Breuil-Kisin twists (prismatic analog of Tate twists).
$\overset{\shortparallel}{BK}$

<u>Def.</u> A BK module on $\mathrm{Spf}\,\mathbb{Z}_p$ is a vector bundle $\mathcal{L}$ on Σ with a morphism $F^*\mathcal{L}\to\mathcal{L}$ whose Coker is supported on $n\Delta_0$ for some n (i.e., an F-crystal which is a locally free finitely generated $\mathcal{O}$-module).

"Conventional" BK module: replace (Σ, F, Δ_0) by $(\mathrm{Spf}\,\mathbb{Z}_p[[x]], \varphi, D)$, where $\varphi(x)=x^p$, $D=\mathrm{Spf}\,\mathbb{Z}_p[[x]]/(x-p)$. One has a morphism $\alpha: (\mathrm{Spf}\,\mathbb{Z}_p[[x]], \varphi) \longrightarrow (\Sigma, F)$ such that $\alpha^{-1}(\Delta_0)=D$; namely, α corresponds to $[x]-p\in W^{(1)}(\mathbb{Z}_p[[x]])$. Get $\alpha^*: \{\text{BK modules}\}\to\{\text{Conventional BK modules}\}$.

<u>Goal</u>: define a BK module $\mathcal{O}_\Sigma\{-1\}$ analogous to $\mathbb{Z}_p(-1)$.

<u>Remark.</u> $\mathcal{O}_\Sigma\{1\}$ doesn't exist, and in the crystalline theory $\mathbb{Z}_p(1)$ doesn't exist. In the crystalline theory the role of $\Sigma=(\mathrm{Spf}\,\mathbb{Z}_p)^\Delta$ is played by $(\mathrm{Spec}\,\mathbb{F}_p)^\Delta=\mathrm{Spf}\,\mathbb{Z}_p$, and an F-crystal on $\mathrm{Spec}\,\mathbb{F}_p$ is just a $\mathbb{Z}_p$-module equipped with a (geometric!) Frobenius endomorphism. $\mathbb{Z}_p(-1)$ is $\mathbb{Z}_p$ with Frobenius acting by p. But $p^{-1}\notin\mathbb{Z}_p$.

$$n\geq 0 \Rightarrow \mathcal{O}_\Sigma\{-n\}:=(\mathcal{O}_\Sigma\{-1\})^{\otimes n}.$$

If M is an F-crystal (e.g., a BK module) then $M\{-n\}:=M\otimes\mathcal{O}_\Sigma\{-n\}$ is called the <u>BK twist</u> of M (this name is used by Bhatt-Scholze).

<u>Pre-definition.</u> $\mathcal{O}_\Sigma\{-1\}:=\bigotimes\limits_{n=0}^{\infty}(F^n)^*\mathcal{O}_\Sigma(\Delta_0)=\bigotimes\limits_{n=0}^{\infty}\mathcal{O}_\Sigma(\Delta_n)$.

<u>Key Fact</u> (to be explained). Let $\mathcal{L}$ be a line bundle on Σ with a trivialization of $p^*\mathcal{L}$, where $p:\mathrm{Spf}\,\mathbb{Z}_p\to\Sigma$ is the image of $p\in W^{(1)}(\mathbb{Z}_p)$. Then $\bigotimes\limits_{n=0}^{\infty}(F^n)^*\mathcal{L}$ makes sense as a line bundle (and behaves reasonably). Assuming "reasonable behavior", we get $F^*\mathcal{O}_\Sigma\{-1\}=\bigotimes\limits_{n=1}^{\infty}(F^n)^*\mathcal{O}_\Sigma(\Delta_0)=\mathcal{O}_\Sigma\{-1\}\otimes\mathcal{O}_\Sigma(-\Delta)$.
So we get $F^*\mathcal{O}_\Sigma\{-1\}\overset{\varphi}{\hookrightarrow}\mathcal{O}_\Sigma\{-1\}$ with $\mathrm{Supp}\,\mathrm{Coker}\,\varphi=\Delta_0$. So $\mathcal{O}_\Sigma\{-1\}$ is a BK module, as promised.

<u>Motivation.</u> Let $x\in\mathrm{Pic}\,\Sigma$ be the class of $\mathcal{O}_\Sigma\{-1\}$. Want $F^*x=x-[\Delta_0]$, i.e., $(1-F^*)x=[\Delta_0]$. Solution: $x=(1+F^*+(F^*)^2+\dots)[\Delta_0]$.

<u>Lemma.</u> $\mathrm{Coker}\,\varphi=\mathcal{O}_\Sigma(\Delta_0)|_{\Delta_0}$ (normal bundle of Δ_0).

