Contents

PRISMATIZATION

In §1 I explain the motivation. In §2 I sketch a "refined" version of the stacky approach to prismatic cohomology. In §3 I explain the details about W_S-modules.

1. Introduction

1.1. **Goals.** I hope that the approach to prismatic cohomology sketched in §2 achieves the following goals.

(i) The Nygaard filtration on prismatic cohomology becomes automatic.

(ii) The definitions of gauge and F-gauge are easy, see §2.3.2. (As explained to me by Peter Scholze, F-gauges should be the coefficients in the prismatic theory.)

(iii) The "Hodge to de Rham" and "Hodge to Hodge-Tate" spectral sequences are hopefully automatic, see §2.6.

1.2. **Some notation and terminology.** Fix a prime p. All schemes are classical for now. A scheme S is said to be p-nilpotent if $p \in H^0(S, \mathcal{O}_S)$ is nilpotent.

Let W be the ring scheme of p-typical Witt vectors over $\mathbb{Z}$. Let $W_S := W \times S$; this is a ring scheme over S. By a W_S-module we mean a commutative affine group scheme over S equipped with an action of the ring scheme W_S.

A g-stack (or simply stack) is an fpqc-stack of groupoids on the category of p-nilpotent schemes. A c-stack is an fpqc-stack of categories on the category of p-nilpotent schemes.

The fully faithful functor from the 2-category of g-stacks to that of c-stacks has a right adjoint (removing non-invertible morphisms). One can consider a c-stack as a g-stack with additional structure; we call it c-structure.

1.3. **Recollections on usual prismatization.** Let S be a p-nilpotent scheme. The stack Σ is defined as follows: an object of $\Sigma(S)$ is a pair (P, ξ), where P is a W_S-module locally isomorphic to W_S and $\xi : P \to W_S$ is a primitive[1] W_S-morphism. A priori, Σ is a c-stack, but it is easy to see that it is a g-stack.

In this situation (P, ξ) is automatically a quasi-ideal[2] in W_S. So given $(P, \xi) \in \Sigma(S)$ we get a ring stack $\mathrm{Cone}(\xi)$ over S. This construction yields a ring stack over Σ, denoted by $(\mathbb{A}^1)^{\Delta}$. Using this ring stack, one defines X^{Δ} for any p-adic scheme X so that for $X = \mathbb{A}^1 := \mathbb{A}^1_{\mathbb{Z}_p}$ one gets the above ring stack and $(\mathrm{Spf}\, \mathbb{Z}_p)^{\Delta} = \Sigma$.

1.4. **A drawback of Σ.** The stack Σ is supposed to parametrize cohomology theories (in some sense). E.g., the points $p \in \Sigma(\mathbb{Z}_p)$ and $V(1) \in \Sigma(\mathbb{Z}_p)$ give rise to de Rham and Hodge-Tate cohomology, respectively. But there is also Hodge cohomology, which is related to de Rham and Hodge-Tate cohomology via spectral sequences. The problem is that Hodge-Tate

[1] Throughout this text, "primitive" really means "primitive of degree 1". If S is the spectrum of a field k this means that ξ maps every (or some) generator of the $W(k)$-module $P(k)$ to an element of $W(k)$ of the form Vu, $u \in W(k)^\times$. For any S, primitivity means that ξ is primitive over every field-valued point of S.

[2] The definition of quasi-ideal is recalled in the proof of Lemma 3.10.10.

cohomology does not correspond to a locus in Σ. To fix this, we define in §2.1.1-2.1.2 a bigger c-stack Σ'.

1.5. **The functors $X \mapsto X^{\Delta'}$ and $X \mapsto X^{\Delta''}$.** These functors are defined in §2.1-2.2 using the strategy of §1.3. However, instead of Σ one uses certain c-stacks Σ' and Σ''. The c-structure ensures that prismatic cohomology is an *effective* F-gauge. (Roughly, the key idea is that a $\mathbb{Z}$-grading on a module is non-negative if and only if the corresponding $\mathbb{G}_m$-action extends to an action of the multiplicative monoid $\mathbb{A}^1$.)

1.6. **Confession.** Before I came up with the definition of Σ', I wanted to work with the c-stack Σ'_+ from §2.8 (and to think of Σ'_+ in terms of §2.8.3(ii)). Accordingly, instead of $X^{\Delta'}$ I wanted to work with the c-stack $X^{\Delta} \times_{\Sigma} \Sigma'_+$ (which is equipped with a canonical map to $X^{\Delta'}$, see formula (2.16)). This approach was directly inspired by the notion of F-gauge from [FJ]. The problem with it is that $(\operatorname{Spec} \mathbb{F}_p)^{\Delta} \times_{\Sigma} \Sigma'_+$ is not what you want. On the other hand, $(\operatorname{Spec} \mathbb{F}_p)^{\Delta'}$ is what you want (see §2.7.1).

2. Outline of refined prismatization

2.1. **Refined prismatization.**

2.1.1. *Admissible W_S-modules.* Let M be a W_S-module. Precomposing the action of W_S on M with $F^n : W_S \to W_S$, we get a new W_S-module structure on the group scheme underlying M; the new W_S-module will be denoted by $M^{(n)}$.

We have a faithfully flat W_S-module homomorphism $F : W_S \twoheadrightarrow W_S^{(1)}$. Its kernel is denoted by $W_S^{(F)}$. By Lemma 3.1.6, $W_S^{(F)}$ canonically identifies with the PD hull of zero in $(\mathbb{G}_a)_S$.

A W_S-module M is said to be *admissible* if for some line bundle $\mathscr{L}$ on S there exists an exact sequence of W_S-modules

$$0 \to \mathscr{L}^\sharp \to M \to M' \to 0, \tag{2.1}$$

where M' is locally isomorphic to $W_S^{(1)}$ and $\mathscr{L}^\sharp := \mathscr{L} \otimes W_S^{(F)}$ (equivalently, $\mathscr{L}^\sharp$ is the PD-hull of the group scheme $\mathscr{L}$ along its zero section). By Lemma 3.10.7, such an exact sequence is unique if it exists; moreover, it is functorial in M.

A W_S-module is said to be *invertible* if it is locally isomorphic to W_S. Such modules are admissible; e.g., if $M = W_S$ then $M' = W_S/W_S^{(F)} = W_S^{(1)}$.

2.1.2. *Definition of* Σ'. Functoriality implies that if M is an admissible W_S-module then any homomorphism $\xi : M \to W_S$ induces a homomorphism $\xi' : M' \to W_S' = W_S^{(1)}$, where M' is as in (2.1). We say that ξ is *primitive* if ξ' is primitive[3]. Note that if M is invertible this is equivalent to primitivity of ξ in the usual sense.

Now define a c-stack Σ' as follows: for any p-nilpotent scheme S, let $\Sigma'(S)$ be the category of pairs (M, ξ), where M is an admissible W_S-module and $\xi : M \to W_S$ is a primitive W_S-morphism.

One checks[4] that Σ' is algebraic over[5] the formal stack $\hat{\mathbb{A}}^1/\mathbb{G}_m$. The morphism $\Sigma' \to \hat{\mathbb{A}}^1/\mathbb{G}_m$ takes (M, ξ) to $\bar{\xi}'$, where

$$\bar{\xi}' : M' \otimes_{W_S^{(1)}} (W_S^{(1)}/V(W_S^{(2)})) \to W_S^{(1)}/V(W_S^{(2)})$$

is induced by $\xi' : M' \to W_S^{(1)}$.

Sometimes we will write (M, ξ_M) instead of (M, ξ) to avoid conflict of notation with other objects denoted by ξ.

2.1.3. *The left fibration* $\Sigma' \to (\mathbb{A}^1/\mathbb{G}_m)_-$. Note that $\xi : M \to W_S$ induces a morphism $v_- : \mathscr{L} \to \mathcal{O}_S$. Thus we get a morphism of c-stacks

$$v_- : \Sigma' \to (\mathbb{A}^1/\mathbb{G}_m)_-, \tag{2.2}$$

where $(\mathbb{A}^1/\mathbb{G}_m)_-$ is the c-stack whose S-points are invertible $\mathcal{O}_S$-modules equipped with a morphism to $\mathcal{O}_S$. The morphism (2.2) is clearly a left fibration in Joyal's sense (see [Lu1, §2.1]). In particular, for any scheme S over $(\mathbb{A}^1/\mathbb{G}_m)_-$, the fiber product of Σ' and S over $(\mathbb{A}^1/\mathbb{G}_m)_-$ is a g-stack.

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[3] Throughout this text, “primitive" means “primitive of degree 1".

[4] One can use diagram (2.12) or Proposition 3.9.1.

[5] Given a morphism of c-stacks $\mathscr{X} \to \mathscr{Y}$, we say that $\mathscr{X}$ is algebraic over $\mathscr{Y}$ if the c-stack $\mathscr{X} \times_{\mathscr{Y}} S$ is algebraic for any morphism $S \to \mathscr{Y}$ with S being a scheme.

2.1.4. *The functor* $X \mapsto X^{\Delta'}$. By Lemma 3.10.10, if $(M,\xi) \in \Sigma'(S)$ then (M,ξ) is a quasi-ideal in W_S, so we get a ring stack $\mathrm{Cone}(\xi)$ over S. This construction yields a c-stack over Σ' denoted by $(\mathbb{A}^1)^{\Delta'}$. The morphism $(\mathbb{A}^1)^{\Delta'} \to \Sigma'$ is clearly a left fibration equipped with a ring structure.

Using this ring stack, one defines a functor $X \mapsto X^{\Delta'}$ from the category of p-adic schemes to the category of left fibrations over Σ' so that $(\mathrm{Spf}\,\mathbb{Z}_p)^{\Delta'} = \Sigma'$.

2.1.5. *The open substacks* $\Sigma_\pm \subset \Sigma'$ *and* $X^{\Delta}_\pm \subset X^{\Delta'}$. Let $\Sigma_-(S)$ be the category of pairs $(M,\xi) \in \Sigma'(S)$ such that the corresponding map $v_- : \mathscr{L} \to \mathcal{O}_S$ is an isomorphism. In other words, Σ_- is the preimage of the open point $\mathbb{G}_m/\mathbb{G}_m \subset (\mathbb{A}^1/\mathbb{G}_m)_-$ with respect to the left fibration (2.2). The substack $\Sigma_- \subset \Sigma'$ is clearly open and affine over Σ'. The restriction of the left fibration (2.2) to Σ_- is still a left fibration.

Let $\Sigma_+(S)$ be the category of pairs $(M,\xi) \in \Sigma'(S)$ such that M is invertible. One can show that the substack $\Sigma_+ \subset \Sigma'$ is open and affine[6] over Σ'. The restriction of the left fibration (2.2) to Σ_+ is *not* a left fibration.[7]

It is easy to see that $\Sigma_+ \cap \Sigma_- = \emptyset$.

For any p-adic scheme X, let $X^{\Delta}_\pm \subset X^{\Delta'}$ be the preimages of the open substacks $\Sigma_\pm \subset \Sigma'$.

