

~~Nov~~ November 6, 2017. Beilinson's talk
 (preparatory for Bhatt's talk on Nov. 13).

A commutative ring. ^{The following approach to Witt vectors goes back to Joyal (1985)}

$F: A \rightarrow A$ is a Frobenius lifting if $F(a) \equiv a^p \pmod{pA}$.

If A has no p -torsion define the Frobenius derivative

$$S: A \rightarrow A, \quad S(a) := \frac{F(a) - a^p}{p}. \quad \text{Then}$$

$$(i) \quad S(a+b) = S(a) + S(b) + \sum_{i=1}^{p-1} \binom{p-1}{i} a^i b^{p-i}$$

$$(ii) \quad S(ab) = a^p S(b) + S(a) b^p + p S(a) S(b), \quad S(1) = 0.$$

Def. For any ring A , a map $S: A \rightarrow A$ satisfying (i)-(ii) is a Frobenius derivation.
 A pair (A, S) is a S -ring.

If $S: A \rightarrow A$ is a Frobenius derivation define $F: A \rightarrow A$, $F(a) := a^p + pS(a)$; then F is a Frobenius lifting. ^{Notation:} Rings S is the category of pairs (A, S) .

~~Rings~~ Example. $\mathbb{Z} \in \text{Rings } S$, $\mathbb{Z}_p \in \text{Rings } S$. Here $S(x) = (x^p - x)/p$.

Lemma. If A is over $\mathbb{Z}/p^n\mathbb{Z}$ then it ~~is~~ cannot carry a Frobenius derivative.

Proof. ~~$\mathbb{Z} \rightarrow A$~~ If (A, S) is a S -ring then $\mathbb{Z} \rightarrow (A, S)$ has to be a S -homomorphism. So

$$S(p^n) = p^{n-1} \cdot \frac{p^n - p^{np}}{p} = p^{n-1} \cdot p^{n-1} \cdot (\text{invertible element})$$

So if $p^n \mapsto 0$ then $p^{n-1} \mapsto 0$, then $p^{n-2} \mapsto 0$, etc.

Lemma. The coproduct in Rings S is the tensor product (it has a canonical Frobenius derivation!). \blacksquare

(-2-)

So $\text{Rings}^{\mathcal{S}}$ has limits and colimits; the forgetful functor $\text{Rings}^{\mathcal{S}} \rightarrow \text{Rings}$ commutes with \varinjlim and \varprojlim . So the forgetful functor has a left adjoint J and a right adjoint W .

Def. W is the Witt vector functor.

$$\begin{aligned} \text{As a set, } W(R) &= \text{Hom}_{\text{Rings}}(\mathbb{Z}[x], W(R)) \\ W(R) &= \varprojlim_{\text{Rings}^{\mathcal{S}}} \text{Hom}_{\text{Rings}}(\mathbb{Z}[x], R) \end{aligned}$$

$$= \text{Hom}_{\text{Rings}}(J\mathbb{Z}[x], R).$$

$$J\mathbb{Z}[x] = ?$$

Have $x, \mathcal{S}x, \mathcal{S}^2x, \dots \in J\mathbb{Z}[x]$.

Claim. $J\mathbb{Z}[x] = \mathbb{Z}[x, \mathcal{S}x, \mathcal{S}^2x, \dots]$.

(A similar statement for $J\mathbb{Q}[x]$ is obvious).

Lemma. Let $(A, \mathcal{S}) \in \text{Rings}^{\mathcal{S}}$. Then $\exists!$ maps $\mathcal{S}_0, \mathcal{S}_1, \dots : A \rightarrow A$ such that

$$F^n(a) = \mathcal{S}_0(a)p^n + p\mathcal{S}_1(a)p^{n-1} + \dots + p^n\mathcal{S}_n(a).$$

Moreover, $\mathcal{S}_0(a) = a$ and $\mathcal{S}_1(a) = \mathcal{S}(a)$ (but \mathcal{S}_2 is different from \mathcal{S}^2 !).

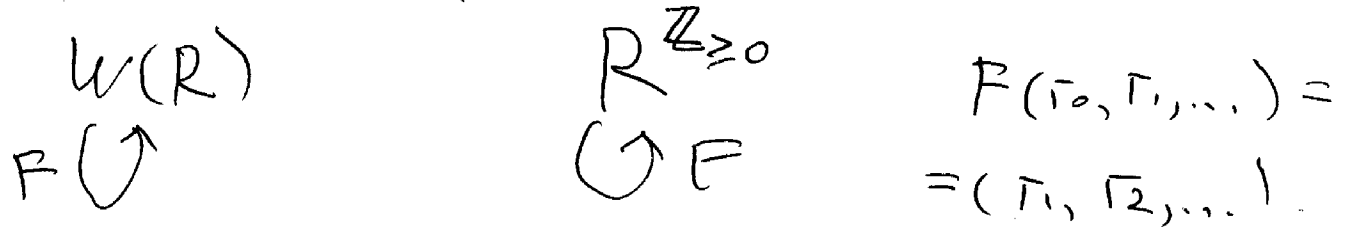
Proof. Proof by induction. (By functoriality, we can compute $\mathcal{S}_{2n+1}(x)$ in the torsion-free situation.)

