November 6, 2017. Beilinson's talk
(preparatory for Bhatt's talk on Nov. 13).

A commutative ring. The following approach to Witt vectors goes back to Joyal (1985)

$F: A \to A$ is a Frobenius lifting if $F(a) \equiv a^p \bmod pA$.

If A has no p-torsion define the Frobenius derivative $\delta: A \to A$, $\delta(a) := \frac{F(a)-a^p}{p}$. Then

(i) $\delta(a+b) = \delta(a) + \delta(b) + \sum_{i=1}^{p-1} p^{-1} \cdot \binom{p}{i} a^i b^{p-i}$

(ii) $\delta(ab) = a^p \delta(b) + \delta(a) b^p + p\,\delta(a)\,\delta(b)$, $\delta(1)=0$.

Def. For any ring A, a map $\delta: A \to A$ satisfying (i)-(ii) is a Frobenius derivation. A pair (A, δ) is a δ-ring.

If $\delta: A \to A$ is a Frobenius derivation define $F: A \to A$, $F(a) := a^p + p\,\delta(a)$; then F is a Frobenius lifting. Notation: Rings^δ is the category of pairs (A, δ).

Example. $\mathbb{Z} \in \mathrm{Rings}^\delta$, $\mathbb{Z}_p \in \mathrm{Rings}^\delta$. Here $\delta(x) = (x - x^p)/p$

Lemma. If A is over $\mathbb{Z}/p^n\mathbb{Z}$ then it can't carry a Frobenius derivative.

Proof. If (A, δ) is a δ-ring then $\mathbb{Z} \to (A, \delta)$ has to be a δ-homomorphism. So $\delta(p^n) = \frac{p^n - p^{np}}{p} = p^{n-1} \cdot$ (invertible element)

So if $p^n \mapsto 0$ then $p^{n-1} \mapsto 0$, then $p^{n-2} \mapsto 0$, etc.

Lemma. The coproduct in Rings^δ is the tensor product (it has a canonical Frobenius derivation!). ■

So Rings^δ has limits and colimits; the forgetful functor $\mathrm{Rings}^\delta \to \mathrm{Rings}$ commutes with $\varinjlim$ and $\varprojlim$. So the forgetful functor has a left adjoint J and a right adjoint W.

Def. W is the Witt vector functor.

$W(R) = ?$ As a set, ~~$W(R)$~~ ~~$W(R) = \mathrm{Hom}_{\mathrm{Rings}}(\mathbb{Z}[x], W(R))$~~

$$W(R) = \mathrm{Hom}_{\mathrm{Rings}}(\mathbb{Z}[x], R) = \mathrm{Hom}_{\mathrm{Rings}^\delta}(J\mathbb{Z}[x], W(R)) =$$

$$= \mathrm{Hom}_{\mathrm{Rings}}(J\mathbb{Z}[x], R).$$

$J\mathbb{Z}[x] = ?$

Have $x, \delta x, \delta^2 x, \ldots \in J\mathbb{Z}[x]$.

Claim. $J\mathbb{Z}[x] = \mathbb{Z}[x, \delta x, \delta^2 x, \ldots]$.

(A similar statement for $J\mathbb{Q}[x]$ is obvious).

Lemma. Let $(A, \delta) \in \mathrm{Rings}^\delta$. Then $\exists!$ maps $\delta_0, \delta_1, \ldots : A \to A$ functorial in A such that

$$F^n(a) = \delta_0(a)^{p^n} + p\,\delta_1(a)^{p^{n-1}} + \ldots + p^n \delta_n(a).$$

Moreover, ~~the maps~~ $\delta_0(a) = a$, $\delta_1(a) = \delta(a)$ (But δ_2 is different from δ^2!)

~~Proof~~ Proof by induction. (By functoriality, we can compute $\delta_{n+1}(x)$ in the torsion-free situation.

<u>Claim.</u> $\delta_n x = \delta^n x + f(x, \delta x, \dots, \delta^{n-1} x)$.

We will identify $W(R)$ with $R^{\mathbb{Z}_{\geq 0}}$ using $\delta_n x$ rather than $\delta^n x$.

How to see the ghost components?

Went a <u>ring</u> homorphism $W(R) \to R^{\mathbb{Z}_{\geq 0}}$.

$$W(R) \quad F \circlearrowright \qquad\qquad R^{\mathbb{Z}_{\geq 0}} \quad \circlearrowright F \qquad F(r_0, r_1, \dots) = (r_1, r_2, \dots)$$

$\varepsilon : W(R) \to R$ by adjunction).

~~$F(r_0$~~ $\varepsilon : R^{\mathbb{Z}_{\geq 0}} \to R$,

$\varepsilon(r_0, r_1, \dots) = r_0$.

$\exists!\ W(R) \to R^{\mathbb{Z}_{\geq 0}}$ preserving F and ε.