One has a tautological isomorphism $X^{\Delta}_+ \xrightarrow{\sim} X^{\Delta}$ (in particular, $\Sigma_+ \xrightarrow{\sim} \Sigma$). One also has a canonical isomorphism $X^{\Delta}_- \xrightarrow{\sim} X^{\Delta}$; in the case $X = \mathrm{Spf}\,\mathbb{Z}_p$ this is the isomorphism $\Sigma_- \xrightarrow{\sim} \Sigma$ given by $(M,\xi) \mapsto (M',\xi')$, where M' and ξ' are as in §2.1.2.

2.1.6. *The canonical morphism* $F' : X^{\Delta'} \to X^{\Delta}$. Recall that X^{Δ}_- is the preimage of the open point $\mathbb{G}_m/\mathbb{G}_m \subset (\mathbb{A}^1/\mathbb{G}_m)_-$ with respect to the canonical left fibration $X^{\Delta'} \to (\mathbb{A}^1/\mathbb{G}_m)_-$. The open point $\mathbb{G}_m/\mathbb{G}_m \subset (\mathbb{A}^1/\mathbb{G}_m)_-$ is the final object of $(\mathbb{A}^1/\mathbb{G}_m)_-$, so we get a canonical morphism

$$F' : X^{\Delta'} \to X^{\Delta}_- = X^{\Delta}, \tag{2.3}$$

whose restriction to X^{Δ}_- equals the identity. It is easy to check that the restriction of (2.3) to X^{Δ}_+ equals $F : X^{\Delta} \to X^{\Delta}$. Thus $F' : X^{\Delta'} \to X^{\Delta}$ is a kind of "interpolation" between $F : X^{\Delta} \to X^{\Delta}$ and $\mathrm{id} : X^{\Delta} \to X^{\Delta}$.

Note that in the particular case $X = \mathrm{Spf}\,\mathbb{Z}_p$ we get a canonical morphism

$$F' : \Sigma' \to \Sigma_- = \Sigma. \tag{2.4}$$

2.2. **Very refined prismatization.**

Define Σ'' by gluing Σ_+ with Σ_- (then we get many new morphisms). For any p-adic scheme X, define $X^{\Delta''}$ by gluing X^{Δ}_+ with X^{Δ}_-; thus we get a functor from the category of p-adic schemes to the category of left fibrations over Σ''.

2.3. **Gauges and F-gauges.**

[6] See Lemma 3.12.2(ii).

[7] Combining §2.8.3(i) and §2.8.2(i), we get a property of Σ_+, which is in some sense opposite to being a left fibration.

2.3.1. *$\mathcal{O}$-modules on c-stacks.* By an $\mathcal{O}$-module on a c-stack $\mathscr{Y}$ we mean a compatible collection of *contra*variant functors $\mathscr{Y}(S) \to \{\mathcal{O}_S\text{-modules}\}$. This is because the $\mathcal{O}$-modules we care about come from cohomology (which is a contravariant functor).

2.3.2. *Definitions.* For a p-scheme X, define an *effective gauge* (resp. an *effective F-gauge*) on X to be an $\mathcal{O}$-module (or a complex of $\mathcal{O}$-modules) on $X^{\Delta'}$ (resp. on $X^{\Delta''}$). Thus an effective F-gauge is an effective gauge whose restrictions to $X^{\Delta}_{\pm}$ are identified with each other.

To define the general (without effectivity) notions of gauge and F-gauge, replace the c-stacks $X^{\Delta'}$ and $X^{\Delta''}$ by the corresponding g-stacks. Using [G, Prop. 3.4.9], one can show that the functors

$$\{\text{effective gauges}\} \to \{\text{gauges}\}, \quad \{\text{effective } F\text{-gauges}\} \to \{F\text{-gauges}\}$$

are fully faithful.

2.3.3. *From gauges to crystals.* Recall that a *crystal* on X is an $\mathcal{O}$-module on X^{Δ}.

Restricting a gauge on X to $X^{\Delta}_{\pm}$, we get crystals $N_{\pm}$ on X. By §2.1.6, in the case of an an effective gauge we also get a canonical morphism $\varphi : F^*N_- \to N_+$. In the case of an effective F-gauge we have $N_+ = N_- = N$, so φ is a morphism $F^*N \to N$, and the pair (N, φ) is an F-crystal.

2.3.4. *Comparing with Fontaine-Jannsen.* If X is the spectrum of a perfect field k of characteristic p, the definitions from §2.3.2 are equivalent to those from [FJ] (the case $k = \mathbb{F}_p$ is explained in §2.7.2, and arbitrary perfect fields are treated similarly).

2.3.5. *Smooth $\mathbb{F}_p$-schemes.* Did somebody[8] define the notion of F-gauge on an *arbitrary* smooth $\mathbb{F}_p$-scheme? If yes then one should compare his (or her) definition with the one from 2.3.2. At least, in the case $X = \mathbb{A}^1_{\mathbb{F}_p}$ this should be doable, see §2.7.4.

2.3.6. *Nygaard filtration.* Given an effective gauge on a p-adic scheme X, one can construct (see §2.8.7 below) the following refinement of the triple (N_+, N_-, φ) from §2.3.3:

(i) a factorization of $\varphi : F^*N_- \to N_+$ as

$$F^*N_- = N_0 \to N_1 \to N_2 \to \ldots \to N_+, \tag{2.5}$$

where N_i's are $\mathcal{O}$-modules on X^{Δ} and N_+ is the (p-completed) direct limit of the N_i's;

(ii) morphisms $N_{i+1} \to N_i(\Delta_0)$ such that the composite maps

$$N_i \to N_{i+1} \to N_i(\Delta_0), \quad N_{i+1} \to N_i(\Delta_0) \to N_{i+1}(\Delta_0)$$

come from the natural morphism $\mathcal{O}_\Sigma \to \mathcal{O}_\Sigma(\Delta_0)$.

One can think of data (i)-(ii) either as an increasing filtration on N_+ or as a decreasing "Nygaard filtration"

$$F^*N_- = N_0 \leftarrow N_1(-\Delta_0) \leftarrow N_2(-2\Delta_0) \leftarrow \ldots \tag{2.6}$$

on F^*N_-.

2.4. The divisor $\Delta'_0 \subset \Sigma'$ and the Breuil-Kisin twists.

[8]Ekedahl? Ogus? Kato?

2.4.1. *The divisor $\Delta_0' \subset \Sigma'$.* The morphism $v_- : \Sigma' \to (\mathbb{A}^1/\mathbb{G}_m)_-$ can be shown[9] to be flat. Let $\Delta_0' \subset \Sigma'$ be the preimage of $\{0\}/\mathbb{G}_m \subset (\mathbb{A}^1/\mathbb{G}_m)_-$; this is an effective Cartier divisor on Σ'. It is easy to see that the isomorphism $\Sigma_+ \xrightarrow{\sim} \Sigma$ induces an isomorphism $\Delta_0' \cap \Sigma_+ \xrightarrow{\sim} \Delta_0$, where Δ_0 is in [D1, p.4]. This justifies the notation Δ_0'. By definition, $\Sigma_- = \Sigma' \setminus \Delta_0'$, so $\Delta_0' \cap \Sigma_- = \emptyset$.

The equality $\Sigma_+ \cap \Sigma_- = \emptyset$ means that Σ_+ is contained in the formal neighborhood of Δ_0'. Note that $\Sigma_+ \not\subset \Delta_0'$ because $\Sigma_+ \cap \Delta_0' = \Delta_0 \neq \Sigma_+$ (or because Δ_0' is a divisor in Σ', while Σ_+ is open in Σ').

2.4.2. *Breuil-Kisin twists.* On the g-stack associated to Σ' we have the line bundle $\mathcal{O}_{\Sigma'}(\Delta_0')$. We claim that $\mathcal{O}_{\Sigma'}(\Delta_0')$ is an $\mathcal{O}$-module on the c-stack Σ' itself. Indeed, $\mathcal{O}_{\Sigma'}(\Delta_0')$ is the pullback of the following $\mathcal{O}$-module on $(\mathbb{A}^1/\mathbb{G}_m)_-$. Recall that an S-point of $(\mathbb{A}^1/\mathbb{G}_m)_-$ is a line bundle $\mathscr{L}$ on S equipped with a morphism $\mathscr{L} \to \mathcal{O}_S$. The $\mathcal{O}$-module on $(\mathbb{A}^1/\mathbb{G}_m)_-$ assigns the *inverse* of $\mathscr{L}$ to such an S-point.

Let $\mathcal{O}_\Sigma\{-1\}$ be as in [D2]. The pullback of the line bundle $\mathcal{O}_\Sigma\{-1\}$ via the morphism $\Sigma' \to \Sigma_- = \Sigma$ from §2.1.6 is also an $\mathcal{O}_{\Sigma'}$-module. Tensoring it by $\mathcal{O}_{\Sigma'}(\Delta_0')$, we get an $\mathcal{O}_{\Sigma'}$-module (or equivalently an effective gauge on $\operatorname{Spf} \mathbb{Z}_p$), which we denote by $\mathcal{O}_{\Sigma'}\{-1\}$.

Restricting $\mathcal{O}_{\Sigma'}\{-1\}$ to Σ_+ (resp. Σ_-) one gets $(F^*\mathcal{O}_\Sigma\{-1\})(\Delta_0)$ (resp. $\mathcal{O}_\Sigma\{-1\}$). So the two restrictions are canonically isomorphic. Therefore $\mathcal{O}_{\Sigma'}\{-1\}$ canonically descends to an $\mathcal{O}_{\Sigma''}$-module (or equivalently, an effective F-gauge on $\operatorname{Spf} \mathbb{Z}_p$), which we denote by $\mathcal{O}_{\Sigma''}\{-1\}$.

For any non-negative $n \in \mathbb{Z}$ we set $\mathcal{O}_{\Sigma'}\{-n\} := (\mathcal{O}_{\Sigma'}\{-1\})^{\otimes n}$, $\mathcal{O}_{\Sigma''}\{-n\} := (\mathcal{O}_{\Sigma''}\{-1\})^{\otimes n}$.

2.5. **More about Δ_0'.**

2.5.1. *Why Δ_0' is important.* It seems that Δ_0' is related to the "Hodge to Hodge-Tate" spectral sequence, see §2.6.3-2.6.4 below.

2.5.2. *The structure on Δ_0'.* It is a c-stack; moreover, one has a left fibration

$$\Delta_0' \to (\{0\}/\mathbb{G}_m)_- \subset (\mathbb{A}^1/\mathbb{G}_m)_-.$$

Since $(\{0\}/\mathbb{G}_m)_-$ is just the classifying c-stack of the mutiplictaive monoid $\mathbb{A}^1$, we can rephrase this structure as follows: Δ_0' is a g-stack over $(\operatorname{Spf} \mathbb{Z}_p)/\mathbb{G}_m$ equipped with an action of $\mathbb{A}^1/\mathbb{G}_m$ (the latter is a monoidal stack over $(\operatorname{Spf} \mathbb{Z}_p)/\mathbb{G}_m$).

