The essence of the prismatic theory is a <u>prismatization functor</u>
$X \mapsto X^{\Delta}$, $\Delta = \mathbb{\Delta} =$ "prism".

X is a ~~p~~ p-adic scheme (for now, $\mathbb{F}_p$-scheme).

X^{Δ} is a p-adic stack with a "lift of Frobenius", i.e.,
$F: X^{\Delta} \to X^{\Delta}$ ~~inducing~~ whose reduction mod p is the usual Fr.

For now, X is over $\mathbb{F}_p$. We'll define X^{Δ} and in this case prove that $X^{\Delta} = X^{\square}$ crystallization defined by Sasha

In the definition of $X^{\square}$ I always assume $(S, \bar{S}) = (\mathrm{Spec}\,\mathbb{Z}_{(p)}, \mathrm{Spec}\,\mathbb{F}_p)$.

$X^{\square}$ is ~~define~~ The definition of $X^{\square}$ involves divided powers; it involves some choices (and a proof of independence of choices).

X^{Δ} ~~is~~ involves Witt vectors and there are no choices in the definition.
Instead, one directly defines $X^{\Delta}(B)$ for any B over $\mathbb{Z}_{(p)}$ (for B over $\mathbb{Z}/p^n\mathbb{Z}$ if you want ~~the~~ to have a construction local w.r.t. X).

We ~~al~~ have already proved such a statement for $X = \mathbb{A}^1_{\mathbb{F}_p}$:

$$(\mathbb{A}^1_{\mathbb{F}_p})^{\square} = \mathrm{Cone}(W \xrightarrow{p} W) \quad \text{(isomorphism of ring stacks).}$$

$$(\mathbb{A}^1_{\mathbb{F}_p})^{\square}(B) = W(B) \overset{L}{\otimes} \mathbb{F}_p := \mathrm{Cone}(W(B) \xrightarrow{p} W(B))$$

Don't erase!

A priori, a pre-stack — ring groupoid.

<u>Def.</u> $X^{\Delta}(B) := \mathrm{Hom}_{\mathbb{F}_p}(A, W(B) \overset{L}{\otimes} \mathbb{F}_p)$ (groupoid to be defined).
(Don't erase!)

Believable: ~~have F~~ lift of Frobenius $F: X^{\Delta} \to X^{\Delta}$ (comes from the Witt Frobenius). Believable: $\mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_p[x], W(B) \overset{L}{\otimes} \mathbb{F}_p) = W(B) \overset{L}{\otimes} \mathbb{F}_p$.

<u>Thm.</u> $X^{\square} = X^{\Delta}$. (Already know for $X = \mathbb{A}^1_{\mathbb{F}_p}$, $A = \mathbb{F}_p[x]$).
Don't erase!

The definition of $\mathrm{Hom}_{\mathbb{F}_p}(A, W(B) \overset{L}{\otimes} \mathbb{F}_p)$ will be ad hoc (without defining the 2-category of ring groupoids & or the ∞-category of derived rings).

Let A, C be rings, let $I \xrightarrow{d} C$ be a quasi-ideal (i.e., I is a C-module, d is C-linear, $dx \cdot y = dy \cdot x$ for $x, y \in I$).
Want to define $\mathrm{Hom}(A, \mathrm{Cone}(I \xrightarrow{d} C))$. (No $\mathbb{F}_p$!)

<u>Idea.</u> Suppose $\mathrm{Ker}\, d = 0$ (so I is an <u>ideal</u>), then $\mathrm{Cone}(I \xrightarrow{d} C) = C/I$.

Given $A \to C/I$ get

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, equal] & \tilde{A} \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I \arrow[r] & C \arrow[r] & C/I \arrow[r] & 0 \end{tikzcd}$$

(morphism of extensions)

<u>Def:</u> Hom $(A, \mathrm{Cone}(I \xrightarrow{d} C))$ is the groupoid of commut. diagrams

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "v"] \arrow[dr, "d"'] & \tilde{A} \arrow[r, "\pi"] \arrow[d, "f"] & A \arrow[r] & 0 \\ & & C & & \end{tikzcd} \qquad (1)$$

Don't erase for a while!

where the upper row is a ring extension, f is a ring homomorphism, and the following (diagram) commutes:

$$\begin{tikzcd} \tilde{A} \arrow[r, "f"] \arrow[dr] & C \arrow[d] \\ & \mathrm{End}\, I \end{tikzcd} \qquad (2)$$

Because $I \subset \tilde{A}$ is an ideal

because $I \xrightarrow{d} C$ is a quasi-ideal ~~quasi-ideal~~

This is indeed a groupoid (rather than set) because $\tilde{A}$ may have nontrivial automorphisms preserving v, f, and $\pi : \tilde{A} \twoheadrightarrow A$.

