Quasi-ideals.

Def. A quasi-ideal in a ring C is $(I, d: I \to C)$, where I is a C-module, d is C-linear, $x\,dy = y\,dx$ for $x, y \in I$.

Example. A quasi-ideal with $\operatorname{Ker} d = 0$ is the same as an ideal ($x\,dy = y\,dx$ automatically).

Example. $I = C$, d = multiplication by $c_0 \in C$ ($x\,dy = y\,dx$ automatically). ((Exercise))

Any quasi-ideal I is a (non-unital) ring w.r.t. $x \circ y := x\,dy = y\,dx$.

If $I \subset C$ is a usual ideal we have the quotient ring C/I. Similar construction for a quasi-ideal $I \xrightarrow{d} C$: the group I acts on C by translations via d (~~now the action is~~ if $\operatorname{Ker} d \neq 0$ the action is not free), $\operatorname{Cone}(I \to C) :=$ quotient groupoid.

Objects: elements of C. $\operatorname{Isom}(c_1, c_2) := \{x \in I \mid c_1 - c_2 = dx\}$.

Composition: adding x's.

The operations $+$ and $\times$ on C induce "operations" on the groupoid, so $\operatorname{Cone}(I \to C)$ is a "ring groupoid". Let us just believe that there is such a notion and use common sense to work with it.

$\operatorname{Ker} d = 0 \Rightarrow$ get the usual quotient.

Variant. Suppose we have $I \xrightarrow{d} C$, but C and I are not sets but schemes over some S, (C is a ring scheme, I is a quasi-ideal scheme). If I is flat over S we can form the quotient stack $\operatorname{Cone}(I \to C) := C/\{\text{action of } I\}$. This is a ring stack (functor $\{S\text{-schemes}\} \to \{\text{ring groupoids}\}$, which is a stack if you forget the ring structure).

Main Theorem.

The goal is to construct an isomorphism between two concrete ring stacks.

$X/\mathbb{F}_p \longmapsto$ "crystallization" $X^{\square}$ (p-adic stack). $(\mathbb{A}^1_{\mathbb{F}_p})^{\square} = ?$

$(\mathbb{A}^1_{\mathbb{F}_p})^{\square}$ is a ring stack. Two explanations:

① $(X \times Y)^{\square} = X^{\square} \times Y^{\square}$ under mild assumptions (In particular, if $X = Y = \mathbb{A}^1_{\mathbb{F}_p}$). So the ring structure on $\mathbb{A}^1_{\mathbb{F}_p}$ induces a ring structure on $(\mathbb{A}^1_{\mathbb{F}_p})^{\square}$.

② $(\mathbb{A}^1_{\mathbb{F}_p})^{\square} := \mathbb{G}_a / \mathbb{G}_a^{\#}$ (stacky quotient) $= \operatorname{Cone}(\mathbb{G}_a^{\#} \to \mathbb{G}_a)$ (as a group stack), and $\mathbb{G}_a^\# \to \mathbb{G}_a$ is a quasi-ideal. We will see this; anyway, it is believable (pretend that $\mathbb{G}_a^\#$ = <u>formal</u> neighborhood of 0 in $\mathbb{G}_a$).

<u>Thm.</u> $(\mathbb{A}^1_{\mathbb{F}_p})^{\mathbb{I}} \cong \mathrm{Cone}\,(W \xrightarrow{p} W)$, where $W :=$ ring scheme of p-typical Witt vectors ~~(as a p-adic scheme)~~ (Isomorphism of ring stacks over $\mathbb{Z}_{(p)}$. In particular, true after p-adic completion, which is what we need)

<u>Remark.</u> The groupoid of B-points of $\mathrm{Cone}\,(W \xrightarrow{p} W)$ is ~~Cone~~
$\mathrm{Cone}\,(W(B) \xrightarrow{p} W(B))$ (Because $H^1(\mathrm{Spec}\, B, W) = 0$).

<u>Remark.</u> (reality check) $\mathbb{A}^1_{\mathbb{F}_p}$ is not merely a ring scheme but a scheme of $\mathbb{F}_p$-algebras (i.e., $1+\dots+1=0$). So $(\mathbb{A}^1_{\mathbb{F}_p})^{\mathbb{I}}$ is an $\mathbb{F}_p$-algebra stack (even though it lives in mixed characteristic). Good news: $\mathrm{Cone}\,(W \xrightarrow{p} W)$ has the same property. ($\mathrm{Cone}\,(\mathbb{Z} \xrightarrow{p} \mathbb{Z})$ maps into it). This property is funny and important. is a ring scheme over $\mathbb{Z}$

Before formulating the Key Lemma, recall that W is equipped with $F: W \to W$, $V: W \to W$. Properties:
Witt vector Frobenius, Verschiebung

$W \otimes \mathbb{F}_p \xrightarrow{F} W \otimes \mathbb{F}_p$ is the usual Frobenius,

F is a ring homomorphism, V is additive and $(Vx)\cdot y = V(x \cdot Fy)$ (identity difficult to remember)
(so V is a module homomorphism, in some sense).