2.5.3. *An explicit description of Δ_0'.* As explained in §2.5.4 below, Δ_0' equipped with the above structure canonically identifies with the quotient $(\mathbb{A}^1)^{\mathrm{dR}}/\mathbb{G}_m$, where

$$(\mathbb{A}^1)^{\mathrm{dR}} := \operatorname{Cone}(\mathbb{G}_a^\sharp \to \mathbb{G}_a) = \operatorname{Cone}(W^{(F)} \to W/VW) = \operatorname{Cone}(W \xrightarrow{p} W).$$

Moreover, the isomorphism

$$\Delta_0' \xrightarrow{\sim} (\mathbb{A}^1)^{\mathrm{dR}}/\mathbb{G}_m, \tag{2.7}$$

identifies Δ_0 with $(\mathbb{G}^m)^{\mathrm{dR}}/\mathbb{G}_m = (\operatorname{Spf} \mathbb{Z}_p)/\mathbb{G}_m^\sharp$ in the usual way; as far as I understand, the isomorphism (2.7) is uniquely determined by this property combined with the structure from §2.5.2.

The restriction of $F' : \Sigma' \to \Sigma$ to Δ_0' is equal to the composite map $\Delta_0' \to \operatorname{Spf} \mathbb{Z}_p \xrightarrow{p} \Sigma$.

[9]See §2.8.2(iii).

2.5.4. *Why* $\Delta'_0 = (\mathbb{A}^1)^{\mathrm{dR}}/\mathbb{G}_m$. If $(M,\xi) \in \Delta'_0(S)$ then $\xi : M \to W_S$ factors through M'. By (3.8), any morphism $M' \to W_S$ factors through $V(W_S^{(1)}) \subset W_S$. But in the definition of admissibility (see §2.1.2) we required ξ' to be primitive. So the morphism $M' \to V(W_S^{(1)})$ is an isomorphism. This isomorphism identifies M' with $W_S^{(1)}$. Thus an object $(M,\xi) \in \Delta'_0(S)$ is the same as a pair consisting of a line bundle $\mathscr{L}$ on S and a W_S-module extension of $W_S^{(1)}$ by $\mathscr{L}^\sharp = \mathscr{L} \otimes W_S^{(F)}$. The stack of such pairs identifies with $(\mathbb{A}^1)^{\mathrm{dR}}/\mathbb{G}_m$ by Proposition 3.9.1.

2.6. The "Hodge to de Rham" and "Hodge to Hodge-Tate" spectral sequences.

2.6.1. *The stacks* $\Sigma'_{\overline{\mathrm{dR}}}$ *and* $X^{\overline{\mathrm{dR}}}$. Recall that $\Sigma'(S)$ is the category of pairs (M,ξ), where M is an admissible W_S-module and $\xi : M \to W_S$ is a primitive W_S-morphism. Now define a c-stack $\Sigma'_{\overline{\mathrm{dR}}}$ as follows: $\Sigma'_{\overline{\mathrm{dR}}}$ is the category of pairs $(M,\xi) \in \Sigma'(S)$ equipped with a splitting $M' \to M$, where M' is as in (2.1). By definition, we have a canonical morphism

$$\Sigma'_{\overline{\mathrm{dR}}} \to \Sigma' \tag{2.8}$$

It is easy to check that the morphism $\Sigma'_{\overline{\mathrm{dR}}} \to (\mathbb{A}^1/\mathbb{G}_m)_-$ is an isomorphism. So the preimage of Σ_- in $\Sigma'_{\overline{\mathrm{dR}}}$ equals $\operatorname{Spf} \mathbb{Z}_p$, and therefore we get a canonical morphism

$$\operatorname{Spf} \mathbb{Z}_p \to \Sigma_- = \Sigma. \tag{2.9}$$

It is easy to check that this is the point $p \in \Sigma(\mathbb{Z}_p)$. By the way, the map (2.9) is not a monomorphism, so (2.8) is not a monomorphism.

Recall that for any p-adic scheme X, the base change of X^{Δ} to $\operatorname{Spf} \mathbb{Z}_p$ corresponding to $p \in \Sigma(\mathbb{Z}_p)$ is denoted by X^{dR}; it is related to the de Rham cohomology of X.

Let $X^{\overline{\mathrm{dR}}}$ be the base change of $X^{\Delta'}$ to $\Sigma'_{\overline{\mathrm{dR}}} = (\mathbb{A}^1/\mathbb{G}_m)_-$. Presumably, it is related to the degeneration of the Rham cohomology to Hodge cohomology, also known as the "Hodge to de Rham" spectral sequence. An alternative name for $X^{\overline{\mathrm{dR}}}$ would be X^{HdR}.

Here is a description of the ring stack $(\mathbb{A}^1)^{\overline{\mathrm{dR}}}$ over $\Sigma'_{\overline{\mathrm{dR}}} = (\mathbb{A}^1/\mathbb{G}_m)_-$. Suppose we are given a morphism $f : S \to (\mathbb{A}^1/\mathbb{G}_m)_-$, i.e., a line bundle $\mathscr{L}$ on S and a map $\mathscr{L} \to \mathcal{O}_S$. Then the f-pullback of $(\mathbb{A}^1)^{\overline{\mathrm{dR}}}$ is the ring stack $\operatorname{Cone}(\mathscr{L}^\sharp \to (\mathbb{G}_a)_S)$ over S. Here $(\mathbb{G}_a)_S := \mathbb{G}_a \times S$ and $\mathscr{L}^\sharp$ is the PD-hull of the zero section in $\mathscr{L}$.

2.6.2. *Remark.* Recall that Σ_+ is the locus where the W_S-module M is invertible. Comparing this with the definition of $\Sigma'_{\overline{\mathrm{dR}}}$, we see that the preimage of Σ_+ in $\Sigma'_{\overline{\mathrm{dR}}}$ is empty.

Recall that $\Sigma_- = \Sigma' \setminus \Delta'_0$. Informally, $\Sigma'_{\overline{\mathrm{dR}}}$ is a "kind of complement" to Σ_+ (except that $\Sigma'_{\overline{\mathrm{dR}}}$ is not a *sub*stack of Σ').

2.6.3. *The stacks* Σ'_{Hdg} *and* X^{Hdg}. Let Σ'_{Hdg} be the preimage of Δ'_0 in $\Sigma'_{\overline{\mathrm{dR}}}$. In other words, Σ'_{Hdg} is the preimage of $(\{0\}/\mathbb{G}_m)_-$ with respect to the isomorphism $\Sigma'_{\overline{\mathrm{dR}}} \xrightarrow{\sim} (\mathbb{A}^1/\mathbb{G}_m)_-$. So we have a canonical isomorphism

$$\Sigma'_{\mathrm{Hdg}} \xrightarrow{\sim} (\{0\}/\mathbb{G}_m)_-. \tag{2.10}$$

Let X^{Hdg} be the base change of $X^{\Delta'}$ to Σ'_{Hdg}. Presumably, it is related to Hodge cohomology. This agrees with the fact that an $\mathcal{O}$-module on Σ'_{Hdg} is the same as a $\mathbb{Z}_+$-graded $\mathcal{O}$-module on $\operatorname{Spf} \mathbb{Z}_p$ (because $(\{0\}/\mathbb{G}_m)_-$ is just the classifying c-stack of the mutiplictaive monoid $\mathbb{A}^1$).

By definition, we have a canonical map $\Sigma'_{\rm Hdg}\to\Delta'_0$. On the other hand, we have canonical isomorphsims $\Sigma'_{\rm Hdg}\buildrel{\sim}\over{\longrightarrow}(\{0\}/\mathbb{G}_m)_-$ and $\Delta'_0\buildrel{\sim}\over{\longrightarrow}(\mathbb{A}^1)^{\rm dR}/\mathbb{G}_m$, see (2.10) and (2.7). In fact, the map

$$(\{0\}/\mathbb{G}_m)_-=\Sigma'_{\rm Hdg}\to\Delta'_0=(\mathbb{A}^1)^{\rm dR}/\mathbb{G}_m$$

comes from $0\in(\mathbb{A}^1)^{\rm dR}(\mathbb{Z}_p)$.

2.6.4. *The "Hodge to Hodge-Tate" spectral sequence.* Let $X^{\rm HHT}$ be the base change of $X^{\Delta'}$ to $\Delta'_0\simeq(\mathbb{A}^1)^{\rm dR}/\mathbb{G}_m$ (presumably, it is related to the "Hodge to Hodge-Tate" spectral sequence). An alternative name for $X^{\rm HHT}$ could be $X^{\overline{\rm HT}}$.

2.7. The stacks $(\operatorname{Spec}\mathbb{F}_p)^{\Delta'}$ and $(\mathbb{A}^1_{\mathbb{F}_p})^{\Delta'}$.

2.7.1. *Description of* $(\operatorname{Spec}\mathbb{F}_p)^{\Delta'}$ *and* $(\operatorname{Spec}\mathbb{F}_p)^{\Delta''}$. A simple argument (see §2.7.3 below) gives the following description of the c-stack $(\operatorname{Spec}\mathbb{F}_p)^{\Delta'}$: an S-point of $(\operatorname{Spec}\mathbb{F}_p)^{\Delta'}$ is a line bundle $\mathscr{L}$ on S equipped with morphisms $\mathcal{O}_S\xrightarrow{v_+}\mathscr{L}\xrightarrow{v_-}\mathcal{O}_S$ such that $v_-v_+=p$. So the g-stack corresponding to $(\operatorname{Spec}\mathbb{F}_p)^{\Delta'}$ is just $(\operatorname{Spf}A)/\mathbb{G}_m$, where A is the p-adic completion of $\mathbb{Z}_p[v_+,v_-]/(v_+v_--p)$ and $\mathbb{G}_m$ acts so that $\deg v_\pm=\pm1$.

The open substack $(\operatorname{Spec}\mathbb{F}_p)^{\Delta}_\pm\subset(\operatorname{Spec}\mathbb{F}_p)^{\Delta'}$ is the locus $v_\pm\neq0$; each of these substacks is isomorphic to $\operatorname{Spf}\mathbb{Z}_p$. Gluing together the two copies of $\operatorname{Spf}\mathbb{Z}_p$, one gets the c-stack $(\operatorname{Spec}\mathbb{F}_p)^{\Delta''}$. The closed substack $(\operatorname{Spec}\mathbb{F}_p)^{\overline{\rm HT}}\subset(\operatorname{Spec}\mathbb{F}_p)^{\Delta'}$ is the locus $v_-=0$ (which is contained in the locus $p=0$).

2.7.2. *$\mathcal{O}$-modules on* $(\operatorname{Spec}\mathbb{F}_p)^{\Delta'}$ *and* $(\operatorname{Spec}\mathbb{F}_p)^{\Delta''}$. It is clear that a gauge on $\operatorname{Spec}\mathbb{F}_p$ in the sense of §2.3.2 is a graded[10] A-module N, i.e., a gauge (or p-gauge) in the sense of [FJ, §1.1]. One can check that effectivity in the sense of §2.3.2 is equivalent to effectivity in the sense of [FJ, §1.1] (i.e., the map $v_-:N_r\to N_{r-1}$ being an isomorphism for all $r\le0$). This justifies the definition of effective gauge from §2.3.2.