We have a map Hom $(A, \mathrm{Cone}(I \xrightarrow{d} C)) \to \mathrm{Hom}(A, C/d(I))$ (the map $\tilde{A} \xrightarrow{f} C \to C/d(I)$ factors through I).

<u>Exercise.</u> Ker $d = 0 \Rightarrow$ the above map is an isomorphism. In particular, the groupoid is a set (by the way, the latter is clear because if Ker $d = 0$ then $\tilde{A} \subset A \times C$).

<u>Remark.</u> If the map $C \to \mathrm{End}\, I$ is injective ~~then~~ (which often happens) then you don't have to specify f (there is at most one f, and it exists iff $\mathrm{Im}(\tilde{A} \to \mathrm{End}\, I) \subset \mathrm{Im}(C \to \mathrm{End}\, I)$ and the ring structure on I as an ideal in $\tilde{A}$ is the same as the ring structure on I as a quasi-ideal in C).

<u>Remark.</u> The notation f, v is not random but parallel to F, V for Witt vectors:

$$fv = d \qquad\qquad FV = p$$

$$\tilde{a} \cdot v(x) = v(f(\tilde{a})x) \qquad\qquad a \cdot Vx = V((Fa) \cdot x)$$

(commutativity of (2)).

~~Consider~~ Let C be a ring, $C \overset{L}{\otimes} \mathbb{F}_p := \mathrm{Cone}(C \xrightarrow{p} C)$, so $\mathrm{Hom}(A, C \overset{L}{\otimes} \mathbb{F}_p)$ is the groupoid of diagrams

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "v"] \arrow[dr, "\varphi"'] & \tilde{A} \arrow[r] \arrow[d, "f"] & A \arrow[r] & 0 \\ & & C & & \end{tikzcd} \qquad \cancel{(1')}$$

Don't erase!

with above properties (commutativity, ~~ex~~ upper row is a ring extensions, the two $\tilde{A}$-actions of C are the same),

Now suppose A is over $\mathbb{F}_p$, i.e., $p_{\tilde{A}} \in v(C)$. Let $p_{\tilde{A}} = v(c)$, then $pc = f(v(c)) = f(p_{\tilde{A}}) = p_C$. So we get a map

$$\mathrm{Hom}(A, C \overset{L}{\otimes} \mathbb{F}_p) \longrightarrow C' := \{c \in C \mid pc = p\}, \; \cancel{1 \in C}$$

$1 \in$

<u>Def.</u> $\mathrm{Hom}_{\mathbb{F}_p}(A, C \overset{L}{\otimes} \mathbb{F}_p) \subset \mathrm{Hom}(A, C \overset{L}{\otimes} \mathbb{F}_p)$ is the preimage of $1 \in C'$.

Don't erase!

Thus we have the following extra requirement: ~~f(1)~~ $v(1) = p_{\tilde{A}}$.

<u>Remark.</u> If $\mathrm{Ker}(C \xrightarrow{p} C) = 0$ then $\mathrm{Hom}_{\mathbb{F}_p}(A, C \overset{L}{\otimes} \mathbb{F}_p) = \mathrm{Hom}(A, C \overset{L}{\otimes} \mathbb{F}_p)$, as it should be: indeed, $C \overset{L}{\otimes} \mathbb{F}_p = C/pC$ is a <u>usual</u> ring (not a ring groupoid or derived ring), so $\mathrm{Hom}_{\mathbb{F}_p}(A, C/pC) = \mathrm{Hom}(A, C/pC)$.

<u>Justifying the definition</u> (more difficult than giving it).

"Scientific" definition:

$$\begin{tikzcd} \mathrm{Hom}_{\mathbb{F}_p}(A, C \overset{L}{\otimes} \mathbb{F}_p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Hom}(A, C \overset{L}{\otimes} \mathbb{F}_p) \arrow[d] \\ \mathrm{Hom}(\mathbb{F}_p, \mathbb{Z} \overset{L}{\otimes} \mathbb{F}_p) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(\mathbb{F}_p, C \overset{L}{\otimes} \mathbb{F}_p) \arrow[d, "\wr"] \\ \mathrm{Hom}(\mathbb{F}_p, \mathbb{F}_p) = \text{"point"} \arrow[r] & 1 \in C' \end{tikzcd}$$

Let $Y = \operatorname{Spec} B$, $Y_0 = Y \otimes \mathbb{F}_p = \operatorname{Spec} A$, where $A = B/pB$.

