(a) “Scheme” means affine scheme. The category $\mathcal{A}ff$ of schemes is closed under limits and colimits.

“pd scheme” means a scheme equipped with a pd ideal, $(S, \mathcal{I}_S, \gamma)$ abbreviated to S; we set then $\bar{S} = \mathrm{Spec}\mathcal{O}_S/\mathcal{I}_S$ and call S a pd-thickening of $\bar{S}$. The category of pd schemes $\mathcal{A}ff^{pd}$ is closed under limits and colimits.

Below we fix a base pd scheme S over $\mathbb{Z}/p^n$ and an S-scheme X such that the pd structure on $\mathcal{I}_S\mathcal{O}_X \subset \mathcal{O}_X$ is well defined. (Say, X is a $\bar{S}$-scheme.) Denote by $X_{\mathrm{crys}} = (X/S)_{\mathrm{crys}}$ the category whose objects are pd thickenings T of X over S (so T is an S-scheme such that the pd structure on $\mathcal{I}_T + \mathcal{I}_S\mathcal{O}_T$ is well defined). X_{crys} is closed under *nonempty* limits (i.e., it has nonempty products and equalizers).

(b) *Alexander-Cech complexes.* We consider presheaves of abelian groups $\mathcal{F}$ on X_{crys}, and call them sheaves. They form an abelian category; for any $T \in X_{crys}$ the functor $\mathcal{F} \mapsto \mathcal{F}(T)$ is exact. We have a left exact functor $\mathcal{F} \mapsto \Gamma(\mathcal{F}) = \Gamma(X_{\mathrm{crys}}, \mathcal{F}) := \lim_{X_{\mathrm{crys}}}\mathcal{F}$ and its derived functor $R\Gamma$.

For $P \in X_{\mathrm{crys}}$ we denote by $\mathcal{F}_P$ the sheaf $T \mapsto \mathcal{F}(P \times T)$; the functor $\mathcal{F} \mapsto \mathcal{F}_P$ is exact. There is an evident map $\mathcal{F}(P) \to \Gamma(\mathcal{F}_P)$. Consider the simplicial object $P^{[\cdot]}$ of X_{crys}. We have the cosimplicial sheaf $\mathcal{F}_{P^{[\cdot]}}$ and the evident coaugmentation map $\mathcal{F} \to \mathcal{F}_{P^{[\cdot]}}$. If we view $\mathcal{F}_{P^{[\cdot]}}$ as a complex of sheaves then this is the map $\mathcal{F} \to H^0\mathcal{F}_{P^{[\cdot]}}$.

Proposition. *Suppose P is such that $Hom(T, P) \neq \emptyset$ for every $T \in X_{\mathrm{crys}}$. Then*

(i) $\mathcal{F}_{P^{[\cdot]}}$ is a cosimplicial resolution of $\mathcal{F}$.

(ii) The evident map $\Gamma(\mathcal{F}) \to H^0\mathcal{F}(P^{[\cdot]})$ is an isomorphism, and the exact functor $\mathcal{F} \mapsto \mathcal{F}(P^{[\cdot]})$ is $R\Gamma$.

(iii) For any $T \in X_{\mathrm{crys}}$ one has $\mathcal{F}(T) \xrightarrow{\sim} R\Gamma(\mathcal{F}_T)$. Thus $\mathcal{F}(P^{[\cdot]}) = \Gamma\mathcal{F}_{P^{[\cdot]}}$.