Claim. $S_n x = S^n x + f(x, Sx, \dots, S^{n-1}x)$.

We will identify $W(R)$ with $R^{\mathbb{Z}_{\geq 0}}$ using $S_n x$ rather than $S^n x$.

How to see the ghost components?

Want a ring homomorphism $W(R) \rightarrow R^{\mathbb{Z}_{\geq 0}}$.



$\varepsilon: W(R) \rightarrow R$ by adjunction.

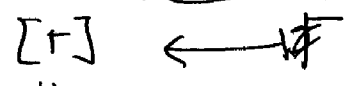
~~$F(\tau_0)$~~ $\varepsilon: R^{\mathbb{Z}_{\geq 0}} \rightarrow R$,

$\varepsilon(\tau_0, \tau_1, \dots) = \tau_0$.

$\exists!$ $W(R) \rightarrow R^{\mathbb{Z}_{\geq 0}}$ preserving F and ε .

Namely, $x \mapsto (\varepsilon x, \varepsilon Fx, \varepsilon F^2x, \dots)$

Facts. ① $W(R) \xrightarrow{\varepsilon} R$



(Teichmüller representative).

This section is multiplicative!

② $V: W(R) \rightarrow W(R)$, $V(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$

Then V is additive and $FV = p$.

③ If R is over \mathbb{F}_p then $F = W(FR)$.

In other words, $F(x_0, x_1, \dots) = (x_0^p, x_1^p, \dots)$.

In particular, $VF = FV$ for F_p -algebras.

Corollary. Let R be an F_p -~~algebra~~. Then

$W(R)$ is \mathbb{Z}_p -flat $\iff F_{\Gamma_R}$ is a monomorphism (i.e., R is reduced).

The map $W(R)/pW(R) \rightarrow R$ is an isomorphism $\iff R$ is semiperfect (i.e., F_{Γ_R} is surjective).

Semiperfect rings and Fontaine's perfectization.

R/F_p , $\tilde{R} \rightarrow R$ universal map from a perfect ring to R . Namely, $\tilde{R} := \varprojlim_{\leftarrow} (R \xrightarrow{F} R \xrightarrow{F} R)$.

Examples. (i) \mathbb{K} perfect field.

$\mathbb{K}[x^{1/p^\infty}]$ is the "traditional" (not Fontaine) perfectization of $\mathbb{K}[x]$. Now set $R := \mathbb{K}[x^{1/p^\infty}] \setminus (x)$.

$\tilde{R} = ?$ Have $\mathbb{K}[x^{1/p^\infty}] \rightarrow \tilde{R}$.

~~\tilde{R}~~ In fact, \tilde{R} is the completion of $\mathbb{K}[x^{1/p^\infty}]$ w.r.t. w.r.t. the ideals (x^n) .

Elements of \tilde{R} have the form $\sum a_i x^i$, $i \in \mathbb{Z}_+ [p^{-1}]$

where $\{i \mid a_i \neq 0\}$ is discrete in \mathbb{R} .
The above Example is a piece of the following one:

(ii) \mathbb{K} perfect field, char $\mathbb{K} = p$, $K := \text{Frac } W(\mathbb{K})$.

$\bar{K} \supset \overline{W(\mathbb{K})}$. Let \mathbb{C} be the completion of \bar{K} .

Set $R := \overline{W(\mathbb{K})} / (p)$; it is semiperfect.

$\tilde{R} :=$ Fontaine's perfectization.

Def. $\mathbb{C}^b :=$ Free \tilde{R} . ("tilt" of \mathbb{C}).

Note: ~~\tilde{R}~~ $\lim_{\leftarrow} (\dots \rightarrow \mathcal{O}_{\mathbb{C}} \xrightarrow{x \mapsto x^p} \mathcal{O}_{\mathbb{C}}) \rightarrow \tilde{R}$ is bijective. So $\lim_{\leftarrow} (\dots \rightarrow \mathcal{O}_{\mathbb{C}} \xrightarrow{x \mapsto x^p} \mathcal{O}_{\mathbb{C}})$ is a ring.

The valuation on $\mathcal{O}_{\mathbb{C}}$ yield a valuation of \tilde{R} . (These valuations take value in \mathbb{Q}).

Thm. (i) \mathbb{C}^b is an alg. closed field complete w.r.t. the valuation.