In our case $X=\operatorname{Spec}\mathbb{F}_p$ the module $N_\pm$ from §2.3.3 is the (p-completed) direct limit of N_n with respect to the maps $v_\pm:N_n\to N_{n\pm1}$ (so in the effective case $N_-=N_0$). Thus in the case $X=\operatorname{Spec}\mathbb{F}_p$ an F-gauge in the sense of §2.3.2 is the same as a φ-gauge in the sense of [FJ, §1.4].

2.7.3. *Some details.* Let us justify the description of $(\operatorname{Spec}\mathbb{F}_p)^{\Delta'}$ from §2.7.1.

$(\operatorname{Spec}\mathbb{F}_p)^{\Delta'}$ is the equalizer of the two morphisms $\Sigma'=(\operatorname{Spf}\mathbb{Z}_p)^{\Delta'}\to(\mathbb{A}^1)^{\Delta'}$ corresponding to $0,p\in\mathbb{A}^1(\mathbb{Z}_p)$. So an S-point of $(\operatorname{Spec}\mathbb{F}_p)^{\Delta'}$ is an object $(M,\xi)\in\Sigma'(S)$ equipped with a section $\sigma:S\to M$ such that $\xi\circ\sigma:S\to W_S$ equals $p\in W(S)$. Interpret σ as a W_S-morphism $f:W_S\to M$. Then f maps $W_S^{(F)}=(\mathbb{G}_a^\sharp)_S$ to $\operatorname{Ker}(M\twoheadrightarrow M')=\mathscr{L}^\sharp$, so we get a morphism $(\mathbb{G}_a^\sharp)_S\to\mathscr{L}^\sharp$ or equivalently, a morphism $v_+:\mathcal{O}_S\to\mathscr{L}$. Since $\xi\circ\sigma=p$, we get $v_-v_+=p$. The map $W_S^{(1)}=W'_S\to M'$ induced by f is an isomorphism, so the extension $0\to\mathscr{L}^\sharp\to M\to M'\to0$ is the pushforward of the canonical extension $0\to W_S^{(F)}\to W_S\to W_S^{(1)}\to0$ via the morphism $W_S^{(F)}=(\mathbb{G}_a^\sharp)_S\xrightarrow{v_+}\mathscr{L}^\sharp$. In other words,

[10] The word "graded" is understood in the p-complete sense.

$M = \mathrm{Coker}(W_S^{(F)} \overset{(v_+,-1)}{\longrightarrow} \mathscr{L}^\sharp \oplus W_S)$, and $\xi : M \to W_S$ comes from the map $\mathscr{L}^\sharp \oplus W_S \overset{(v_-,p)}{\longrightarrow} W_S$. Recall that if $\mathscr{L} = \mathcal{O}_S$ then $\mathscr{L}^\sharp = W_S^{(F)}$.

The above argument can be rephrased in terms of §2.8 as follows: $(\mathrm{Spec}\,\mathbb{F}_p)^{\mathbb{A}'}$ canonically identifies with the fiber product of Σ'_+ and $(\mathrm{Spec}\,\mathbb{F}_p)^{\mathbb{A}} = \mathrm{Spf}\,\mathbb{Z}_p$ over $\Sigma_+ = \Sigma = (\mathrm{Spf}\,\mathbb{Z}_p)^{\mathbb{A}}$.

2.7.4. *$\mathcal{O}$-modules on $(\mathbb{A}^1_{\mathbb{F}_p})^{\mathbb{A}'}$.* We claim that:

(i) in the situation of §2.7.3 one can rewrite $\mathrm{Cone}(M \overset{\xi}{\longrightarrow} W_S)$ as $\mathrm{Cone}(H \overset{\alpha}{\longrightarrow} (\mathbb{G}_a)_S)$, where (H, α) is a flat scheme of quasi-ideals in $(\mathbb{G}_a)_S$;

(ii) H and its Cartier dual H^* can be described explicitly, and the description of H^* is very simple (see below);

(iii) the explicit description of H^* allows one to describe effective gauges on the scheme $\mathbb{A}^1_{\mathbb{F}_p}$ (i.e., $\mathcal{O}$-modules on $(\mathbb{A}^1_{\mathbb{F}_p})^{\mathbb{A}'}$) as graded C-modules N satisfying certain conditions[11], where C is a concrete noncommutative topological graded algebra over the ring A from §2.7.1; the graded A-algebra C is described below.

Here is an explanation of (i). The morphism $f : W_S \to M$ from §2.7.3 gives a faithfully flat homomorphism from $\mathrm{Cone}(W_S \overset{p}{\longrightarrow} W_S)$ to $\mathrm{Cone}(M \overset{\xi}{\longrightarrow} W_S)$. On the other hand, by Proposition 3.4.1, $\mathrm{Cone}(W_S \overset{p}{\longrightarrow} W_S) = \mathrm{Cone}((\mathbb{G}_a^\sharp)_S \to (\mathbb{G}_a)_S)$, so we get a faithfully flat homomorphism from $(\mathbb{G}_a)_S$ to $\mathrm{Cone}(W_S \overset{p}{\longrightarrow} W_S)$. The composite map $(\mathbb{G}_a)_S \twoheadrightarrow \mathrm{Cone}(M \overset{\xi}{\longrightarrow} W_S)$ realizes $\mathrm{Cone}(M \overset{\xi}{\longrightarrow} W_S)$ as $\mathrm{Cone}(H \to (\mathbb{G}_a)_S)$ for some explicit S-flat quasi-ideal H in $(\mathbb{G}_a)_S$ (see §2.7.5 below for more details).

The graded A-algebra C mentioned in (iii) is as follows. First, consider the A-algebra generated by elements x, D of degree 0 and L of degree -1 with defining relations

$$[D, x] = 1,\ v_+ \cdot L = D^p,\ [D, L] = 0,\ [L, x] = v_- \cdot D^{p-1}.$$

(here x is the coordinate on $\mathbb{A}^1$). Then complete this algebra with respect to the (2-sided) ideal generated by p and L.

Here is a description of the formal group H^* mentioned in (ii), assuming that $S = \mathrm{Spf}\,A$ and A is as in in §2.7.1: the coordinate ring of H^* is $A[[D, L]]/(D^p - v_+ L)$, and the coproduct Δ in this coordinate ring is given by

$$\Delta(D) = D \otimes 1 + 1 \otimes D, \quad \Delta(L) = L \otimes 1 + 1 \otimes L + v_- \cdot \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} L^i \otimes L^{p-i}.$$

2.7.5. *Computing H and H^*.* We assume that $S = \mathrm{Spf}\,A$, so $\mathscr{L} = \mathcal{O}_S$, $\mathscr{L}^\sharp = W_S^{(F)}$. In §2.7.4 we defined a quasi-ideal H in $(\mathbb{G}_a)_S$. One shows straightforwardly that H is the middle cohomology[12] of the complex

$$0 \to W_S^{(F)} \overset{(v_+,-V)}{\longrightarrow} W_S^{(F)} \oplus W_S^{(F^2)} \overset{(v_-,F)}{\longrightarrow} W_S^{(F)} \to 0, \qquad W_S^{(F^2)} := \mathrm{Ker}(F^2 : W_S \to W_S), \tag{2.11}$$

[11] The first condition is that N is p-complete and $N \otimes (\mathbb{Z}/p^r\mathbb{Z})$ is a discrete C-module for every $r \in \mathbb{N}$. The second condition is that the map $v_- : N_r \to N_{r-1}$ is an isomorphism for all $r \leq 0$.

[12] The other cohomology groups are zero.

and the canonical homomorphism $H\to(\mathbb{G}_a)_S$ comes from the maps

$$W_S^{(F)}\oplus W_S^{(F^2)}\twoheadrightarrow W_S^{(F^2)}\hookrightarrow W_S\twoheadrightarrow W_S/V(W_S^{(1)})=(\mathbb{G}_a)_S\,.$$

So H^* is the middle cohomology of the Cartier dual of (2.11). The latter is very explicit because the Cartier dual of $W_S^{(F^n)}$ is just the formal completion of $W_S/V^n(W_S^{(n)})=(W_n)_S$.

2.8. **The stack Σ'_+.**

2.8.1. *What will be constructed.* In §2.8.3 we will construct a c-stack Σ'_+, and in §2.8.4 we will construct a diagram

$$\Sigma_+\times(\mathbb{A}^1/\mathbb{G}_m)_-\longleftarrow\Sigma'_+\longrightarrow\tilde{\Sigma}'\longrightarrow\Sigma'. \tag{2.12}$$

This is a diagram of left fibrations over the c-stack $(\mathbb{A}^1/\mathbb{G}_m)_-$, which has the following properties:

(a) the composite maps $\Sigma'_+\to\Sigma'\xrightarrow{F'}\Sigma_-$ and $\Sigma'_+\to\Sigma_+\times(\mathbb{A}^1/\mathbb{G}_m)_-\to\Sigma_+\hookrightarrow\Sigma'\xrightarrow{F'}\Sigma_-$ are the same (here F' is the morphism (2.4)); so one can consider (2.12) as a diagram of stacks over Σ_-;

(b) the morphism $\tilde{\Sigma}'\longrightarrow\Sigma'$ is a gerbe banded by $\mathbb{G}_m^\sharp=(W^\times)^{(F)}$;

(c) the morphism $\Sigma'_+\longrightarrow\tilde{\Sigma}'$ is a torsor with respect to a flat group scheme over $\tilde{\Sigma}'$; the group scheme is fpqc-locally isomorphic to $\mathbb{G}_a^\sharp\times\tilde{\Sigma}'$;

(d) the morphism $\Sigma'_+\to\Sigma'$ is an isomorphism over the open substack $\Sigma_+\subset\Sigma'$;

(e) the morphism $\Sigma'_+\to\Sigma_+$ is faithfully flat;

(f) the morphism $\Sigma'_+\to(\mathbb{A}^1/\mathbb{G}_m)_-$ is faithfully flat;

(g) the morphism $\Sigma'_+\to\Sigma_+\times(\mathbb{A}^1/\mathbb{G}_m)_-$ is not flat but very easy to understand.

2.8.2. *Corollaries.* (i) By §2.8.1(b-c), the morphisms $\Sigma'_+\to\tilde{\Sigma}'\to\Sigma'$ are faithfully flat.

(ii) Each of the stacks from (2.12) is faithfully flat over Σ_-: this follows from (i) and §2.8.1(a,e).

(iii) Each of the stacks from (2.12) is faithfully flat over $(\mathbb{A}^1/\mathbb{G}_m)_-$: this follows from (i) and §2.8.1(f).

2.8.3. *Definition of Σ'_+.* Here are three equivalent definitions.

(i) $\Sigma'_+(S)$ is the category of triples consisting of an object $(M,\xi_M)\in\Sigma'(S)$, an object $(P,\xi_P)\in\Sigma_+(S)$, and a morphism $(P,\xi_P)\to(M,\xi_M)$.

(ii) An object of $\Sigma'_+(S)$ is an object $(P,\xi_P)\in\Sigma_+(S)$ with an additional piece of data. To define it, note that ξ_P gives rise to a line bundle $\mathscr{L}_P:=P/V(P')$ and a morphism $\bar{\xi}_P:\mathscr{L}_P\to\mathcal{O}_S$. The additional piece of data is a factorization of $\bar{\xi}_P$ as

$$\mathscr{L}_P\xrightarrow{v_+}\mathscr{L}\xrightarrow{v_-}\mathcal{O}_S \tag{2.13}$$

for some line bundle $\mathscr{L}$.