<u>Proof.</u> It suffices to show that once $p^*\mathcal{O}_\Sigma(\Delta_0)$ is trivialized then $[(F^n)^*\mathcal{O}_\Sigma(\Delta_0)]|_{\Delta_0}$ (for $n>0$) is canonically trivial. Reason: commutativity of

the diagram

$$\begin{tikzcd} \Delta_0 \arrow[r, hook] \arrow[dr] & \Sigma \arrow[r, "F^n"] & \Sigma \\ & \operatorname{Spf} \mathbb{Z}_p \arrow[ur, "p"'] & \end{tikzcd}$$

for $n > 0$. To check commutativity, we have to show that if $u \in W^\times$ then $F^n(Vu) = p \cdot (\text{"canonical" unit})$. Indeed, $F^n(Vu) = p \cdot F^{n-1}(u)$, $F^{n-1}(u) \in W^\times$. ∎

To transform the pre-definition into a definition, I have to formulate and prove

<u>The contracting property of $F:\Sigma\to\Sigma$.</u>

Let C be a category (e.g., a groupoid). Let $F: C\to C$ be a functor.

<u>Def.</u> $C^F := \{(c,\alpha) \mid c\in C,\ \alpha: c \xrightarrow{\sim} F(c)\}$ — category of F-fixed points.

Remark. The functor $C^F\to C$ is faithful but <u>not fully</u> faithful. (So C^F is <u>not</u> a full subcategory of Σ).

<u>Def.</u> $C[F^{-1}] := \varinjlim\,(C \xrightarrow{F} C \xrightarrow{F} \dots)$

Localization of C w.r.t. F (the universal way to make F invertible).

~~Exercise~~ <u>Def.</u> $F:\Sigma\to\Sigma$ is <u>contracting</u> if $C[F^{-1}] =$ point.

category with one object and one morphism.

<u>Exercise.</u> If F is contracting then $C^F =$ point.

<u>Beginning of the proof.</u> Let $c\in C$. Then c and $F(c)$ have isomorphic images in $C[F^{-1}]$. So $\exists n: F^n(c)\simeq F^{n+1}(c)$. So $F(F^n(c))\simeq F^n(c)$. So $C^F\neq\emptyset$. ~~(I could stop here.)~~

<u>End of the proof.</u> a) If C is a set ~~the~~ finishing the proof is easy.

b) In general, we have to show that if $c, c'\in C^F$ then $\mathrm{Hom}_{C^F}(c,c') =$ point. Let $S := \mathrm{Hom}_C(c,c')$. We have $F: S\to S$ and $\mathrm{Hom}_{C^F}(c,c') = S^F$. We know that $S[F^{-1}] := \varinjlim\,(S\xrightarrow{F} S\to\dots)$ is a point. So S^F is a point by a). ∎

<u>Remark.</u> Without assuming that F is contracting, one ~~probably~~ has $C^F = C[F^{-1}]^F$.

<u>Proposition.</u> ~~Let be a (discrete) ring in which p is nilpotent. Then~~ $F:\Sigma(R)\to\Sigma(R)$ is contracting for all $R\in \mathrm{Rings}_p$.

<u>Corollary.</u> $\Sigma^F = \mathrm{Spf}\,\mathbb{Z}_p$. (The corresponding morphism $\mathrm{Spf}\,\mathbb{Z}_p\to\Sigma$ is $p:\mathrm{Spf}\,\mathbb{Z}_p\to\Sigma$. Indeed, $\Sigma^F(\mathbb{Z}_p)$ has a single element).

<u>Exercise.</u> (i) The map $\overset{\mathrm{Spf}\,\mathbb{Z}_p}{\overset{\|}{\Sigma}}{}^{F} \to \Sigma$ is <u>not</u> a monomorphism. (Not surprising).

(ii) It has "schematically dense image", i.e., if $\mathcal{Y} \subset \Sigma$ is a closed substack such that ~~$\mathcal{Y}$~~ $\mathcal{Y}(\mathbb{Z}_p) \ni p$ then $\mathcal{Y} = \Sigma$.

(ii) $\Rightarrow$ $H^0(\Sigma, \mathcal{O}_\Sigma) = \mathbb{Z}_p$.

Let us explain the "key fact": if $\mathcal{L}$ is a line bundle with $p^*\mathcal{L}$ trivialized then $\bigotimes_{n=0}^{\infty} (F^n)^*\mathcal{L}$ makes sense. Idea: $\prod_{n=0}^{\infty} x_n$ makes sense if $x_n = 1$ for $n > n_0$. We are in a slightly different situation (but familiar from automorphic form theory).

<u>Construction.</u> Let $R \in \mathrm{Rings}_p$, $f : \mathrm{Spec}\, R \to \Sigma$. Want to define $\bigotimes_{n=0}^{\infty} f^*(F^n)^*\mathcal{L}$. Note: $f^*(F^n)^* = (F^n \circ f)^*$. Have $f \in \Sigma(R)$, $F : \Sigma(R) \to \Sigma(R)$, $F^n \circ f = F^n(f)$. Recall: $p^*\mathcal{L}$ is trivialized. ~~So~~ But the Proposition tells us that $F^n(f)$ "$=$" p for $n \gg 0$. More precisely, $\exists \ell : F^\ell(f) \simeq p$. Choosing $\alpha_\ell : F^\ell(f) \xrightarrow{\sim} p$, get $\alpha_n : F^n(f) \xrightarrow{\sim} p$ for $n \geq \ell$. A different choice of α_ℓ gives the same α_n for $n \geq \ell'$. ■

<u>Trivializing $p^*\mathcal{O}_\Sigma(-\Delta_0)$</u> (this is necessary to define $\mathcal{O}_\Sigma\{-1\}$).

