* $\mathcal{H}_n = D(G_n)^{N_n\times N_n,(\psi,-\psi)}$ *is the appropriate Hecke category for the pair* $(G_n,(N_n,\psi))$.

Proof. Note that this functor is G_n-equivariant and that its essential image factors through G_n (by the above analysis). Therefore, by Theorem 0.3.1, it suffices to show that it is an equivalence on (N_n,ψ)-invariants, which is clear.

□

Corollary 0.3.3. *Observe that* $D(G_n)_{reg}$ *admits a unique monoidal structure such that the localization functor* $D(G_n) \to D(G_n)_{reg}$ *is monoidal.*

Then $D(G_n)_{reg}$ *and* $\mathcal{H}_n$ *(as defined in the previous corollary) are Morita equivalent, with bimodule* $D(G_n)^{N_n,\psi}$ *defining this equivalence.*

The remainder of this section is devoted to the proof of Theorem 0.3.1.

0.4. **Example:** $n = 2$ **case.** First, we prove Theorem 0.3.1 in the $n = 2$ case. This case is simpler than the general case, and contains one of the main ideas in the proof of the general case.

Note that by Fourier transform along $\mathfrak{g} \otimes \mathbb{G}_a \subseteq G_2$, an action of G_2 on $\mathcal{C}$ is equivalent to the datum of G on $\mathcal{C}$, and an action of $(D(\mathfrak{g}), \overset{!}{\otimes})$ on $\mathcal{C}$ as an object of $D(G)\text{–}\mathsf{mod}$ (where G acts on $D(\mathfrak{g})$ by the adjoint action). In the sheaf of categories language [Gai], we obtain:

$$D(G_2)\text{–}\mathsf{mod} \simeq \mathsf{ShvCat}_{/(\mathfrak{g}/G)_{dR}}.$$

The functor of (N_2,ψ)-invariants then corresponds to global sections of the sheaf of categories over $(f+\mathfrak{b}/N)_{dR}$, i.e., the de Rham space of the Kostant slice. Recall that the Kostant slice $f+\mathfrak{b}/N$ is an affine scheme and maps smoothly to $\mathfrak{g}/G$ with image $\mathfrak{g}^{reg}/G$.

First, recall from [Gai] Theorem 2.6.3 that as the Kostant slice is a scheme (not a stack), $(f+\mathfrak{b}/N)_{dR}$ is 1-affine. In particular, its global sections functor is conservative.

Therefore, it suffices to note that pullback of sheaves of categories along the map $(f+\mathfrak{b}/N)_{dR} \to (\mathfrak{g}^{reg}/G)_{dR}$ is conservative. However, in the diagram:

$$\begin{array}{ccc} f+\mathfrak{b}/N & \longrightarrow & \mathfrak{g}^{reg}/G \\ \downarrow & & \downarrow \\ (f+\mathfrak{b}/N)_{dR} & \longrightarrow & (\mathfrak{g}^{reg}/G)_{dR} \end{array}$$

pullback for sheaves of categories along the vertical maps is conservative for formal reasons (e.g., write de Rham as the quotient by the infinitesimal groupoid), and conservativeness of pullback along the upper arrow follows from descent of sheaves of categories along smooth (or more generally fppf) covers, c.f. [Gai] Appendix A. This implies that pullback along the bottom arrow is conservative as well.

Remark 0.4.1. It follows from the above analysis that the Hecke algebra $\mathcal{H}_2$ (in the notation of Corollary 0.3.2) is equivalent to D-modules on the group scheme of regular centralizers.

0.5. **Heisenberg groups.** We will deduce the general case of Theorem 0.3.1 from the representation theory of Heisenberg groups, which we digress to discuss now.

Let V be a finite-dimensional vector space. We do not distinguish between V and the additive group scheme $V \otimes_k \mathbb{G}_a$ in this section.

Let $H = H(V)$ denote the corresponding *Heisenberg group*; by definition, H is the semidirect product:

$$V \ltimes (V^\vee \times \mathbb{G}_a)$$

where V acts on $V^\vee \times \mathbb{G}_a$ via:

$$v \cdot (\lambda, c) = (\lambda, c + \lambda(v))$$

where $(v, \lambda, c) \in V \times V^\times \times \mathbb{G}_a$.

Remark 0.5.1. Recall that H only depends on the symplectic vector space $W = V \oplus V^\vee$, not on the choice of polarization $V \subseteq W$. But the above presentation is convenient for our purposes.

0.6. Observe that $\mathbb{G}_a \subseteq H$ is central. In particular, $D(\mathbb{A}^1) \overset{\text{Fourier}}{\simeq} D(\mathbb{G}_a)$ maps centrally to H, where we use $D(\mathbb{A}^1)$ to indicate that we consider the $\overset{!}{\otimes}$-monoidal structure and $D(\mathbb{G}_a)$ to indicate the convolution monoidal structure.