For safety, B is a $\mathbb{Z}_{(p)}$-algebra.

Promise (of a construction). $\exists$ canonical morphism $Y \to Y_0^\Delta$, i.e., an object of the groupoid $Y_0^\Delta(B)$. (Don't erase!)

(I don't require p to be nilpotent in B; however, one probably needs this to define X^Δ without assuming X affine.)

Remark. We do have $Y \to Y_0^\square$. Indeed, Y is a PD-thickening ~~of~~ of Y_0, ~~Fo~~ and for any PD-thickening $Z \supset Y_0$ one has a canonical map $Z \to Y_0^\square$ (if Z is PD-smooth this is clear by the definition of Y_0; in general, ~~embed~~ choose $Z \hookrightarrow Z'$, where Z' is a PD-smooth PD-thickening of Y_0, then check that the ~~~~ ~~map~~ composed map $Z \to Z' \to Y_0^\square$ doesn't depend on the choice of Z').

Particular case $Y = \cancel{X}\, Y_0$: get $Y_0 \cancel{X} \to \cancel{X}\, Y_0^\Delta$ for any ~~X~~ $Y_0/\mathbb{F}_p$.

Construction.

$A = B/pB$, want an object of $\operatorname{Hom}_{\mathbb{F}_p}(A, W(B) \overset{L}{\otimes} \mathbb{F}_p)$, i.e., a diagram

$$\begin{tikzcd} 0 \arrow[r] & W(B) \arrow[r, "V"] \arrow[dr, "p"'] & \tilde{A} \arrow[r] \arrow[d, "f"] & A \arrow[r] & 0 \\ & & W(B) & & \end{tikzcd} \qquad (1')$$

(Don't erase!)

with properties mentioned above, e.g., $V(1) = p_{\tilde{A}}$.

Particular case $A = B$, i.e., $p_B = 0$: let $\tilde{A} = W(\cancel{A} B)$, let (1') be the diagram

$$\begin{tikzcd} 0 \arrow[r] & W(B) \arrow[r, "V"] \arrow[dr, "p"'] & W(B) \arrow[r] \arrow[d, "F"] & B \arrow[r] & 0 \\ & & W(B) & & \end{tikzcd}$$

Key point: $V(1) = p$ because B is over $\mathbb{F}_p$: $V(1) = VF(1)$

$p = FV(1)$ (only because B is over $\mathbb{F}_p$)

General case: the problem is to descend $0 \to W(B) \xrightarrow{V} W(B) \to B \to 0$ to $0 \to W(B) \xrightarrow{v} \tilde{A} \to A \to 0$. Construction: $\tilde{A} := W(B)/I$, $I = (p - V(1))$. Then ~~we have~~ $\mathrm{Im}\,(I \to \cancel{W(B)/VW(B)} \to B) = pB$, So we have $W(B) \xrightarrow{v} \tilde{A} \to A \to 0$. Have $F(I)=0$, so f is defined. ~~So~~ Injectivity $\mathrm{Ker}\, v \overset{?}{\doteq} 0$. $I \cap V\, W(B) \overset{?}{\doteq} 0$. ~~So we~~ This amounts to the following

Lemma. Let $x \in W(B)$, $x(p - V(1)) \in V \cdot W(B)$. Then $x(p - V(1)) = 0$.

(This could be an exercise. Hints: $p - V(1) \in \mathrm{Ker}\,(W(\mathbb{Z}) \to W(\mathbb{F}_p))$, $F(p - V(1)) = 0$.)