$FV = p$ (But $VF \neq FV$ unless we are over $\mathbb{F}_p$).

$V: W \to W$ is a closed embedding, VW is an ideal, $W/VW = \mathbb{G}_a$ (By difficult identity)
$0 \to W \xrightarrow{V} W \to \mathbb{G}_a \to 0$. ~~faithfully and~~ flat (in particular, F is surjective

$F: W \to W$ is ~~surjective~~ (as a morphism of <u>schemes</u>, not functors).
True for $F: W_n \to W_{n-1}$ because true after $\otimes \mathbb{F}_p$ and after $\otimes \mathbb{Z}[\frac{1}{p}]$ ~~(Clear mod p, the p-adic case follows)~~ ($W_{(n)}, W_{(n-1)}$ are $\mathbb{Z}$-flat).

So $W^{(F)} := \mathrm{Ker}\,(W \xrightarrow{F} W)$ is a flat group scheme.

$W^{(F)} \subset W$ is an ideal (so W acts).

<u>Easy Lemma.</u> The action of W on $W^{(F)}$ factors through $W/VW = \mathbb{G}_a$.

<u>Proof.</u> $(Vx)\cdot y = 0$ if $Fy = 0$. ∎
$\quad\; \parallel$
$V(x \cdot Fy)$ (By the "difficult identity")

<u>Cor.</u> $W^{(F)} \xrightarrow{\subset W \twoheadrightarrow} \mathbb{G}_a$ is a quasi-ideal.

$\mathbb{G}_a^\# \to \mathbb{G}_a$ is (also) a quasi-ideal.

<u>Key Lemma.</u> Over $\mathbb{Z}_{(p)}$ these quasi-ideals are canonically isomorphic:

$$
\begin{tikzcd}
\mathbb{G}_a^\# \arrow[r, "\sim"] \arrow[dr] & W^{(F)} \arrow[d] \\
& \mathbb{G}_a = W/VW
\end{tikzcd}
$$

<u>Proof of Thm (assuming the lemma).</u> $(\mathbb{A}^1_{\mathbb{F}_p})^{\mathbb{I}} = \mathrm{Cone}(\mathbb{G}_a^\# \to \mathbb{G}_a) =$

$= \mathrm{Cone}(W^{(F)} \to W/VW) = \mathrm{Cone}(VW \to W/W^{(F)}) =$

$= \mathrm{Cone}(VW \xrightarrow{F} W) = \mathrm{Cone}(W \xrightarrow{FV} W) = \mathrm{Cone}(W \xrightarrow{p} W)$. ∎

<u>Exercise.</u> ~~Construct~~ check that $\mathbb{G}_a^\# \otimes \mathbb{F}_p$ is isomorphic to $W^{(F)} \otimes \mathbb{F}_p$ ~~by hands~~ as schemes.
(Possible because modulo p we know how F acts on $W = \mathbb{A}^\infty$)

~~Remark. The Key Lemma holds over $\mathbb{Z}_{(p)}$ (not merely after p-adic completion).~~ I will prove the Key Lemma using Witt's definition of W. It is easier to do it using Joyal's definition (to be discussed by Anthony Wang).

<u>Remark.</u> The Key Lemma holds over $\mathbb{Z}$, ~~if $W^{(F)}$ but~~ after you change the definition of $W^{(F)}$ ~~&~~ as follows:

$$W^{(F)} := \bigcap_{n>1} \mathrm{Ker}(W^{Big} \xrightarrow{F_n} W^{Big}) = \bigcap_{\ell \in \{\text{primes}\}} \mathrm{Ker}(W^{Big} \xrightarrow{F_\ell} W^{Big})$$

~~Now $\mathbb{G}_a = W/\sum_{n>1} V_n W$~~ Recall: $F_m F_n = F_{mn}$, $F_1 = \mathrm{id}$.

<u>Exercise.</u> Over $\mathbb{Z}_{(p)}$, the two versions of $W^{(F)}$ are the same.

<u>Proof of Key Lemma.</u> $\mathbb{G}_a^\# := \mathrm{Spec}\, \mathbb{Z}[\frac{x^n}{n!}]$. ~~Let x_n~~

$$\mathbb{Z}[\tfrac{x^n}{n!}] = \bigoplus_{n=0}^{\infty} \mathbb{Z} x_n, \quad x_n := \frac{x^n}{n!}. \quad \text{Relations:}$$

$$x_0 = 1, \quad x_m x_n = \binom{m+n}{m} x_{m+n}. \qquad (*)$$

Let R be a ring, then $\mathbb{G}_a^\#(R) = \{$sequences $x_n \in R$ satisfying $(*)\}$.

Group structure on $\mathbb{G}_a^\#(R)$:

$$\{x_n\} + \{y_n\} = \{z_n\}, \text{ where } z_n = \sum_{k+\ell} x_k y_\ell$$

R-module structure on $\mathbb{G}_a^\#(R)$: $a \cdot \{x_n\} = \{a^n x_n\}$.