Let $D(H)\text{–}\mathsf{mod}_{reg} \subseteq D(H)\text{–}\mathsf{mod}$ denote the subcategory where $D(\mathbb{A}^1)$ acts through its localization $D(\mathbb{A}^1 \backslash 0)$, i.e., where all Fourier coefficients are non-zero.

Theorem 0.6.1. *The functor:*

$$D(H)\text{–}\mathsf{mod}_{reg} \xrightarrow{\mathcal{C} \mapsto \mathcal{C}^V} D(\mathbb{A}^1 \backslash 0)\text{–}\mathsf{mod}$$

is an equivalence.

Corollary 0.6.2. *The functor of V-invariants is conservative on $D(H)\text{–}\mathsf{mod}_{reg}$.*

Proof of Theorem 0.6.1. Note that by duality, V acts on $V \times \mathbb{A}^1$; explicitly, this is given by the formula:

$$v \cdot (w, c) := (w - c \cdot v, c)$$

By Fourier transform along $V^\vee \times \mathbb{G}_a \subseteq H$, we see that a $D(H)$-module structure on $\mathcal{C}$ is equivalent to giving a $D(V)$ (under convolution) action on $\mathcal{C}$, and an additional $(D(V \times \mathbb{A}^1), \overset{!}{\otimes})$-action in the category $D(V)\text{–}\mathsf{mod}$.

Using the sheaf of categories language [Gai], this is equivalent to the data of a sheaf of categories on $(V_{dR} \times \mathbb{A}^1_{dR})/V_{dR}$, where we are quotienting using the above action. The corresponding object of $H\text{–}\mathsf{mod}$ lies in $H\text{–}\mathsf{mod}_{reg}$ if and only if the sheaf of categories is pushed forward from:

$$(V_{dR} \times \mathbb{A}^1_{dR} \backslash 0)/V_{dR} = \mathbb{A}^1_{dR} \backslash 0.$$

Therefore, we obtain an equivalence of the above type. Geometrically, this equivalence is given by taking global sections of a sheaf of categories, which for $(V_{dR} \times \mathbb{A}^1_{dR})/V_{dR}$ corresponds to taking (strong) V-invariants for the corresponding H-module category.

□

0.7. **Proof of Theorem 0.3.1.** We now return to the setting of Theorem 0.3.1.

In what follows, for $\mathfrak{h}$ a nilpotent Lie algebra, we let $\exp(\mathfrak{h})$ denote the corresponding unipotent algebraic group.

Let $N_n^m = \exp(t^{n-m}\mathfrak{n}[[t]]/t^n\mathfrak{n}[[t]]) \subseteq N_n$ for $1 \leqslant m \leqslant n$. E.g., for $m = 1$ we recover the group $\mathfrak{n} \otimes \mathbb{G}_a \subseteq G_n$.

We will show by induction on m that the functor of (N_n^m, ψ)-invariants is conservative on $G_n\text{–}\mathsf{mod}_{reg}$.

Let us begin with the base case $m = 1$. Here this follows by the argument of §0.4. Indeed, we have a homomorphism $G_2 \to G_n$ which identifies $G \subseteq G_2, G_n$ and $\mathfrak{g} \otimes \mathbb{G}_a \subseteq G_2, G_n$. Restricting along this homomorphism, we obtain that ψ-invariants for $N \cdot N_n^1 \subseteq N_n$ is conservative, and a fortiori, (N_n^1, ψ)-invariants is such.

0.8. Now assume the conservativeness for $1 \leqslant m-1$ and let us show it for $m \leqslant n$.

We will do this by another induction. Let $\mathfrak{g} = \oplus_s \mathfrak{g}_s$ be the principal grading, so e.g. $e_i \in \mathfrak{g}_1$ and $\mathfrak{n} = \oplus_{s \geqslant 1} \mathfrak{g}_s$. For $r \geqslant 1$, let $\mathfrak{n}_{\geqslant r} := \oplus_{s \geqslant r} \mathfrak{g}_s$.

Now define:

$$N_n^{m,r} := \exp\left(t^{n-m+1}\mathfrak{n}[[t]] + t^{n-m}\mathfrak{n}_{\geqslant r}[[t]]/t^n\mathfrak{n}[[t]]\right) \subseteq N_n^m \subseteq N_n.$$

We will show by descending induction on $r \geqslant 1$ that $(N_n^{m,r}, \psi)$-invariants is conservative. Note that this result is clear from our hypothesis on m for $r \gg 0$, since then $\mathfrak{n}_{\geqslant r} = 0$ and $N_n^{m,r} = N_n^{m-1}$. Moreover, a proof for all r implies the next step in the induction with respect to m, since $N_n^{m,1} = N_n^m$, so would complete the proof of Theorem 0.3.1.