Proof. $F(p - V(1)) = 0$, so $x \cdot (p - V(1))$ depends only on $x \bmod V \cdot W(B)$. So we can assume that $x = [b]$, $b \in B$. We have $p - V(1) \in \mathrm{Ker}\,(W(\mathbb{Z}) \to W(\mathbb{F}_p))$. Write $p - V(1) = \sum_i V^i [c_i]$, $c_i \in p\mathbb{Z}$, then $[b](p - V(1)) = \sum_i V^i [b^{p^i} c_i]$. The assumption $x(p - V(1)) \in V \cdot W(B)$ means that $pb = 0$. This implies that $b^{p^i} c_i = 0$ because $c_i \in p\mathbb{Z}$. ∎

Remark. The numbers c_i can be computed inductively from the equations $c_0 = p$, $c_0^{p^n} + p c_1^{p^{n-1}} + \dots + p^n c_n = 0$ for $n > 0$ ~~$n \geq 0$~~. So $c_1 = -p^{p-1}$, $c_2 = -p^{p^2-2} + (-1)^{p-1} p^{p^2-p-1}$.

$\boxed{\text{Don't erase}}$

Thm. $X^\Delta = X^{\square}$ for $X/\mathbb{F}_p$ (X affine).

Know for $X = \mathbb{A}^1_{\mathbb{F}_p}$. Follows for $X = (\mathbb{A}^1_{\mathbb{F}_p})^S$. Source of pain: the functors don't quite commute with limits.

$X = \operatorname{Spec} A \Rightarrow X^\Delta(B) := \operatorname{Hom}_{\mathbb{F}_p}(A, W(B)\tilde{\otimes}\mathbb{F}_p)$, $B/\mathbb{Z}_{(p)}$

To define $X^\square$, choose $X \hookrightarrow Y$, Y formally smooth$/\mathbb{Z}_{(p)}$

Let $H_{X\hookrightarrow Y} :=$ PD-hull of X in Y ($\gamma_n(p) = \frac{p^n}{n!}$). Then
$X^\square := H_{X\hookrightarrow Y}/\Gamma$, where Γ is the groupoid with Nerve$(\Gamma) = H_{X\hookrightarrow Y^{[\cdot]}}$.
$X \hookrightarrow Y^{[n]}$, $[n] := \{0, 1, \dots, n\}$, so we have

The simplicial scheme $H_{X\hookrightarrow Y^{[\cdot]}}$. ~~is the nerve of a~~ flat groupoid Γ acting on $H_{X\hookrightarrow Y}$. $X^\square := H_{X\hookrightarrow Y}/\Gamma$ (quotient stack).

Corollaries (of the def. of $X^\square$). ① Have $H_{X\hookrightarrow Y} \to X_\square$ (clear if Y is formally smooth, the general case follows).

② Take $X = Y_0 := Y \otimes \mathbb{F}_p$, then $H_{X\hookrightarrow Y} = Y$, so we get a canonical morphism $Y \to (Y_0)_\square$.

Thus, given $X \hookrightarrow Y$, get a commutative diagram

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr] & H_{X\hookrightarrow Y} \arrow[r] \arrow[d] & Y \arrow[d] \\
 & X^\square \arrow[r] & Y_0^\square
\end{tikzcd}$$

③ The square is Cartesian (easy if Y is an affine space, the general case follows).

Strategy: Prove ①–③ for X^Δ {instead of $X^\square$}.

Last time: $Y \to (Y_0)_\Delta$. Let us construct ①, ③ for X^Δ.

$\boxed{\text{Don't erase}}$

Define $H'_{X\hookrightarrow Y}$ by the Cartesian square

$$\begin{tikzcd}
H'_{X\hookrightarrow Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\
X^\Delta \arrow[r] & Y_0^\Delta
\end{tikzcd}$$

Prop. $H'_{X\hookrightarrow Y} \xrightarrow{\sim} H_{X\hookrightarrow Y}$ (functorial in $X \hookrightarrow Y$).

($H_{X\hookrightarrow Y} :=$ PD-hull of X in Y).

Proof of the Theorem assuming the Proposition.

The Proposition allows us ~~to~~ to not distinguish H' from H. So for any $X \hookrightarrow Y$, $Y/\mathbb{Z}_{(p)}$, we have $H_{X\hookrightarrow Y}$ ~~and~~ and the map $H_{X\hookrightarrow Y} \to X^{\Delta}$ (so $H_{X\hookrightarrow Y}$ is over X^{Δ}). By functoriality, we have a simplicial scheme $H_{X\hookrightarrow Y^{[\cdot]}}$ over X^{Δ}. So $\forall n$ we have $f_n : H_{X\hookrightarrow Y^{[n]}} \to H_{X\hookrightarrow Y} \times_{X^{\Delta}} H_{X\hookrightarrow Y} \times \ldots \times_{X^{\Delta}} H_{X\hookrightarrow Y}$.