Things greatly simplify if you introduce the Generating function $f(t) := \sum_{n=0}^{\infty} x_n t^n \in 1 + tR[[t]]$.

Then $\mathbb{G}_a^\#(R) = \{f \in R[[t]]^\times \mid f(t_1+t_2) = f(t_1) f(t_2)\}$, group structure: multiplication. ($f(0)=1$ automatically)

R-module structure on $\mathbb{G}_a^{\#}(R)$: $(af)(t)=f(at)$, $a\in R$, $f\in R[[t]]^\times$.

<u>Reformulation.</u> (As a group) $\mathbb{G}_a^{\#}=\underline{Hom}(\hat{\mathbb{G}}_a,\mathbb{G}_m)=\left\{\begin{matrix}\text{Cartier dual}\\ \text{of } \hat{\mathbb{G}}_a\end{matrix}\right\}$ (This is well known, of course).

<u>Good "news".</u> As a group, $W^{Big}(R) \overset{\text{"identifies with"}}{=} Ker(R[[t]]^\times \to R^\times)$.

So $\mathbb{G}_a^{\#}\subset W^{Big}$.

<u>Remains:</u> $\mathbb{G}_a^{\#}=W^{(F)}$, where $W^{(F)}:=\bigcap\limits_{l>1} F_l$.

<u>Sketch of the proof.</u>

<u>Exercise.</u> $W^{(F)}$ is $\mathbb{Z}$-flat. (Treat each p separately using p-typical Witt vectors.)

$\mathbb{G}_a^{\#}$ is clearly $\mathbb{Z}$-flat.

<u>Remains:</u> $\mathbb{G}_a^{\#}\otimes\mathbb{Q}=W^{(F)}\otimes\mathbb{Q}$. (Exercise?)

$W\xrightarrow{\text{ghost}}\mathbb{G}_a^{\mathbb{N}}$, $W\otimes\mathbb{Q}\xrightarrow{\sim}\mathbb{G}_a^{\mathbb{N}}\otimes\mathbb{Q}$

$f\mapsto$ coefficients of $\pm t\frac{f'}{f}$ (the sign depends on the identification $W(R)\xrightarrow{\sim} Ker(R[[t]]^\times\to R^\times)$. I prefer the plus sign.

Action of F_l on $\mathbb{G}_a^{\mathbb{N}}$: $(x_n)\mapsto(x_{ln})$
(x_n) is killed by F_l iff $x_n=0$ for all $n\in l\cdot\mathbb{N}$.

$\text{ghost}(W^{(F)}\otimes\mathbb{Q})=\left\{\begin{matrix}\text{sequences } (x_n) \text{ such that}\\ x_n=0\ \forall n>1\end{matrix}\right\}$.

So if R is over $\mathbb{Q}$ then

$W^{(F)}(R)=\{f\in R[[t]]^\times \mid f(0)=1,\ \frac{f'}{f} \text{ is constant}\}$

$\mathbb{G}_a^{\#}(R)=\{f\in R[[t]]^\times \mid f(t_1+t_2)=f(t_1)f(t_2)\}=\{e^{at}\}$

<u>Question.</u> Is there a direct proof of the equality $\mathbb{G}_a^{\#}=W^{(F)}$ (by manipulating with identities rather than using flatness)? This way I can only prove that $\mathbb{G}_a^{\#}\subset W^{(F)}$.

We have proved that $(\mathbb{A}^1_{\mathbb{F}_p})^{⧄} = \mathcal{R}$, where $\mathcal{R} := \mathrm{Cone}(W \xrightarrow{p} W)$.
Now let $X = \mathrm{Spec}\, A$. $X^{⧄} = ?$

$X^{⧄}(B) = ?$ Here $A/\mathbb{F}_p$, B is over $\mathbb{Z}/p^n\mathbb{Z}$.

Exercise. Prove that $X^{⧄}(B) = \mathrm{Hom}_{\mathbb{F}_p}(A, \mathcal{R}(B))$ using two false assumptions:

1) $\mathcal{R}$ is a ring scheme (rather than stack),
2) $X \mapsto X^{⧄}$ commutes with limits.

Hint: $X = \varprojlim (\mathbb{A}^n_{\mathbb{F}_p} \underset{0}{\overset{f}{\rightrightarrows}} \mathbb{A}^n_{\mathbb{F}_p})$ (X is defined by the equation $f(x)=0$. So X is an equalizer).

Next time: we'll show that the answer is correct ~~despite~~ (despite the false assumptions)

$X^{\Delta}(B) := \mathrm{Hom}_{\mathbb{F}_p}(A, \mathcal{R}(B))$

X^{Δ} — prismatization

Hom — to be defined (because $\mathcal{R}(B)$ is a ring groupoid).

We'll prove that $X^{⧄} = X^{\Delta}$.

~~Th~~ $X^{⧄}$ is defined as a quotient of a scheme by a flat groupoid, & such a presentation depends on a choice, and one had to check independence of this choice. ~~$X^{⧄}$~~

~~X^{Δ} is defined wil~~ X^{Δ} is defined by directly specifying $X^{\Delta}(B)$ (no choices).