Proof. (i) means that $\mathcal{F}(P^{[\cdot]} \times T)$ is a resolution of $\mathcal{F}(T)$ for every $T \in X_{\mathrm{crys}}$. This comes since the augmented simplicial object $P^{[\cdot]} \times T/T$ is contractible; one constructs a contraction choosing a point in $\mathrm{Hom}(T, P)$. The first claim of (ii) is straightforward. To check the second one we check that our functor is effaceable. It is enough to find an injective map $\mathcal{F} \to \mathcal{G}$ such that $H^{>0}\mathcal{G}(P^{[\cdot]}) = 0$. The canonical map with $\mathcal{G} = \mathcal{F}_P$ satisfies this by (i). Now (iii) follows from (i) and (ii). □

(c) *A cosimplicial digression.* Recall that one has the Dold-Puppe equivalence between the category of cosimplicial abelian groups and coconnective complexes of abelian groups. It assigns to a cosimplicial abelian group A the normalized complex $N(A)$ whose components $N(A)^m = \cap\mathrm{Ker}(\sigma_i : A^m \to A^{m-1})$. If A is a cosimplicial ring then $N(A)$ is a dg ring with respect to the Alexander-Whitney $\cup$ product: for $a \in N(A)^m, b \in N(A)^n$ one has $a \cup b := \delta(a)\tau(b)$ where τ where $\delta : [m] \hookrightarrow [m+n]$, $\tau : [n] \hookrightarrow [m+n]$ are the monotone embeddings with images $[0, m]$ and $[m, m+n]$. If A is an associative ring then so is $N(A)$; a similar fact about commutativity is not necessary true.

Definition. A cosimplicial ring A is said to be *of shuffle type* if the next property holds: for every monotone embeddings $\delta : [m] \hookrightarrow [\ell]$, $\tau : [n] \hookrightarrow [\ell]$ with $m + n > \ell$ one has $\delta(N(A)^m)\tau(N(A)^n) = 0$.

Below $E(m, \ell)$ is the set of increasing embeddings $\delta : [m] \hookrightarrow [\ell]$ with $\delta(0) = 0$.

Lemma. *(i) It is enough to check the condition of the definition for $\delta \in E(m, \ell)$, $\tau \in E(n, \ell)$.*

(ii) Suppose A is of shuffle type. If $\delta : [m] \hookrightarrow [m+n]$, $\tau : [n] \hookrightarrow [m+n]$ are monotonous embeddings such that $\delta([m]) \cup \tau([n]) = [m+n]$ then for $a \in N(A)^m$, $b \in N(A)^n$ one has $\delta(a)\tau(b) = \pm a \cup b \in N(A)^{m+n} \subset A^{m+n}$ with the sign $\pm$ depending on δ and τ. □

Proposition. *The restriction of N to the subcategory* Shuff *of shuffle-type cosimplicial rings is an equivalence of categories* Shuff $\xrightarrow{\sim}$ (dg rings). *It identifies the category of commutative shuffle-type rings with commutative dg rings.*

Sketch of a proof. We construct the inverse functor. Let A be a cosimplicial abelian group and $\cup$ be a product on $N(A)$; we need to construct a shuffle product on A such that the corresponding Alexander-Whitney product equals $\cup$. Recall the Dold-Puppe decomposition $A^\ell = \oplus_{m,\delta\in E(m,\ell)} {}^\delta N(A)^m$ where ${}^\delta N(A)^m$ is a copy on $N(A)^m$ embedded into $N(A)^\ell$ via δ. The shuffle product $\delta(a)\tau(b)$ for $a \in N(A)^m$, $b \in N(A)^n$ vanishes if $\delta([m]) \cap \tau([n]) \neq \{0\}$; otherwise it equals $\pm\gamma(a \cup b)$ where $\gamma \in E(m+n,\ell)$ is the embedding with image $\delta([m]) \cup \tau([n])$ and $\pm$ is the sign of the permutation s of $[1, m+n]$ such that γs equals δ on $[1,m]$ and is $x \mapsto \tau(x-m)$ on $[m+1, m+n]$. □

We call the equivalence of the proposition *the Dold-Puppe correspondence.*

(d) Let R be a commutative ring. Let A be a cosimplicial shuffle-type R-algebra and N be a commutative dg R-algebra that are identified by the Dold-Puppe correspondence. Then N is generated by $N^0 = A^0$ as a dg algebra iff A is generated by A^0 as a cosimplicial algebra. Notice that we have a canonical map of cosimplicial algebras $A^{0\otimes[\cdot]} \to A$, and the latter property means that it is surjective, i.e., Spec A is a closed subscheme of $(\mathrm{Spec}\, A^0)^{[\cdot]}$. In particular for an S-scheme T we have a closed simplicial subscheme $T^\Omega_\cdot$ of $T^{[\cdot]}$ that corresponds to the dg algebra $\Omega_{T/S}$.

