Format of prismatic theory

Prismatization functor $X \mapsto X^{\Delta}$

Input: p-adic (formal) scheme

Output: formal stack with a lift of Frobenius $F: X^{\Delta} \to X^{\Delta}$

The theory is cleaner if everything is derived.

(Complexes of) $\mathcal{O}$-modules on X^{Δ} are called (prismatic) crystals on X

An F-crystal is a pair $(M, F^*M \to M)$

crystal

maybe required to be an isogeny

These are p-adic analogs of $\mathbb{Z}_\ell$-sheaves.

$f: X \to Y$ gives $f^{\Delta}: X^{\Delta} \to Y^{\Delta}$

$f_*^{\Delta}: \{\text{F-crystals on } X\} \to \{\text{F-crystals on } Y\}$,

$f_*^{\Delta}(\mathcal{O}_{X^{\Delta}})$ is an F-crystal on Y.

(I could skip the rest of this page)

If X is over $\mathbb{F}_p$ this is crystalline cohomology.

In this case X^{Δ} is as discussed before (up to derived issues).

$(\operatorname{Spec} \mathbb{F}_p)^{\Delta} = \operatorname{Spf} \mathbb{Z}_p$ (formal scheme rather than stack!)

For $f: X \to \operatorname{Spec} \mathbb{F}_p$, $f_*^{\Delta}(\mathcal{O}_{X^{\Delta}})$ is a complex of $\mathbb{Z}_p$-modules with action of Frobenius.

For $f: X \to \operatorname{Spf} \mathbb{Z}_p$, $f_*^{\Delta}(\mathcal{O}_{X^{\Delta}})$ lives on $(\operatorname{Spf} \mathbb{Z}_p)^{\Delta}$

stack, not scheme!

Can "feel" $f_*^{\Delta}(\mathcal{O}_{X^{\Delta}})$ (or any F-crystal on X) by looking at its pullbacks to various schemes S over $(\operatorname{Spf} \mathbb{Z}_p)^{\Delta}$

a "prism" is such a scheme.

Breuil-Kisin modules are such pullbacks (in a slightly more general situation, e.g., $\mathbb{Z}_p$ is replaced by O_K, where $[K:\mathbb{Q}_p] < \infty$).

The existing literature doesn't emphasize the fact that we are dealing with $\mathcal{O}$-modules on stacks.

NB. The ~~format of~~ stacky format of prismatic theory was suggested by Bhatt and Lurie during a walk in Cambridge, MA. No literature is available. I am explaining my understanding of the subject, whose correctness is not guaranteed. (A saying from Arinkin's old webpage: "The formula is incorrect but easy to use".)

Definition of $\Sigma := (\mathrm{Spf}\,\mathbb{Z}_p)^{\Delta}$

W := Witt vectors (as a p-adic formal scheme ~~over $\mathbb{F}_p$~~).
Let $x_0, x_1, \ldots$ be the usual ("Witt") coordinates. ($\sum_{i=0}^{\infty} V^i[x_i]$)
As formal scheme, $W = \mathrm{Spf}$ (p-adic completion of $\mathbb{Z}_p[x_0, x_1, \ldots]$).
Let R be a ring in which p is nilpotent. Define $W^{(1)}(R) \subset W(R)$,

$$W^{(1)}(R) := \left\{ \sum_{i=0}^{\infty} V^i[x_i] \mid x_0 \text{ nilpotent}, x_1 \in R^{\times} \right\}.$$

As a formal scheme, $W^{(1)} = \mathrm{Spf}\, A$, where A is the completion of $\mathbb{Z}_p[x_0, x_1, \ldots][x_1^{-1}]$ w.r.t. the ideal (p, x_0).
(The topology on A is not p-adic, so I wouldn't call $W^{(1)}$ a p-adic scheme.) Two "small parameters": p and x_0.
Equivalently, $W^{(1)}$ is obtained from W by formal completion along the locus $x_0 = 0$ and removing $x_1 = 0$.

Remark. $p \in W^{(1)}(\mathbb{Z}_p) \subset W(\mathbb{Z}_p)$; indeed, the image of p in $W(\mathbb{F}_p)$ equals $V(1) = (0, 1, 0, \ldots)$
Points of $W^{(1)}$ are "deformations of p".

Def. $\Sigma := W^{(1)}/W^{\times}$ ($W^{\times}$ acts on W by multiplication).
There are also more "economic" but less canonical presentations of Σ as a quotient (choose a transversal to the subscheme $x_0 = p = 0$).

Definition of X^{Δ} in general.

$$\{\text{Derived p-adic affine schemes}\} \to \{\text{Derived stacks over } \Sigma\}$$
$$X \longmapsto X^{\Delta} \quad \text{prismatization}$$

Characterized by two properties:
① Commutation with (projective) limits. The final object is a particular case of limit, so In particular, $(\mathrm{Spf}\,\mathbb{Z}_p)^{\Delta} = \Sigma$

Then $(\mathbb{A}^1_{\mathbb{Z}_p})^{\Delta}$ is a ring stack over Σ, and it remains to ~~describe~~ specify it. Equivalently, to ~~descr~~ specify $(\mathbb{A}^1_{\mathbb{Z}_p})^{\Delta} \times_{\Sigma} W^{(1)}$ as a ring stack over $W^{(1)}$ equipped with $W^{\times}$-action.

② The fiber of $(\mathbb{A}^1_{\mathbb{Z}_p})^{\Delta} \times_{\Sigma} W^{(1)}$ over $\alpha \in W^{(1)}$ is $\mathrm{Cone}(W \xrightarrow{\alpha} W)$. If $u \in W^{\times}$ then $\mathrm{Cone}(W \xrightarrow{u\alpha} W) = \mathrm{Cone}(W \xrightarrow{\alpha} W)$
(α is a "deformation of p").

Exercise. $(\mathrm{Spec}\,\mathbb{F}_p)^{\Delta} = \mathrm{Spf}\,\mathbb{Z}_p$ (no stackiness!).
(Write $\mathrm{Spec}\,\mathbb{F}_p = \varprojlim \left(\mathrm{Spf}\,\mathbb{Z}_p \underset{0}{\overset{p}{\rightrightarrows}} \mathbb{A}^1_{\mathbb{Z}_p} \right)$).
More generally, if $X/\mathbb{F}_p$ is an l.c.i. then X^{Δ} is as in previous talks.