(iii) An object of $\Sigma'_+(S)$ consists of an object $(M,\xi_M)\in\Sigma'(S)$, an invertible W_S-module P, and a morphism $P\to M$ inducing an isomorphism $P'\xrightarrow{\sim}M'$.

2.8.4. *Construction of diagram* (2.12). All the arrows are forgetful maps. Here are more details.

Think of Σ'_+ in terms of §2.8.3(ii). Then the morphism $\Sigma'_+ \to \Sigma_+ \times (\mathbb{A}^1/\mathbb{G}_m)_-$ forgets the map $\mathscr{L}_P \to \mathscr{L}$ (but remembers $\mathscr{L}$ and the map $\mathscr{L} \to \mathcal{O}_S$).

Define $\tilde{\Sigma}'(S)$ as follows: an object of $\tilde{\Sigma}'(S)$ consists of an object $(M, \xi_M) \in \Sigma'(S)$, an invertible W_S-module P, and an isomorphism $P' \xrightarrow{\sim} M'$. The map $\tilde{\Sigma}' \longrightarrow \Sigma'$ is clear. The map $\Sigma'_+ \longrightarrow \tilde{\Sigma}'$ is also clear if one thinks of Σ'_+ in terms of §2.8.3(iii).

2.8.5. *The open substacks* $\Sigma_{++}, \Sigma_{+-} \subset \Sigma'_+$. Let $\Sigma_{++}, \Sigma_{+-} \subset \Sigma'_+$ be the preimages of the open substacks $\Sigma_+, \Sigma_- \subset \Sigma'$. Then Σ_{++} and Σ_{+-} are disjoint open substacks of Σ'_+.

In terms of (2.13), the substack Σ_{++} (resp. Σ_{+-}) is the locus where v_+ (resp. v_-) is invertible. So the morphism $\Sigma'_+ \to \Sigma_+$ (see the left arrow in (2.12)) induces isomorphisms

$$\Sigma_{++} \xrightarrow{\sim} \Sigma_+ = \Sigma, \quad \Sigma_{+-} \xrightarrow{\sim} \Sigma_+ = \Sigma.$$

By §2.8.1(a) and §2.1.6, the composite maps

$$\Sigma = \Sigma_{++} \xrightarrow{\pi_+} \Sigma_+ \xrightarrow{F'} \Sigma_- = \Sigma \tag{2.14}$$

$$\Sigma = \Sigma_{+-} \xrightarrow{\pi_-} \Sigma_- \xrightarrow{F'} \Sigma_- = \Sigma \tag{2.15}$$

equal $F : \Sigma \to \Sigma$. As already mentioned in §2.8.1(d), π_+ is an isomorphism, so it is not quite necessary to distinguish Σ_{++} from Σ_+. On the other hand, $F' : \Sigma_- \to \Sigma_-$ is the identity, so the map π_- from (2.15) is *not* an isomorphism; in fact, after identifying Σ_{+-} and Σ_- with Σ, it becomes the morphism $F : \Sigma \to \Sigma$.

2.8.6. *$\mathcal{O}$-modules on Σ'_+ and Σ'.* Using §2.8.3(ii), one can describe $\mathcal{O}$-modules on Σ'_+ as graded $\mathcal{O}$-modules on Σ with an additional structure (this is parallel to §2.7.2). On the other hand, one could try to use §2.8.1(b-c) to describe $\mathcal{O}$-modules on Σ' as $\mathcal{O}$-modules on Σ'_+ with an additional structure.

2.8.7. *More details on the Nygaard filtration.* Let us sketch the construction of data (i)-(ii) from §2.3.6.

For any p-adic scheme X, we have a canonical morphism

$$X_+^{\mathbb{A}} \times_{\Sigma_+} \Sigma'_+ \to X^{\mathbb{A}'} \tag{2.16}$$

of left fibrations over Σ'_+; this morphism is tautological if you think of Σ'_+ in terms of §2.8.3(i).

Now suppose we are given an effective gauge on X, i.e., an $\mathcal{O}$-module on $X^{\mathbb{A}'}$. Let N'_+ be its pullback via (2.16). Using the explicit description of the morphism $\Sigma'_+ \to \Sigma_+$ that comes from §2.8.3(ii), one interprets N'_+ as data (i)-(ii) from §2.3.6. More precisely, $N_i = q_* N'_+(i \cdot \mathfrak{D}_+)$ and $N_i(-i \cdot \Delta_0) = q_* N'_+(-i \cdot \mathfrak{D}_-)$, where N_i is as in §2.3.6, q is the map $X_+^{\mathbb{A}} \times_{\Sigma_+} \Sigma'_+ \to X_+^{\mathbb{A}} = X^{\mathbb{A}}$, and $\mathfrak{D}_\pm \subset \Sigma'_+$ is the effective divisor $v_\pm = 0$; here $v_\pm$ is as in (2.13).

3. W_S-MODULES

In most of this section we work with arbitrary schemes (rather than schemes over $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$).

3.1. The group scheme $\mathbb{G}_a^\sharp$.

3.1.1. *Definition of* $\mathbb{G}_a^\sharp$. Let $\mathbb{G}_a^\sharp := \operatorname{Spec} A$, where $A \subset \mathbb{Q}[x]$ is the subring generated by the elements

$$u_n := x^{p^n}/p^{\frac{p^n-1}{p-1}}, \quad n \geq 0.$$

It is easy to see that the ideal of relations between the u_n's is generated by the relations $u_n^p = pu_{n+1}$.

Since $p^{\frac{p^n-1}{p-1}} \in (p^n)! \cdot \mathbb{Z}_p^\times$, there is a unique homomorphism $\Delta : A \to A \otimes A$ such that $\Delta(x) = x \otimes 1 + 1 \otimes x$. The pair (A, Δ) is a Hopf algebra over $\mathbb{Z}$. So $\mathbb{G}_a^\sharp$ is a group scheme over $\mathbb{Z}$.

3.1.2. *Remarks.* (i) $\mathbb{G}_a^\sharp \otimes \mathbb{Z}_{(p)}$ is just the PD-hull of zero in $\mathbb{G}_a \otimes \mathbb{Z}_{(p)}$.

(ii) The embedding $\mathbb{Z}[x] \hookrightarrow A$ induces a morphism of group schemes

$$\mathbb{G}_a^\sharp \to \mathbb{G}_a. \tag{3.1}$$

(iii) The morphism (3.1) induces an isomorphism $\mathbb{G}_a^\sharp \otimes \mathbb{Q} \to \mathbb{G}_a \otimes \mathbb{Q}$.

Lemma 3.1.3. *Let $u_n \in A$ be as in §3.1.1. If $n > 0$ then $\Delta(u_n) - u_n \otimes 1 - 1 \otimes u_n$ is not divisible by any prime.*

Proof. As a $\mathbb{Z}$-module, A has a basis formed by elements of the form $\prod_i u_i^{a_i}$, where $0 \leq a_i < p$ and almost all numbers a_i are zero. The coefficient of $u_0 \otimes \prod_{i=0}^{n-1} u_i^{p-1}$ in $\Delta(u_n)$ equals 1. □

3.1.4. $\mathbb{G}_a^\sharp$ *as a quasi-ideal in* $\mathbb{G}_a$. There is a unique action of the ring scheme $\mathbb{G}_a$ on $\mathbb{G}_a^\sharp$ inducing the usual action of $\mathbb{G}_a \otimes \mathbb{Q}$ on $\mathbb{G}_a^\sharp \otimes \mathbb{Q} = \mathbb{G}_a \otimes \mathbb{Q}$. Thus $\mathbb{G}_a^\sharp$ is a $\mathbb{G}_a$-module. Moreover, the morphism (3.1) makes $\mathbb{G}_a^\sharp$ into a quasi-ideal in $\mathbb{G}_a$.

3.1.5. $W^{(F)}$ *as a quasi-ideal in* $\mathbb{G}_a$. Let W be the ring scheme over $\mathbb{Z}$ formed by p-typical Witt vectors. Let $W^{(F)} := \operatorname{Ker}(F : W \to W)$. The action of W on $W^{(F)}$ factors through $W/VW = \mathbb{G}_a$. The composite map

$$W^{(F)} \hookrightarrow W \twoheadrightarrow W/VW = \mathbb{G}_a$$

is a morphism of $\mathbb{G}_a$-modules, which makes $\mathbb{G}_a^\sharp$ into a quasi-ideal in $\mathbb{G}_a$.

Lemma 3.1.6. *$\mathbb{G}_a^\sharp$ and $W^{(F)}$ are isomorphic as quasi-ideals in $\mathbb{G}_a$. Such an isomorphism is unique.*

Proof. Uniqueness is clear. To construct the isomorphism, $\mathbb{G}_a^\sharp \xrightarrow{\sim} W^{(F)}$, we will use the approach to W developed by Joyal [J85] (an exposition of this approach can be found in [B16] and [BG, §1]).

Let B be the coordinate ring of W. Let $F^* : B \to B$ be the homomorphism corresponding to $F : W \to W$. The map $W \otimes \mathbb{F}_p \to W \otimes \mathbb{F}_p$ induced by F is the usual Frobenius, so there is a map $\delta : B \to B$ such that $F^*(b) = b^p + p\delta(b)$ for all $b \in B$ (of course, the map δ is neither additive nor multiplicative).

The pair (B, δ) is a δ-ring in the sense of [J85]. The main theorem of [J85] says that B is the *free δ-ring* on y_0, where $y_0 \in B$ corresponds to the canonical homomorphism $W \to W/VW = \mathbb{G}_a$. This means that as a ring, B is freely generated by the elements $y_n := \delta^n(y_0)$, $n \geq 0$. We have $F^*(y_n) = y_n^p + py_{n+1}$. The closed subscheme

$$\{0\} \subset W = \operatorname{Spec} B$$

identifies with $\operatorname{Spec} B/(y_0, y_1, \ldots)$. This implies that the closed subscheme $W^{(F)} \subset W$ identifies with $\operatorname{Spec}(B/I)$, where the ideal $I \subset B$ is generated by $y_n^p + py_{n+1}$, $n \geq 0$. On the other hand, B/I identifies with the ring A from §3.1.1 via the epimorphism $B \twoheadrightarrow A$ that takes y_n to $(-1)^n u_n$. $\square$

3.2. **The group schemes $W^\times$ and $(W^\times)^{(F)}$.**

Lemma 3.2.1. *Let R be a ring in which p is nilpotent. Then*

(i) a Witt vector $\alpha \in W(R)$ is invertible if and only if its 0th component is;

(ii) $\alpha \in W(R)$ is invertible if and only if $F(\alpha)$ is.

Proof. The ideal $\operatorname{Ker}(W(R) \to W(R/pR))$ is nilpotent. So we can assume that R is an $\mathbb{F}_p$-algebra.

To prove (i), it suffices to show that for any $x \in W(R)$ one has $1 + Vx \in W(R)^\times$. Indeed, since $VF = FV = p$ we have $(Vx)^n = p^{n-1}V(x^n) = V^n(F^{n-1}x)$, so Vx is topologically nilpotent.