Namely, $x \mapsto (\varepsilon x, \varepsilon F x, \varepsilon F^2 x, \dots)$

<u>Facts.</u> ①
$$\begin{array}{ccc} W(R) & \overset{\varepsilon}{\underset{}{\rightleftarrows}} & R \\ & & \rotatebox{90}{$\in$} \\ [r] & \longleftarrow & r \\ \| & & \\ (r, 0, 0, \dots) & & \end{array}$$
(Teichmuller representation).

This section is multiplicative!

② $V : W(R) \to W(R)$, $V(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$

Then V is additive and $FV = p$.

③ If R is over $\mathbb{F}_p$ then $F = W(\mathrm{Fr}_R)$.

In other words, $F(x_0, x_1, \dots) = (x_0^p, x_1^p, \dots)$.

In particular, $VF = FV$ for $\mathbb{F}_p$-algebras.

<u>Corollary.</u> Let R be an $\mathbb{F}_p$-~~torsion~~ algebra. Then

$W(R)$ is $\mathbb{Z}_p$-flat $\Leftrightarrow$ Fr_R is a monomorphism (i.e., R is reduced).

The map $W(R)/pW(R) \to R$ is an isomorphism $\Leftrightarrow$ R is semiperfect (i.e., Fr_R is surjective).

<u>Semiperfect rings and Fontaine's perfectization.</u>

$R/\mathbb{F}_p$, $\tilde{R} \to R$ universal map from a perfect ring to R. Namely, $\tilde{R} := \varprojlim(\dots \xrightarrow{\mathrm{Fr}} R \xrightarrow{\mathrm{Fr}} R)$.

<u>Examples.</u> (i) k perfect field.

$k[x^{1/p^\infty}]$ is the "traditional" (not Fontaine) perfectization of $k[x]$. Now set $R := k[x^{1/p^\infty}]/(x)$.

$\tilde{R} = ?$ Have $k[x^{1/p^\infty}] \to \tilde{R}$.

In fact, $\tilde{R}$ is the completion of $k[x^{1/p^\infty}]$ ~~wrt.~~ w.r.t. the ideals (x^n).

Elements of $\tilde{R}$ have the form $\sum\limits_{i \in \mathbb{Z}_+[p^{-1}]} a_i x^i$, where $\{i \mid a_i \neq 0\}$ is discrete in $\mathbb{R}$.

The above Example is a piece of the following one:

(ii) k perfect field, char $k = p$, $K := \operatorname{Frac} W(k)$.

$\overline{K} \supset \overline{W(k)}$. Let $\mathbb{C}$ be the completion of $\overline{K}$.

Set $R := \overline{W(k)}/(p)$; it is semiperfect.

$\tilde{R} :=$ Fontaine's perfectization.

<u>Def.</u> $\mathbb{C}^\flat := \mathrm{Frac}\, \tilde{R}$. ("tilt" of $\mathbb{C}$).

Note: the map ~~$\tilde{R} :=$~~ $\varprojlim (\dots \to \mathcal{O}_C \xrightarrow{x \mapsto x^p} \mathcal{O}_C) \to \tilde{R}$ is bijective. So $\varprojlim (\dots \to \mathcal{O}_C \to \mathcal{O}_C)$ is a ring.

The valuation on $\mathcal{O}_{\mathbb{C}}$ yield a valuation of $\tilde{R}$. (These valuations take value in $\mathbb{Q}$).

<u>Thm.</u> (i) $\mathbb{C}^\flat$ is an alg. closed field complete w.r.t. the valuation.

(ii) Let $\mathbb{Q}_p(1)$ be the Tate module of $\mathbb{C}$. Then we have (an embedding $\mu_{p^\infty} = (\mathbb{Q}_p/\mathbb{Z}_p)(1) \hookrightarrow R^\times$ and therefore an embedding) $\mathbb{Q}_p(1) \hookrightarrow \tilde{R}^\times$.

Let $\varepsilon \in \mathbb{Z}_p(1)$ be a generator, (so $\varepsilon \in \mathrm{Ker}(\tilde{R}^\times \twoheadrightarrow R^\times)$). Then we have $k((\varepsilon-1)) \hookrightarrow \mathbb{C}^\flat$. This subfield of $\mathbb{C}^\flat$ doesn't depend on the choice of ε.

(iii) ~~$k((\varepsilon-1))$~~ $\mathbb{C}^\flat$ is the completion of the algebraic closure of $k((\varepsilon-1))$.

(iv). Let $G := \mathrm{Gal}(\overline{K}/K)$, $\chi : G \to \mathbb{Z}_p^\times$ cyclotomic character.