Lemma. Suppose $Y = (\mathbb{A}^1_{\mathbb{Z}_{(p)}})^S$. Then

(i) $\pi : H_{X\hookrightarrow Y} \to X^{\Delta}$ is faithfully flat,

(ii) f_n is an isomorphism $\forall n$.

Deducing the Theorem from the Lemma:

General remark. Let S be a scheme, $\mathcal{Y}$ a stack, $\pi : S \to \mathcal{Y}$ a morphism. Consider the simplicial scheme $S \leftleftarrows S \times_{\mathcal{Y}} S \Lleftarrow S \times_{\mathcal{Y}} S \times_{\mathcal{Y}} S \ldots$ It is the ~~is a~~ nerve of a groupoid acting on S; denote it by Γ_π. If π is flat then Γ_π is flat; if π is faithfully flat then $S/\Gamma_\pi \xrightarrow{\sim} \mathcal{Y}$. (End of the remark.)

By (i) and the remark, $X^{\Delta} = H_{X\hookrightarrow Y}/\Gamma_\pi$, where $\pi : H_{X\hookrightarrow Y} \to X^{\Delta}$.

By definition, $X^{\square} = H_{X\hookrightarrow Y}/\Gamma'$, where $\mathrm{Nerve}(\Gamma') = H_{X\hookrightarrow Y^{[\cdot]}}$.

By (ii), $\Gamma' \xrightarrow{\sim} \Gamma_\pi$. ■

Proof of the Lemma. Reduction to the case $X = Y_0$: $H_{X\hookrightarrow Y^{[\cdot]}} = H_{Y_0 \hookrightarrow Y^{[\cdot]}} \times_{Y_0^{\Delta}} X^{\Delta}$, so we can assume that $X = Y_0$. Then $Y_0^{\Delta} = Y_0^{\square}$, and the lemma is clear from the definition of $Y_0^{\square}$. ■

Remarks for myself. 1 The lemma holds if Y is formally smooth over $\mathbb{Z}_{(p)}$ ~~(not ne~~ (which is weaker than being an affine space).

2 Statement (ii) of the lemma implies that the simplicial scheme $H_{X\hookrightarrow Y^{[\cdot]}}$ is the nerve of a groupoid. So the formal smoothness assumption is probably necessary even for statement (ii).

Proof of the Proposition (which says that $H_{X\hookrightarrow Y} = H'_{X\hookrightarrow Y}$).

Have $Y \supset Y_0 \supset X$, $Y = \operatorname{Spec} C$, $X = \operatorname{Spec} A$.

$$0 \to I \to C \to A \to 0 \quad \text{over } \mathbb{Z}_{(p)}, \qquad p \in I$$

Let B be a ring. We have $H(X,Y) \to Y \leftarrow H'(X,Y)$ and therefore $H(B) \longrightarrow \operatorname{Hom}(C,B) \longleftarrow H'(B)$. Let $\alpha \in \operatorname{Hom}(C,B)$. Consider the fibers $H_\alpha(B) \subset H(B)$, $H'_\alpha(B) \subset H'(B)$ (these are groupoids). (these are sets) (which turn out to be sets)

Claim. $H_\alpha(B) = H''_\alpha(B) = H'_\alpha(B)$, where

$H''_\alpha(B) :=$ set of C-linear maps $\gamma : I \to W^{(F)}(B)$ s.t. $\gamma(p) = p - V(1)$ and the diagram

$$\begin{array}{ccc} I & \xrightarrow{\gamma} & W^{(F)}(B) \\ \cap & & \downarrow \\ C & \xrightarrow{\alpha} & B \end{array} \qquad (*)$$

commutes.

Note: $p - V(1) \in W^{(F)}(B)$ Because $F(p - V(1)) = F(p) - FV(1) = p - p = 0$.

Proof of $H_\alpha(B) = H''_\alpha(B)$ (more interesting; no W in the definition of PD-hull)

(Hint: the notation γ is related to the divided powers γ_i)

By the construction (rather than definition) of PD hull, an element of $H_\alpha(B)$ is a collection of $\gamma_n : I \to B$, $n \geq 0$, s.t.

1) $\gamma_n(x+y) = \sum_{i=0}^{n} \gamma_i(x)\gamma_{n-i}(y)$, 2) $\gamma_n(cx) = c^n \gamma_n(x)$, $c \in C$.