Statement (ii) follows from (i) because $F : W \otimes \mathbb{F}_p \to W \otimes \mathbb{F}_p$ is the usual Frobenius. $\square$

Remark 3.2.2. For *any* ring R one can show by induction that an element of $W_n(R)$ is invertible if and only if all of its ghost components are.

3.2.3. *The group scheme $(W^\times)^{(F)}$.* Let

$$(W^\times)^{(F)} := \operatorname{Ker}(F : W^\times \to W^\times),$$

where $W^\times$ is the multiplicative group of the ring scheme W. Then $(W^\times)^{(F)}$ identifies with the multiplicative group of the non-unital ring scheme[13] $W^{(F)}$.

On the other hand, let $\mathbb{G}_m^\sharp$ be the multiplicative group of the non-unital ring scheme $\mathbb{G}_a^\sharp$ (the ring structure on $\mathbb{G}_a^\sharp$ comes from the quasi-ideal structure described in §3.1.4). Note that $\mathbb{G}_m^\sharp \otimes \mathbb{Z}_{(p)}$ is the PD-hull of 1 in $\mathbb{G}_m \otimes \mathbb{Z}_{(p)}$.

Lemma 3.1.6 provides an isomorphism $\mathbb{G}_a^\sharp \xrightarrow{\sim} W^{(F)}$. It is an isomorphism between quasi-ideals in $\mathbb{G}_a$ and therefore a ring homomorphism. So it induces an isomorphism of group schemes

$$(W^\times)^{(F)} \xrightarrow{\sim} \mathbb{G}_m^\sharp. \tag{3.2}$$

3.3. **Faithful flatness of $F : W \to W$ and $F : W^\times \to W^\times$.** Joyal's description of W (see the proof of Lemma 3.1.6) shows that the morphism $F : W \to W$ is faithfully flat.

Here is another proof. It suffices to check faithful flatness of $F : W_{n+1} \to W_n$ for each n. This reduces to proving faithful flatness of the two maps

$$F : W_{n+1} \otimes \mathbb{Z}[1/p] \to W_n \otimes \mathbb{Z}[1/p], \quad F : W_{n+1} \otimes \mathbb{F}_p \to W_n \otimes \mathbb{F}_p.$$

[13] By definition, the multiplicative group of a non-unital ring A is $\operatorname{Ker}(\tilde{A}^\times \to \mathbb{Z}^\times)$, where $\tilde{A} := \mathbb{Z} \oplus A$ is the ring obtained by formally adding the unit to A.

The first map can be treated using ghost components. The second map is just the composite of the projection $W_{n+1} \otimes \mathbb{F}_p \to W_n \otimes \mathbb{F}_p$ and the usual Frobenius.

The same argument proves faithful flatness of $F : W^\times \to W^\times$.

3.4. **The Picard stack** $\mathrm{Cone}(\mathbb{G}_a^\sharp \to \mathbb{G}_a)$ **in terms of** W.

Proposition 3.4.1. *One has a canonical isomorphism of Picard stacks over* $\mathbb{Z}$

$$\mathrm{Cone}(\mathbb{G}_a^\sharp \to \mathbb{G}_a) \xrightarrow{\sim} \mathrm{Cone}(W \xrightarrow{p} W)$$

Proof. By Lemma 3.1.6, $\mathrm{Cone}(\mathbb{G}_a^\sharp \to \mathbb{G}_a) = \mathrm{Cone}(W^{(F)} \to W/VW)$. We have

$$\mathrm{Cone}(W^{(F)} \to W/VW) = \mathrm{Cone}(VW \to W/W^{(F)}) = \mathrm{Cone}(VW \xrightarrow{F} W),$$

where the second equality follows from §3.3. But $\mathrm{Cone}(VW \xrightarrow{F} W) = \mathrm{Cone}(W \xrightarrow{FV} W)$ and $FV = p$. □

3.5. **Generalities on** W_S**-modules.**

Let $W_S := W \times S$; this is a ring scheme over S. By a W_S-module we mean a commutative affine group scheme over S equipped with an action of the ring scheme W_S.

3.5.1. Hom_W *and* $\underline{\mathrm{Hom}}_W$. If M and N are W_S-modules we write $\mathrm{Hom}_W(M, N)$ for the group of all W_S-morphisms $M \to N$.

Let $\mathcal{A}$ be the category of fpqc-sheaves of abelian groups on the category of S-schemes. Sometimes it is convenient to embed the category of W_S-modules into the bigger category of objects of $\mathcal{A}$ equipped with a W_S-action. Given W_S-modules M and N, one defines an object $\underline{\mathrm{Hom}}_W(M, N)$ in the bigger category; namely, $\underline{\mathrm{Hom}}_W(M, N)$ is the contravariant functor

$$S' \mapsto \mathrm{Hom}_W(M \times_S S', N \times_S S').$$

In some important cases this functor turns out to be representable; then $\underline{\mathrm{Hom}}_W(M, N)$ is a W_S-module.

3.5.2. *The functor* $M \mapsto M^{(n)}$. Let $n \in \mathbb{Z}$, $n \geq 0$. Let M be a W_S-module. Precomposing the action of W_S on M with $F^n : W_S \to W_S$, we get a new W_S-module structure on the group scheme underlying M; the new W_S-module will be denoted by $M^{(n)}$.

3.6. **Examples of** W_S**-modules.**

Define W_S-modules $(\mathbb{G}_a)_S$ and $(\mathbb{G}_a^\sharp)_S$ as follows:

$$(\mathbb{G}_a)_S := \mathbb{G}_a \times S, \quad (\mathbb{G}_a^\sharp)_S := \mathbb{G}_a^\sharp \times S,$$

where the ring scheme W acts on $\mathbb{G}_a$ via the canonical ring epimorphism $W \twoheadrightarrow W/VW = \mathbb{G}_a$. Applying §3.5.2 to the W_S-modules W_S and $(\mathbb{G}_a)_S$, we get W_S-modules $W_S^{(n)}$ and $(\mathbb{G}_a)_S^{(n)}$ for each integer $n \geq 0$.

We have a W_S-module homomorphism $F : W_S \to W_S^{(1)}$, which is a faithfully flat map by §3.3. Its kernel is denoted by $W_S^{(F)}$. By Lemma 3.1.6, we have a canonical isomorphism $W_S^{(F)} \xrightarrow{\sim} (\mathbb{G}_a^\sharp)_S$.

In addition to the exact sequence

$$0 \to W_S^{(F)} \to W_S \xrightarrow{F} W_S^{(1)} \to 0. \tag{3.3}$$

we have the exact sequence

$$0 \to W_S^{(1)} \xrightarrow{V} W_S \to (\mathbb{G}_a)_S \to 0. \tag{3.4}$$

3.7. **Duality between exact sequences** (3.3) **and** (3.4)**.** The goal of this subsection is to prove Proposition 3.7.3.

Lemma 3.7.1. *(i) If $n > 0$ then* $\mathrm{Hom}_W(W_S^{(F)}, (\mathbb{G}_a)_S^{(n)}) = 0$.

(ii) The W_S-module morphisms $W_S^{(F)} \hookrightarrow W_S \twoheadrightarrow (\mathbb{G}_a)_S$ induce an isomorphism

$$H^0(S, \mathcal{O}_S) = \mathrm{Hom}_W((\mathbb{G}_a)_S, (\mathbb{G}_a)_S) \xrightarrow{\sim} \mathrm{Hom}_W(W_S^{(F)}, (\mathbb{G}_a)_S).$$

Proof. By Lemma 3.1.6, we can replace $W_S^{(F)}$ by $(\mathbb{G}_a^\sharp)_S$. We can assume that S is affine, $S = \operatorname{Spec} R$. Let A and u_n be as in §3.1.1. Recall that $(\mathbb{G}_a^\sharp)_S = \operatorname{Spec}(A \otimes R)$.

Let $f \in \mathrm{Hom}_W(W_S^{(F)}, (\mathbb{G}_a)_S^{(n)})$. Since f commutes with the action of Teichmüller elements of the Witt ring, we see that the function $f \in A \otimes R$ is homogeneous of degree p^n. So $f = cu_n$ for some $c \in R$. If $n > 0$ then $c = 0$ by Lemma 3.1.3. □

Lemma 3.7.2. *(i) The multiplication pairing*

$$W_S \times W_S \to W_S \tag{3.5}$$

kills $W_S^{(F)} \times V(W_S^{(1)}) \subset W_S \times W_S$.

(ii) The kernel of the morphism $W_S \to \underline{\mathrm{Hom}}_W(V(W_S^{(1)}), W_S)$ induced by (3.5) *equals $W_S^{(F)}$.*

(iii) The kernel of the morphism $W_S \to \underline{\mathrm{Hom}}_W(W_S^{(F)}, W_S)$ induced by (3.5) *equals $V(W_S^{(1)})$.*

Proof. Statement (i) is clear. To prove (ii), use the section $V(1)$ of the S-scheme $V(W_S^{(1)})$. Statement (iii) follows from (i) and the equality

$$\mathrm{Ker}((\mathbb{G}_a)_S \to \underline{\mathrm{Hom}}_W(W_S^{(F)}, W_S^{(F)}) = 0;$$

this equality is clear because $W_S^{(F)} = (\mathbb{G}_a^\sharp)_S$. □

By Lemma 3.7.2(i), the pairing (3.5) and the exact sequences (3.3)-(3.4) yield W_S-bilinear pairings

$$W_S^{(1)} \times W_S^{(1)} \to W_S, \tag{3.6}$$

$$(\mathbb{G}_a)_S \times W_S^{(F)} \to W_S. \tag{3.7}$$

The pairing (3.6) is symmetric; in fact, this is just the multiplication $W_S^{(1)} \times W_S^{(1)} \to W_S^{(1)}$ followed by $V : W_S^{(1)} \hookrightarrow W_S$. The pairing (3.7) is the composite

$$(\mathbb{G}_a)_S \times W_S^{(F)} \to W_S^{(F)} \hookrightarrow W_S,$$

where the first map is the action of $(\mathbb{G}_a)_S$ on $W_S^{(F)}$.

Proposition 3.7.3. *The pairings* (3.6) *and* (3.7) *induce isomorphisms*

$$W_S^{(1)} \xrightarrow{\sim} \underline{\mathrm{Hom}}_W(W_S^{(1)}, W_S), \tag{3.8}$$

$$W_S^{(F)} \xrightarrow{\sim} \underline{\mathrm{Hom}}_W((\mathbb{G}_a)_S, W_S), \tag{3.9}$$

$$(\mathbb{G}_a)_S \xrightarrow{\sim} \underline{\mathrm{Hom}}_W(W_S^{(F)}, W_S). \tag{3.10}$$

Proof. The statements about $\underline{\mathrm{Hom}}_W(W_S^{(1)}, W_S)$ and $\underline{\mathrm{Hom}}_W((\mathbb{G}_a)_S, W_S)$ are easy because $W_S^{(1)}$ and $(\mathbb{G}_a)_S$ appear as *quotients* of W_S. More precisely, they are equivalent to Lemmas 3.7.2(iii) and 3.7.2(ii), respectively.