0.9. For $r \geqslant 1$, assume the conservativeness (in the regular setting) of $(N_n^{m,r+1}, \psi)$-invariants; we will deduce it for $N_n^{m,r}$. The idea is to make a Heisenberg group act on $(N_n^{m,r+1}, \psi)$-invariants so that invariants with respect to a Lagrangian gives $(N_n^{m,r}, \psi)$-invariants.

Step 1. Define $\mathfrak{h}_0 \subseteq \mathrm{Lie}(G_n) = \mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]]$ as:

$$t^{m-1}\mathfrak{g}_{1-r} \oplus \mathrm{Lie}(N_n^{m,r}).$$

Observe that $\mathfrak{h}_0$ is a Lie subalgebra. Indeed, $t^{m-1}\mathfrak{g}_{1-r}$ commutes with $t^{n-m+1}\mathfrak{g}[[t]] \bmod t^n\mathfrak{g}[[t]]$, and $[t^{m-1}\mathfrak{g}_{1-r}, t^{n-m}\mathfrak{n}_{\geqslant r}] \subseteq t^{n-1}\mathfrak{n}$.

In this way, we see that[1] $\mathrm{Lie}(N_n^{m,r+1})$ is a normal Lie subalgebra of $\mathfrak{h}_0$, and that for $\xi \in \mathfrak{h}_0$ and $\varphi \in \mathrm{Lie}(N_n^{m,r+1})$, $\psi([\xi, \varphi]) = 0$.

Moreover, $\mathfrak{h}_0$ is nilpotent so exponentiates to a group $H_0 \subseteq G_n$.[2] Combining this with the above, we see that H_0 acts on $(N_n^{m,r+1}, \psi)$-invariants for any category with an action of G_n.

Step 2. Let $\mathfrak{g}'_{1-r} \subseteq \mathfrak{g}_{1-r}$ denote $\mathrm{Ad}_f^{2r-1}(\mathfrak{g}_r)$. Observe that the pairing:

$$\psi([-,-]) : \mathfrak{g}_r \otimes \mathfrak{g}_{1-r} \to k \tag{0.9.1}$$

induces a perfect pairing between $\mathfrak{g}_r$ and $\mathfrak{g}'_{1-r}$. Indeed, the diagram:

$$\begin{array}{ccccc}
\mathfrak{g}_r \otimes \mathfrak{g}_r & \xrightarrow{\mathrm{id} \otimes \mathrm{Ad}_f^{2r-1}} & \mathfrak{g}_r \otimes \mathfrak{g}_{1-r} & \xrightarrow{\mathrm{id} \otimes \mathrm{Ad}_f} & \mathfrak{g}_r \otimes \mathfrak{g}_{-r} \\
 & & & \searrow^{\psi([-,-])} & \downarrow{\scriptstyle -\kappa(-,-)} \\
 & & & & k
\end{array}$$

commutes[3], and $\mathrm{Ad}_f^{2r} : \mathfrak{g}_r \to \mathfrak{g}_{-r}$ is an isomorphism by $\mathfrak{sl}_2$-representation theory.

Define $\mathfrak{h}'_0 \subseteq \mathfrak{h}_0$ as:

$$t^{m-1}\mathfrak{g}'_{1-r} \oplus \mathrm{Lie}(N_n^{m,r}).$$

Again, $\mathfrak{h}'_0$ integrates to a group H'_0.

Step 3. Finally, recall that the adjoint action of H_0 fixes $N_n^{m,r+1} \subseteq H_0$ and preserves its character ψ to $\mathbb{G}_a$. Let $K \subseteq N_n^{m,r+1}$ be the kernel of ψ: clearly K is normal in H_0.

One immediately observes that $H := H'_0/K$ is a Heisenberg group. The central $\mathbb{G}_a$ is induced by the map:

$$\mathbb{G}_a = N_n^{m,r+1}/K \to H'_0/K = H.$$

The vector space defining the Heisenberg group is $t^{n-m}\mathfrak{g}_r$, and its dual is embedded as $t^{m-1}\mathfrak{g}'_{1-r} = H'_0/K$.

Now observe that our Heisenberg group H acts on $\mathcal{C}^{N_n^{m,r+1},\psi}$ for any $\mathcal{C}$ acted on by G_n, with its central $\mathbb{G}_a$ acting through the exponential character. Now the result follows from Corollary 0.6.2.

References

[Gai] Dennis Gaitsgory. Sheaves of categories and the notion of 1-affineness. In T. Pantev, C. Simpson, B. Toën, M. Vaquié, and G. Vezzosi, editors, *Stacks and Categories in Geometry, Topology, and Algebra:*, Contemporary Mathematics. American Mathematical Society, 2015.

[1] The same is true for r instead of $r+1$, but the statement with the character is not.

[2] The embedding exponentiates because $\mathfrak{h}_0 \subseteq \mathfrak{n} + t\mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]]$, i.e., the Lie algebra of a unipotent subgroup of G_n. Here we use that $m \geqslant 2$.

[3] Proof: write $\psi(-)$ as $\kappa(f,-)$ and use Ad-invariance of κ.