Lemma. *One has $T^\Omega_0 = T$, T^Ω_1 is the subscheme of $T^{[1]} = T \times_S T$ whose ideal is the suare of the ideal J of the diagonal. If $2 \in \mathcal{O}_T$ is invertible then $T^\Omega_\cdot$ is the largest simplicial subscheme of $T^{[\cdot]}$ with these properties; if not, one adds extra equations $\partial_0(\nu)\partial_2(\nu) = 0$ where $\nu \in J/J^2 \subset \mathcal{O}(T^\Omega_1)$, $\partial_i : [1] \to [2]$ are the face maps.* □

(e) We return to the setting of (a). For $T \in X_{\mathrm{crys}}$ its *pd de Rham algebra* $\Omega_{\mathrm{pd}}(T/S)$ is the quotient of $\Omega(T/S)$ modulo the relations $d(\gamma^i f) = \gamma^{i-1}(f)df$, $f \in \mathcal{I}_T$. Let $T^{\Omega_{\mathrm{pd}}}_\cdot$ be a closed simplicial pd subscheme of $T^{[\cdot]}$ defined by a cosimplicial ideal $\mathcal{J}(T^{[\cdot]}) \subset \mathcal{O}(T^{[\cdot]})$ generated by $J^{\{2\}}$ where $J \subset \mathcal{O}(T \times T)$ is the (pd) ideal of the diagonal $T \hookrightarrow T \times T$ and $J^{\{2\}}$ its second divided power (which is the ideal generated by $\gamma^{>1}(J)$). This is a simplicial object of X_{crys}.

Lemma. *$\Omega_{\mathrm{pd}}(T/S)$ corresponds to $\mathcal{O}(T^{\Omega_{\mathrm{pd}}}_\cdot)$ by Dold-Puppe.* □

If $\mathcal{F}$ is an $\mathcal{O}$-crystal on X_{crys} then the dg $\Omega_{\mathrm{pd}}(T/S)$ -module $N(\mathcal{F}(T^{\Omega_{\mathrm{pd}}}_\cdot))$ equals $\Omega_{\mathrm{pd}}(T/S) \otimes_{\mathcal{O}(T)} \mathcal{F}(T)$ as a plain graded $\Omega_{\mathrm{pd}}(T/S)$ -module; we call it the pd de Rham complex of T with coefficients in $\mathcal{F}$ and denote by $\Omega_{\mathrm{pd}}(T/S, \mathcal{F})$.

(f) An object $P \in X_{\mathrm{crys}}$ is said to be *pd smooth* if every closed embedding $P \hookrightarrow T$ in X_{crys} admits left inverse, i.e., P is a retraction of T. For example every *coordinate thickening*, i.e., the pd-hull of a closed embedding $X \hookrightarrow \mathbb{A}^I$, is pd smooth. Since every T admits a closed embedding into a coordinate thickening, pd smooth thickenings are the same as retracts of coordinate thickenings. A pd smooth P satisfies the condition of the proposition in (b).

As in (b), for $P \in X_{\mathrm{crys}}$ and a sheaf $\mathcal{F}$ we have the cosimplicial sheaf $\mathcal{F}_{P_\cdot^{\Omega_{\mathrm{pd}}}}$ and the coaugmentation map $\mathcal{F} \to \mathcal{F}_{P_\cdot^{\Omega_{\mathrm{pd}}}}$.