Then $(\mathbb{C}^\flat)^{\mathrm{Ker}\,\chi} = \widehat{k((\varepsilon-1))^{\mathrm{perf}}}$, so

$$\mathrm{Ker}\ \chi \xrightarrow{\sim} \mathrm{Gal}\left(k((\varepsilon-1))^{\mathrm{sep}} / k((\varepsilon-1)) \right).$$

<u>Side remark.</u> (the "field of norms"). Let $E \subset \overline{K}$ be a Galois extension of K such that

(i) k_E/k is finite, where k_E is the residue field

(ii) $\mathrm{Gal}(E/K)$ is an infinite p-adic Lie group.

Then one defines a natural subfield $\mathcal{N}_E < \mathbb{C}^\flat$ such that $\mathcal{N}_E \simeq k_E((t))$, G preserves $\mathcal{N}_E$ and acts on it through $Gal(E/K)$.
For details, see the 1983 article by Wintenberger (which goes back to his joint works with Fontaine).

The ring $A_{inf} = A_{inf}(R)$, where R is semiperfect.

Def. $A_{inf}(R) := W(\tilde{R})$. ($A_{inf}$ is the universal pro-nilpotent thickening of R, see p. 7.) By definition, we have $F : A_{inf}(R) \to A_{inf}(R)$.

Examples. (i) $R = k[x^{1/p^\infty}]/(x)$.

Then we have $[x] \in A_{inf}$, and

$$A_{inf} = \Big\{ \sum_{i \in \mathbb{Z}_+[\frac{1}{p}]} a_i [x]^i \;\Big|\; a_i \in W(k), \text{ satisfying the following condition} \Big\}.$$

Condition: the set $\{(v(a_i), i)\} \subset \mathbb{Z}_+ \times \mathbb{Z}_+[\frac{1}{p}]$ is discrete in $\mathbb{R}^2$.

Equivalently: $\forall n$ $\{ i \mid a_i \not\equiv 0 \bmod p^n \}$ is discrete in $\mathbb{R}$.

Note that $\dim A_{inf} = 2$.

Maps $A_{inf} := W(\tilde{R}) \twoheadrightarrow \tilde{R} =: O_{\mathbb{C}^\flat}$, $A_{inf} = W(\tilde{R}) \twoheadrightarrow W(k)$, $A_{inf} \to O_{\mathbb{C}}$ (Because A_{inf} is the universal thickening of $O_{\mathbb{C}}/p$)

(ii) $R = O_{\mathbb{C}}/p$. The 3 natural divisors in $\operatorname{Spec} A_{inf}$ (one of them is not a Cartier divisor):

$$\begin{tikzcd}
 & O_{\mathbb{C}} \arrow[dr] & \\
A_{inf} \arrow[ur] \arrow[r, two heads] \arrow[dr] & O_{\mathbb{C}^\flat} \arrow[r] & k \\
 & W(k) \arrow[ur] &
\end{tikzcd}$$

not Cartier divisor (the ideal is not principal)

$\operatorname{Ker}(A_{inf} \to O_{\mathbb{C}^\flat}) = (p)$

$\operatorname{Ker}(A_{inf} \to O_{\mathbb{C}}) = (\omega)$,

where $\omega = \dfrac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1} = \sum_{i=0}^{p-1} [\varepsilon^{i/p}]$

$F : A_{inf} \to A_{inf}$ preserves ~~the ker~~ the quotients $O_{\mathbb{C}^\flat}$ and $W(k)$ but not $O_{\mathbb{C}}$.

The local field at the top divisor $\operatorname{Spec} O_C$ is denoted by B_{dR}. As usual, "local field at divisor" means "completion of the field of fraction w.r.t. the valuation corresponding to the divisor".
Why "inf"? inf = infinitesimal. More precisely:

<u>Lemma</u>. The map $A_{inf} \twoheadrightarrow R$ is a ~~very~~ universal (pro)nilpotent thickening of R: given an epimorphism $T \to R$ with nilpotent kernel there is a unique homomorphism $A_{inf} \to T$ such that the diagram commutes.

<u>The ring $A_{crys} = A_{crys}(R)$.</u>

$I := \operatorname{Ker}(A_{inf} \twoheadrightarrow R)$. Inside $A_{inf}[\frac{1}{p}]$ consider the ring generated by the divided powers of I, then take the p-adic completion. Equivalently, for each $x \in I$ we add $f(x)$, $f^2(x)$, $f^3(x)$, where $f(x) := \frac{x^p}{p}$. ~~(So the~~

(In other words, we force the Frobenius to be trivial modulo p.).

Note that ~~if I i~~ it is enough to take x to be ~~the~~ elements of some system of generators of I ~~; moreover~~, (or even generators modulo p).

Explanation of the notation "crys":
A_{crys} is the universal p-adic divided power thickening of R. So $A_{crys} = \Gamma((\operatorname{Spec} R/W(R))_{crys}, \mathcal{O})$.