$\Delta_0 \subset \Sigma$ is defined by $\xi_0 = 0$. So $p^*\Delta_0 \subset \mathrm{Spf}\,\mathbb{Z}_p$ is defined by $p = 0$. So $p^*\mathcal{O}_\Sigma(-\Delta_0)$ is the ideal $p\mathbb{Z}_p \subset \mathbb{Z}_p$. As a generator of $p\mathbb{Z}_p$, choose p.

<u>On Pic Σ.</u>

Have $F^* : \mathrm{Pic}\,\Sigma \to \mathrm{Pic}\,\Sigma$. (There is no F_* because $F : \Sigma \to \Sigma$ is not finite.) So $\mathrm{Pic}\,\Sigma$ is a $\mathbb{Z}[F^*]$-module.

<u>Fact.</u> This module is I-adically complete, where $I = (p, F^*)$. So $\mathrm{Pic}\,\Sigma$ is a $\mathbb{Z}_p[[F^*]]$-module.

<u>Question.</u> Is it ~~freely generated by~~

Conjecture. $\mathrm{Pic}\,\Sigma$ is the free $\mathbb{Z}_p[[F^*]]$-module generated ~~by~~ by $\mathcal{O}_\Sigma(\Delta_0)$.

~~**Remark.** There is no $F_*: \mathrm{Pic}\,\Sigma \to \mathrm{Pic}\,\Sigma$. Reason: F is not finite. (F is ind-finite = integral. F is also flat.)~~

Exercise. ~~An~~ Any ~~F-crystal~~ invertible BK module on $\mathrm{Spf}\,\mathbb{Z}_p$ ~~which is an~~ invertible ~~$\mathcal{O}_\Sigma$-module~~ is isomorphic to $\mathcal{O}_\Sigma\{-n\}$ for some $n \geq 0$.

Proof of the Proposition on p. ~~2~~3.

Claim. Let $R \in \mathrm{Rings}_{p}$. Then $F: \Sigma(R) \to \Sigma(R)$ is contracting, i.e., $\varinjlim\, (\Sigma(R) \xrightarrow{F} \Sigma(R) \xrightarrow{F} \ldots) = \text{point}$.

① At the level of objects: $\forall x \in \Sigma(R)\, \exists n: F^n(x) \simeq p$.

$\Sigma := W^{(1)}/W^\times$, so it suffices to show that $\forall \xi \in W^{(1)}(R)\, \exists n: F^n\xi = pu$ for some $u \in W(R)^\times$.

$\xi = \sum_i V^i[\xi_i]$, $\xi_i \in R$, ξ_0 nilpotent, $\xi_1 \in R^\times$. So $\xi = [\xi_0] + Vy$, $y \in W(R)^\times$.

Then $F^n \xi = [\xi_0^{p^n}] + pF^{n-1}y$. $n \gg 0 \Rightarrow \xi_0^{p^n} = 0$, $F^n\xi = pu$, $u = F^{n-1}y \in W(R)^\times$.

② It remains to study $F: \mathrm{Aut}\,p \to \mathrm{Aut}\,p$, where $\mathrm{Aut}\,p := \{u \in W(R)^\times \mid pu = p\}$.

Want: if $u \in W(R)^\times$, $pu = p$ then $\exists n: F^n u = 1$.

Claim. $F^n(u-1) \in W(p^nR) = \mathrm{Ker}\,(W(R) \to W(R/p^nR))$.

This is enough because $p \in R$ is nilpotent. Proof by induction on n:

Case $n=1$. $p(u-1) = 0$, so $FV(u-1) = 0$. In $W(R/pR)$ we have $FV = VF$, so $VF(u-1) = 0$, so $F(u-1) = 0$.

Induction step. Suppose $F^n(u-1) \in W(p^nR)$. Then $VF^{n+1}(u-1) = V(1) \cdot F^n(u-1) = (V(1) - p) \cdot F^n(u-1)$ because $p(u-1) = 0$. But $V(1) - p \in W(pR)$, $F^n(u-1) \in W(p^nR)$, $W(pR) \cdot W(p^nR) \subset W(p^{n+1}R)$.

So $F^{n+1}(u-1) \in W(p^{n+1}R)$. ■

Just in case: if $i \leq j$ then $V^i[a] \cdot V^j[b] = V^i([a] \cdot F^iV^j[b]) = p^iV^j[ab^{p^{j-i}}]$.