Theorem. *Suppose P is pd smooth and $\mathcal{F}$ is an $\mathcal{O}$-crystal. Then*
(i) $\mathcal{F}_{P_\cdot^{\Omega_{\mathrm{pd}}}}$ is a cosimplicial resolution of $\mathcal{F}$.
(ii) One has $\Omega_{\mathrm{pd}}(P/S, \mathcal{F}) \xrightarrow{\sim} N(\Gamma \mathcal{F}_{P_\cdot^{\Omega_{\mathrm{pd}}}}) \xrightarrow{\sim} R\Gamma(\mathcal{F}_{P_\cdot^{\Omega_{\mathrm{pd}}}}) \xrightarrow{\sim} R\Gamma(\mathcal{F})$.
(iii) The resuling quasi-isomorphism $R\Gamma(\mathcal{F}) \to \Omega\mathcal{F}(P)$ can be realized, using the proposition in (b), as the restriction map $\mathcal{F}(P^{[\cdot]}) \to \mathcal{F}(P_\cdot^{\Omega_{\mathrm{pd}}})$.

Proof. Assume (i). The first isomorphism is $\mathcal{F}(P_\cdot^{\Omega_{\mathrm{pd}}}) \xrightarrow{\sim} \Gamma(\mathcal{F}_{P_\cdot^{\Omega_{\mathrm{pd}}}})$ which follows from the first assertion (iii) of the proposition in (b), the second quasi-isomorphism $\Gamma(\mathcal{F}_{P_\cdot^{\Omega_{\mathrm{pd}}}}) \xrightarrow{\sim} R\Gamma(\mathcal{F}_{P_\cdot^{\Omega_{\mathrm{pd}}}})$ follows from loc. cit, the last isomorphism follows from (i). Claim (iii) follows, say, from the last assertion in loc. cit.

It remains to prove (i). We want to check that for $T \in X_{\mathrm{crys}}$ the complex $\mathcal{F}(P_\cdot^{\Omega_{\mathrm{pd}}} \times T)$ is a resolution of $\mathcal{F}(T)$. Since there is a section of $P \times T/T$, the map $\mathcal{F}(T) \to H^0\mathcal{F}(P_\cdot^{\Omega_{\mathrm{pd}}} \times T)$ is injective. Let $P \hookrightarrow P'$ be an embedding of P into coordinate P'. Since P is a retract of P', the complex $\mathcal{F}(P_\cdot^{\Omega_{\mathrm{pd}}} \times T)$ is a direct summand of $\mathcal{F}(P'^{\Omega_{\mathrm{pd}}}_\cdot \times T)$, and so our claim for P follows from that about P'. Replacing P by P' we can assume that P is a coordinate thickening. Thus P is the relative pd-hull of $\mathbb{A}^I \times T/T$ at a section $X \to \mathbb{A}^I \times T$ over $X \subset T$. Extending it to a section s over T and applying the substraction of s automorphism of $\mathbb{A}^I \times T/T$, we can assume that our section is zero. Then $P \times T$ is the product of pd schemes $\mathbb{G}_a^{\sharp I}$ and T where $\mathbb{G}_a^{\sharp}$ is the pd-hull of $\mathbb{G}_a$ at 0. Thus $\mathcal{F}(P_\cdot^{\Omega_{\mathrm{pd}}} \times T) = \mathcal{O}((\mathbb{G}_a^{\sharp I})_\cdot^{\Omega_{\mathrm{pd}}}) \otimes \mathcal{F}(T) = \Omega_{\mathrm{pd}}(\mathbb{G}_a^{\sharp})^{\otimes I} \otimes \mathcal{F}(T) \xleftarrow{\sim} \mathcal{F}(T)$, the first equality comes since $\mathcal{F}$ is an $\mathcal{O}$-crystal, the last one since $\Omega_{\mathrm{pd}}(\mathbb{G}_a^{\sharp}) \xleftarrow{\sim} \mathbb{Z}$. □