To prove the statement about $\underline{\mathrm{Hom}}_W(W_S^{(F)}, W_S)$, use Lemma 3.7.1 and the filtration

$$W_S \supset V(W_S^{(1)}) \supset V^2(W_S^{(2)}) \supset \dots, \tag{3.11}$$

whose successive quotients are the W_S-modules $(\mathbb{G}_a)_S^{(n)}$, $n \geq 0$. □

3.8. More computations of Hom_W.

Proposition 3.8.1. *(i) The action of $(\mathbb{G}_a)_S$ on $W_S^{(F)}$ induces an isomorphism*

$$(\mathbb{G}_a)_S \xrightarrow{\sim} \underline{\mathrm{Hom}}_W(W_S^{(F)}, W_S^{(F)}).$$

(ii) $\underline{\mathrm{Hom}}_W(W_S^{(F)}, W_S^{(1)}) = 0$.

(iii) The morphism $F : W_S \to W_S^{(1)}$ induces isomorphisms

$$\underline{\mathrm{Hom}}_W(W_S^{(1)}, W_S^{(1)}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_W(W_S, W_S^{(1)}) = W_S^{(1)}, \tag{3.12}$$

$$\underline{\mathrm{Hom}}_W(W_S^{(1)}, W_S^{(F)}) \xrightarrow{\sim} \mathrm{Ker}(W_S^{(F)} \to (\mathbb{G}_a)_S) = \mathrm{Ker}((\mathbb{G}_a^\sharp)_S \to (\mathbb{G}_a)_S). \tag{3.13}$$

Proof. Statement (i) follows from (3.10). Statement (ii) is deduced from Lemma 3.7.1(i) using the filtration (3.11). The isomorphism (3.12) follows from the fact that the W_S-module $W_S^{(1)}$ is killed by $W_S^{(F)} \subset W_S$. The isomorphism (3.13) follows from (3.10). □

3.8.2. *Remarks.* (i) Although the map $\mathbb{G}_a^\sharp \otimes \mathbb{Z}[1/p] \to \mathbb{G}_a \otimes \mathbb{Z}[1/p]$ is an isomorphism, it is easy to see from §3.1.1 that $\mathrm{Ker}(\mathbb{G}_a^\sharp \to \mathbb{G}_a) \neq 0$.

(ii) By Proposition 3.4.1, the r.h.s. of (3.13) can be rewritten as $\mathrm{Ker}(W_S \xrightarrow{p} W_S)$.

3.9. **Extensions of $W_S^{(1)}$ by $W_S^{(F)}$.** Given W_S-modules M and N, let $\underline{\mathrm{Ex}}_W(M, N)$ denote the Picard stack over S whose S'-points are extensions of $N \times_S S'$ by $M \times_S S'$. The following statement strengthens formula (3.13).

Proposition 3.9.1. *One has a canonical isomorphism*

$$\underline{\mathrm{Ex}}_W(W_S^{(1)}, W_S^{(F)}) \xrightarrow{\sim} \mathrm{Cone}(W_S^{(F)} \to (\mathbb{G}_a)_S) = \mathrm{Cone}((\mathbb{G}_a^\sharp)_S \to (\mathbb{G}_a)_S). \tag{3.14}$$

In particular, the stack $\underline{\mathrm{Ex}}_W(W_S^{(1)}, W_S^{(F)})$ is algebraic.

Proof. Let S' be an S-scheme. Pushing forward the canonical extension

$$0 \to W_{S'}^{(F)} \to W_{S'} \xrightarrow{F} W_{S'}^{(1)} \to 0 \tag{3.15}$$

via a morphism $W_{S'}^{(F)} \to W_{S'}^{(F)}$, one gets a new extension of $W_{S'}^{(1)}$ by $W_{S'}^{(F)}$. Thus one gets an isomorphism $\underline{\mathrm{Ex}}_W(W_S^{(1)}, W_S^{(F)}) \xrightarrow{\sim} \mathrm{Cone}(W_S^{(F)} \to \underline{\mathrm{Hom}}_W(W_S^{(F)}, W_S^{(F)}))$. By Proposition 3.8.1(i), the canonical map $(\mathbb{G}_a)_S \to \underline{\mathrm{Hom}}_W(W_S^{(F)}, W_S^{(F)})$ is an isomorphism. □

Combining Propositions 3.9.1 and 3.4.1, we get a canonical isomorphism

$$\underline{\mathrm{Ex}}_W(W_S^{(1)}, W_S^{(F)}) \xrightarrow{\sim} \mathrm{Cone}(W_S \xrightarrow{p} W_S). \tag{3.16}$$

We will also give its direct construction (see §3.9.4 below). It is based on the following

Lemma 3.9.2. *For every scheme S, every exact sequence $0 \to W_S^{(F)} \xrightarrow{i} M \xrightarrow{\pi} W_S^{(1)} \to 0$ Zariski-locally on S admits a rigidification of the following type: a W_S-morphism*

$$r : M \to W_S$$

such that $r|_{W_S^{(F)}} = \mathrm{id}$. All such rigidifications form a torsor over $\underline{\mathrm{Hom}}_W(W_S^{(1)}, W_S) \simeq W_S^{(1)}$.

Proof. The lemma is a consequence of the following fact, which can be easily deduced from Proposition 3.8.1(i): every extension of $W_S^{(1)}$ by W_S splits Zariski-locally on S.

Here is a slightly more direct proof. We already know that $\underline{\mathrm{Hom}}_W(W_S^{(1)}, W_S) \simeq W_S^{(1)}$, see (3.8). Since every $W_S^{(1)}$-torsor is Zariski-locally trivial, it suffices to prove that r exists fpqc-locally. So we can assume that there exists a W_S-morphism $\sigma : W_S \to M$ such that $\pi \circ \sigma = F$. A choice of σ realizes our exact sequence as a pushforward of the canonical exact sequence

$$0 \to W_S^{(F)} \to W_S \xrightarrow{F} W_S^{(1)} \to 0 \tag{3.17}$$

via some $h : W_S^{(F)} \to W_S^{(F)}$. Constructing r is equivalent to extending h to a morphism $W_S \to W_S$. This is possible by Proposition 3.8.1(i). □

Corollary 3.9.3. *Every extension of $W_S^{(1)}$ by $W_S^{(F)}$ can be Zariski-locally on S obtained as a pullback of* (3.17) *via some $\zeta \in \mathrm{End}_W(W_S^{(F)})$.* □

3.9.4. *Direct construction of* (3.16). By Lemma 3.9.2 and Corollary 3.9.3,

$$\underline{\mathrm{Ex}}_W(W_S^{(1)}, W_S^{(F)}) = \mathrm{Cone}(\underline{\mathrm{Hom}}_W(W_S^{(1)}, W_S) \xrightarrow{g} \underline{\mathrm{Hom}}_W(W_S^{(1)}, W_S^{(1)})),$$

where the map g comes from $F : W_S \to W_S^{(1)}$. Using (3.8), (3.12), and the formula $FV = p$, one identifies g with the map $W_S^{(1)} \xrightarrow{p} W_S^{(1)}$.

3.10. **Admissible W_S-modules.**

Definition 3.10.1. A W_S-module is said to be *invertible* if it is locally isomorphic to W_S.

Remark 3.10.2. In the above definition the word "locally" can be understood in either Zariski or fpqc sense (the W^*-torsors are the same).

3.10.3. *Notation.* Let $\mathscr{L}$ be a line bundle on S. Then $\mathscr{L}$ is a module over the ring scheme $(\mathbb{G}_a)_S$, and we set

$$\mathscr{L}^\sharp := \mathscr{L} \otimes_{(\mathbb{G}_a)_S} W_S^{(F)} = \mathscr{L} \otimes_{(\mathbb{G}_a)_S} (\mathbb{G}_a^\sharp)_S.$$

If S is a $\mathbb{Z}_{(p)}$-scheme then $\mathscr{L}^\sharp$ is the PD-hull of the group scheme $\mathscr{L}$ along its zero section.

Lemma 3.10.4. *(i) The functor $M \mapsto M^{(1)}$ from §3.5.2 induces an equivalence between the category of invertible W_S-modules and the category of W_S-modules locally isomorphic to $W_S^{(1)}$.*

(ii) The functor $\mathscr{L} \mapsto \mathscr{L}^\sharp$ induces an equivalence between the category of line bundles $\mathscr{L}$ on S and the category of W_S-modules locally isomorphic to $W_S^{(F)}$. The inverse functor is $M \mapsto \underline{\mathrm{Hom}}_W(W_S^{(F)}, M)$.

Proof. Statement (i) follows from (3.12). Statement (ii) follows from Proposition 3.8.1(i). □

3.10.5. *Remark.* Similarly to Remark 3.10.2, in the above lemma the word "locally" can be understood in either Zariski or fpqc sense.

Definition 3.10.6. A W_S-module M is said to be *admissible* if there exists an exact sequence of W_S-modules

$$0 \to M_0 \to M \to M' \to 0, \tag{3.18}$$

where M_0 is locally isomorphic to $W_S^{(F)}$ and M' is locally isomorphic to $W_S^{(1)}$.

Lemma 3.10.7. *The exact sequence* (3.18) *is essentially unique if it exists. Moreover, it is functorial in M.*

Proof. Follows from Proposition 3.8.1(ii) □

3.10.8. *Remarks.* (i) By the previous lemma, admissibility of a W_S-module is a local property.

(ii) By Lemma 3.10.4(ii), the exact sequence (3.18) can be rewritten as

$$0 \to \mathscr{L}^\sharp \to M \to M' \to 0, \tag{3.19}$$

where $\mathscr{L}$ is a line bundle on S. Here $\mathscr{L} = \mathscr{L}_M := \underline{\operatorname{Hom}}_W(W_S^{(F)}, M_0) = \underline{\operatorname{Hom}}_W(W_S^{(F)}, M)$.

(iii) The exact sequence (3.3) shows that any invertible W_S-module M is admissible. In this case

$$M' = M \otimes W_S^{(1)} \tag{3.20}$$

and $\mathscr{L} = M \otimes (\mathbb{G}_a)_S$. Formula (3.20) can be rewritten in the spirit of Lemma 3.10.4(i) as $M' = N^{(1)}$, where $N = M \otimes_{W_S,F} W_S$.

(iv) If S is a $\mathbb{Z}[p^{-1}]$-scheme then all admissible W_S-modules are invertible because for every open $S' \subset S$ one has $\underline{\operatorname{Ex}}_W(W_{S'}^{(1)}, W_{S'}^{(F)}) = 0$ by (3.14) or (3.16).

(v) Let $S = \operatorname{Spec} k$, where k is a field of characteristic p. Then the admissible W_S-module $W_S^{(F)} \oplus W_S^{(1)}$ is not invertible because $W_S^{(F)}$ is not reduced as a scheme.