3) $\gamma_n(p) = \frac{p^n}{n!}$, 4) $\gamma_0(x) = 1$, $\gamma_1 = \alpha|_I$, 5) $\gamma_m(x)\cdot\gamma_n(x) = \binom{m+n}{m}\cdot\gamma_{m+n}(x)$

Consider the generating function $\gamma(x) := \sum_i \gamma_i(x) t^i \in B[[t]]^*$.

Have $\gamma : I \longrightarrow \operatorname{Ker}(B[[t]]^\times \to B^\times)$

$$\begin{array}{ccc} I & \longrightarrow & \operatorname{Ker}(B[[t]]^\times \to B^\times) \\ \downarrow & & \| \\ W^{(F)}(B) & \subset & W(B) \\ \| & & \\ \operatorname{Hom}(\hat{G}_a, G_m) & & \end{array}$$

$\gamma(I) \subset W^{(F)}(B)$ by 5).

γ is C-linear by 1)-2)

Diagram $(*)$ commutes by 4)

By 3), $\gamma(p) = e^{pt} \in B[[t]]^*$, and $p - V(1) \in W^{(F)}(\mathbb{Z})$ corresponds to $e^{pt} \in \mathbb{Z}[[t]]^\times$ (!) Because the ghost components of $p - V(1)$ are $(p, 0, 0, \dots)$. ∎

Proof of the equality $H'_\alpha(B) = H''_\alpha(B)$ (skipped during the talk).

$$\begin{tikzcd} H'_{X\hookrightarrow Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y = \operatorname{Spec} C \arrow[d] \\ X^\Delta \arrow[r] \arrow[d, phantom, "\parallel"] & Y_0^\Delta \arrow[d, phantom, "\parallel"] \\ (\operatorname{Spec} A)^\Delta & (\operatorname{Spec}(C/pC))^\Delta \end{tikzcd}$$

$\alpha \in \operatorname{Hom}(C,B) = Y(B)$

$\downarrow$

$\beta \in Y_0^\Delta(B)$

$H'_\alpha(B) :=$ preimage of β in $X^\Delta(B)$.

Think of $\beta \in Y_0^\Delta(B) = (\operatorname{Spec} C/pC)^\Delta(B) = \operatorname{Hom}(\operatorname{Spec} B, (\operatorname{Spec} C/pC)^\Delta)$ as the composed map $\operatorname{Spec} B \to (\operatorname{Spec} B/pB)^\Delta \to \operatorname{Spec} C/pC$ (rather than $\operatorname{Spec} B \to \operatorname{Spec} C \to (\operatorname{Spec} C/pC)^\Delta$). Then we see that β corresponds to the diagram

$$\begin{tikzcd} 0 \arrow[r] & W(B) \arrow[r, "V"] \arrow[dr, "p"'] & R \arrow[r] \arrow[d, "F"] & C/pC \arrow[r] & 0 \\ & & W(B) & & \end{tikzcd}$$

where $R := (W(B)/(p - V(1))) \times_{B/pB} C/pC$. Recall that $X = \operatorname{Spec} A = \operatorname{Spec}(C/I)$, where $I \ni p$. So $H'_\alpha(B)$ is the set of C-module splittings $I/pC \to R$ **whose** image is contained in $\operatorname{Ker}(R \xrightarrow{F} W(B))$.

A splitting $I/pC \to R$ is the same as a lift of the map $I/pC \hookrightarrow C/pC \xrightarrow{\bar\alpha} B/pB$ to $\bar\gamma : I/pC \to W(B)/(p - V(1))$. Since the square

$$\begin{tikzcd} W(B) \arrow[r, two heads] \arrow[d] & B \arrow[d] \\ W(B)/(p-V(1)) \arrow[r, two heads] & B/pB \end{tikzcd}$$

is Cartesian, this is the same as a lift of the map $I \hookrightarrow C \xrightarrow{\alpha} B$ to $\gamma : I \to W(B)$ such that $\gamma(p) = p - V(1)$.

We want $\gamma(I) \subset W^{(F)}(B)$ because we want the composed map $I/pC \to R \xrightarrow{F} W(B)$ to be zero.

Thus $H'_\alpha(B)$ is the set of C-module homomorphisms $\gamma : I \to W^{(F)}(B)$ such that $\gamma(p) = p - V(1)$ and the composed maps $I \xrightarrow{\gamma} W^{(F)}(B) \to B$ and $I \hookrightarrow C \xrightarrow{\alpha} B$ are equal to each other.