Lemma 3.10.9. *Let M be an admissible W_S-module. Then* (3.18) *induces an exact sequence*

$$0 \to \underline{\operatorname{Hom}}_W(M', W_S) \to \underline{\operatorname{Hom}}_W(M, W_S) \to \underline{\operatorname{Hom}}_W(M_0, W_S) \to 0.$$

Proof. This is a reformulation of Lemma 3.9.2. □

Lemma 3.10.10. *Let M be an admissible W_S-module and $\xi : M \to W_S$ a W_S-morphism. Then (M, ξ) is a quasi-ideal in W_S.*

Proof. We have to show that for every S-scheme S' one has

$$\xi(\alpha)\beta - \xi(\beta)\alpha = 0 \quad \text{for all } \alpha, \beta \in M(S'). \tag{3.21}$$

We can assume that M is an extension of $W_S^{(1)}$ by $W_S^{(F)}$.

By (3.10), the identity (3.21) holds if α and β are sections of $W_S^{(F)}$. So considering the l.h.s. of (3.21) when α is a section of $W_S^{(F)}$ and β is arbitrary, we get a W_S-bilinear pairing $W_S^{(F)} \times W_S^{(1)} \to W_S$. But all such pairings are zero by (3.12) and Proposition 3.8.1(ii).

Thus the l.h.s. of (3.21) defines a W_S-bilinear pairing $B : W_S^{(1)} \times W_S^{(1)} \to W_S$. It is strongly skew-symmetric (i.e.,the restriction of B to the diagonal is zero). So using the epimorphism $F : W_S \twoheadrightarrow W_S^{(1)}$, we see that $B = 0$. □

3.11. The c-stack Adm and the g-stacks $\mathrm{Adm}_{\mathscr{L}}$.

3.11.1. *Definition of* Adm. For a scheme S, let $\mathrm{Adm}(S)$ be the category whose objects are admissible W_S-modules and whose morphisms are those W_S-linear maps $M_1 \to M_2$ that induce an *iso*morphism $M_1' \xrightarrow{\sim} M_2'$. We have a functor

$$\mathrm{Adm}(S) \to \{\text{line bundles on } S\}, \quad M \mapsto \mathscr{L}_M,$$

where $\mathscr{L}_M$ is as in Remark 3.10.8(ii). This functor is a left fibration in Joyal's sense (see [Lu1, §2.1]); equivalently, it makes $\mathrm{Adm}(S)$ into a category cofibered in groupoids over the category of lines bundles on S.

By Lemma 3.10.7 or Remark 3.10.8(i), the assignment $S \mapsto \mathrm{Adm}(S)$ is a c-stack for the fpqc topology (not merely a c-prestack).

3.11.2. *Definition of* $\mathrm{Adm}_{\mathscr{L}}$. Now fix a scheme S and a line bundle $\mathscr{L}$ on S. Define a g-stack $\mathrm{Adm}_{\mathscr{L}}$ over S as follows: for an S-scheme S', let $\mathrm{Adm}_{\mathscr{L}}(S')$ be the groupoid of objects $M \in \mathrm{Adm}(S')$ equipped with an isomorphism $\mathscr{L}_M \xrightarrow{\sim} \mathscr{L} \times_S S'$.

The *g*-stack $\mathrm{Adm}_{\mathscr{L}}$ depends functorially on $\mathscr{L}$, and the assignment $\mathscr{L} \mapsto \mathrm{Adm}_{\mathscr{L}}$ commutes with base change $\tilde{S} \to S$. We can think of the c-stack Adm as such collection of g-stacks $\mathrm{Adm}_{\mathscr{L}}$.

Proposition 3.11.3. *(i) The c-stack* Adm *is algebraic.*

(ii) The g-stacks $\mathrm{Adm}_{\mathscr{L}}$ *are algebraic.*

Proof. Statements (i) and (ii) are equivalent. Statement (ii) follows from algebraicity of the stack $\underline{\mathrm{Ex}}_W(W_S^{(1)}, W_S^{(F)})$, see Proposition 3.9.1 or formula (3.16). □

3.12. The diagram $\mathrm{Adm}_+ \to \widetilde{\mathrm{Adm}} \to \mathrm{Adm}$.

3.12.1. *The stack* Inv. Let $\mathrm{Inv}(S)$ be the groupoid of invertible W_S-modules. The g-stack Inv is algebraic: this is just the classifying stack of $W^\times$.

Lemma 3.12.2. *(i) If S is p-nilpotent then the functor* $\mathrm{Inv}(S) \to \mathrm{Adm}(S)$ *is fully faithful.*

(ii) For every $n \in \mathbb{N}$, the morphism $\mathrm{Inv} \otimes \mathbb{Z}/p^n\mathbb{Z} \to \mathrm{Adm} \otimes \mathbb{Z}/p^n\mathbb{Z}$ *is an affine open immersion.*

Proof. If R is a ring in which p is nilpotent and $w \in W(R)$ is such that $F(w) \in W(R)^\times$ then $w \in W(R)^\times$ by Lemma 3.2.1(ii). Statement (i) follows.

To deduce (ii) from (i), we have to show that for any p-nilpotent scheme S and any $M \in \mathrm{Adm}(S)$, the corresponding fiber product $S \times_{\mathrm{Adm}} \mathrm{Inv}$ is an open subscheme of S which is affine over S. By Corollary 3.9.3, we can assume that M is the extension of $W_S^{(1)}$ by $W_S^{(F)}$ obtained as a pullback of the canonical exact sequence

$$0 \to W_S^{(F)} \to W_S \xrightarrow{F} W_S^{(1)} \to 0 \tag{3.22}$$

via some $\zeta \in \mathrm{End}_W(W_S^{(1)}) = W^{(1)}(S)$. We will show that in this situation

$$S \times_{\mathrm{Adm}} \mathrm{Inv} = V(\zeta_0), \tag{3.23}$$

where ζ_0 is the 0th component of the Witt vector $\zeta \in W^{(1)}(S)$ and $V(\zeta_0) \subset S$ is the open subscheme $\zeta_0 \neq 0$. It suffices to prove (3.23) if ζ_0 is either invertible or zero.

Suppose that ζ_0 is invertible. Then ζ is invertible by Lemma 3.2.1(i). So the ζ-pullback of (3.22) is isomorphic to (3.22). Therefore M is invertible.

It remains to show that if $\zeta_0 = 0$ and S is the spectrum of a perfect field of characteristic p then M is not invertible. Indeed, perfectness implies that ζ is divisible by p, so the ζ-pullback of (3.22) splits by (3.16). But $W_S^{(F)} \oplus W_S^{(1)}$ is not invertible, see Remark 3.10.8(v). $\square$

3.12.3. *The stacks* Adm_+ *and* $\widetilde{\mathrm{Adm}}$. Let $\mathrm{Adm}_+(S)$ be the category whose objects are triples (P, M, f), where $P \in \mathrm{Inv}(S)$, $M \in \mathrm{Adm}(S)$, and f is an $\mathrm{Adm}(S)$-morphism $P \to M$; by a morphism $(P_1, M_1, f_1) \to (P_2, M_2, f_2)$ we mean a pair (g, h), where $g : P_1 \to P_2$ is an isomorphism, $h : M_1 \to M_2$ is an $\mathrm{Adm}(S)$-morphism, and $hf_1 = f_2 g$.

Let $\widetilde{\mathrm{Adm}}(S)$ be the category whose objects are triples (P, M, ϕ), where $P \in \mathrm{Inv}(S)$, $M \in \mathrm{Adm}(S)$, and $\phi : P' \xrightarrow{\sim} M'$ is an isomorphism; by a morphism $(P_1, M_1, \phi_1) \to (P_2, M_2, \phi_2)$ we mean a pair (g, h), where $g : P_1 \to P_2$ is an isomorphism, $h : M_1 \to M_2$ is an $\mathrm{Adm}(S)$-morphism, and $h'\phi_1 = \phi_2 g'$.

The stacks Adm_+ and $\widetilde{\mathrm{Adm}}$ are algebraic because Adm and Inv are (see Proposition 3.11.3 and §3.12.1). We have the forgetful morphisms

$$\mathrm{Adm}_+ \to \widetilde{\mathrm{Adm}} \to \mathrm{Adm}\,. \tag{3.24}$$

They are left fibrations.

3.12.4. *The morphism* $\mathrm{Adm}_+ \to \mathrm{Inv}$. Let $(P, M, f) \in \mathrm{Adm}_+(S)$. Then we have line bundles $\mathscr{L} = \mathscr{L}_M := \underline{\mathrm{Hom}}_W(W^{(F)}, M)$ and $\mathscr{L}_P := P/V(P') \simeq \underline{\mathrm{Hom}}_W(W^{(F)}, P)$. Moreover, f induces a morphism $\varphi : \mathscr{L}_P \to \mathscr{L}$. Note that the exact sequence $0 \to \mathscr{L}^\sharp \to M \to M' \to 0$ is just the pushforward of the exact sequence $0 \to \mathscr{L}_P^\sharp \to P \to P' \to 0$ with respect to $\varphi^\sharp : \mathscr{L}_P^\sharp \to \mathscr{L}^\sharp$. So we can think of $\mathrm{Adm}_+(S)$ as follows: an object of $\mathrm{Adm}_+(S)$ is a triple $(P, \mathscr{L}, \varphi : \mathscr{L}_P \to \mathscr{L})$, where $P \in \mathrm{Inv}(S)$ and $\mathscr{L}$ is a line bundle on S; a morphism $(P_1, \mathscr{L}_1, \varphi_1) \to (P_2, \mathscr{L}_2, \varphi_2)$ is a pair (g, h), where $g : P_1 \to P_2$ is an isomorphism, $h : \mathscr{L}_1 \to \mathscr{L}_2$ is a morphism, and the corresponding diagram

$$\begin{array}{ccc} \mathscr{L}_{P_1} & \xrightarrow{\varphi_1} & \mathscr{L}_1 \\ \downarrow & & \downarrow{\scriptstyle h} \\ \mathscr{L}_{P_2} & \xrightarrow{\varphi_2} & \mathscr{L}_2 \end{array}$$

commutes.

Thus the morphism $\mathrm{Adm}_+ \to \mathrm{Inv}$ is very simple. Indeed, one can think of $\varphi : \mathscr{L}_P \to \mathscr{L}$ as a section of the line bundle $\mathscr{N} = \mathscr{L}_P^{\otimes -1} \otimes \mathscr{L}$, so Adm_+ identifies with the product of Inv and the stack whose S-points are line bundles on S equipped with a section.

Proposition 3.12.5. *(i) The morphisms* (3.24) *are faithfully flat.*

(ii) The morphism $\widetilde{\mathrm{Adm}} \to \mathrm{Adm}$ *is a* $\mathbb{G}_m^\sharp$*-gerbe.*

(iii) The morphism $\mathrm{Adm}_+ \to \widetilde{\mathrm{Adm}}$ *is an* H*-torsor, where* H *is the following group scheme over* $\widetilde{\mathrm{Adm}}$*: an* S*-point of* H *is a quadruple* (P, M, ϕ, σ)*, where* $(P, M, \phi) \in \widetilde{\mathrm{Adm}}(S)$ *and* σ *is a section of the group scheme* $(\mathscr{L}_P^{\otimes -1} \otimes \mathscr{L}_M)^\sharp$.

Proof. Statement (ii) follows from the isomorphism (3.2). Statement (iii) is clear. Statement (i) follows from (ii) and (iii). $\square$

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