(ii) Let $\mathbb{Q}_p(1)$ be the Tate module of \mathbb{C} .
 (on embedding $\chi_{\text{cyc}} = (\mathbb{Q}_p/\mathbb{Z}_p)(1) \hookrightarrow \mathbb{R}^*$ and therefore an embedding $\mathbb{C}^* \hookrightarrow \mathbb{R}^*$)
 Then ~~we~~ we have $\mathbb{Q}_p(1) \hookrightarrow \mathbb{R}^*$.
 (so $\exists \epsilon \in \ker(\mathbb{R}^* \rightarrow \mathbb{R}^*)$).

Let $\epsilon \in \mathbb{Z}_p(1)$ be a generator, Then we have $k((\epsilon-1)) \hookrightarrow \mathbb{C}^b$. This subfield of \mathbb{C}^b doesn't depend on the choice of ϵ .

(iii) $k((\epsilon-1)) \subset \mathbb{C}^b$ is the completion of the algebraic closure of $k((\epsilon-1))$.

(iv). Let $G := \text{Gal}(\bar{K}/K)$, $\chi: G \rightarrow \mathbb{Z}_p^*$ cyclotomic character.

Then $(\mathbb{C}^b)^{\text{Ker } \chi} = \widehat{k((\epsilon-1))}^{\text{perf}}$, so
 $\text{Ker } \chi \cong \text{Gal}(k((\epsilon-1))^{\text{sep}} / k((\epsilon-1)))$.

Side remark. (the "field of norms"). ~~Let~~ Let $E \subset \bar{K}$ be a Galois extension of K such that

- (i) k_E/k is finite, where k_E is the residue field
- (ii) $\text{Gal}(E/K)$ is an ^{infinite} p -adic Lie group.

Then one defines a natural subfield $\mathcal{W}_E \subset \mathbb{C}^b$ such that $\mathcal{W}_E \cong k_E((t))$, G preserves \mathcal{W}_E and acts on it through $\text{Gal}(E/k)$.

For details, see the 1983 article by Wintenberger (which goes back to his joint works with Fontaine).

The ring $A_{\text{inf}} = A_{\text{inf}}(R)$, where R is semiperfect.

Def. $A_{\text{inf}}(R) = W(R)$. (A_{inf} is the universal pro-nilpotent thickening of R , see p.7.) By definition, we have $F: A_{\text{inf}}(R) \rightarrow A_{\text{inf}}(R)$.

Examples. (i) $R = k[x^{1/p^\infty}]/(x)$.

Then we have $[x] \in A_{\text{inf}}$, and

$$A_{\text{inf}} = \left\{ \sum_{i \in \mathbb{Z}[\frac{1}{p}]} a_i [x]^i \mid a_i \in W(R), \text{ satisfying the following condition} \right\}$$

Condition: the set $\{(v(a_i), i)\} \subset \mathbb{Z}_+ \times \mathbb{Z}_+[\frac{1}{p}]$ is discrete in \mathbb{R}^2 .

Equivalently: $\{i \mid a_i \not\equiv 0 \pmod{p^N}\}$ is discrete in \mathbb{R} .

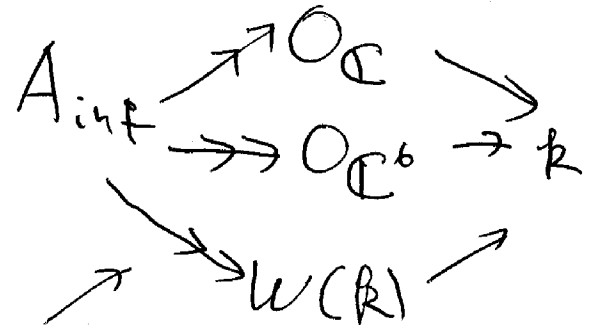
Note that $\dim A_{\text{inf}} = 2$.

(Because A_{inf} is the universal thickening of $O_{\mathbb{C}/p}$)

Maps $A_{\text{inf}} := W(R) \rightarrow \tilde{R} := O_{\mathbb{C}^b}$, $A_{\text{inf}} := W(R) \rightarrow W(R)$, $A_{\text{inf}} \rightarrow O_{\mathbb{C}}$

(ii) $R = O_{\mathbb{C}}/p$. The 3 natural divisors

in $\text{Spec } A_{\text{inf}}$ (one of them is not a Cartier divisor):



$$\text{Ker}(A_{\text{inf}} \rightarrow O_{\mathbb{C}^b}) = (p)$$

$$\text{Ker}(A_{\text{inf}} \rightarrow O_{\mathbb{C}}) = (\omega)$$

$$\text{where } \omega = \frac{[t]-1}{[t^{1/p}]-1} = \sum_{i=0}^{p-1} [t^{i/p}]$$

not Cartier divisor (the ideal is not principal)

$F: A_{\text{inf}} \rightarrow A_{\text{inf}}$ preserves the quotients $O_{\mathbb{C}^b}$ and $W(R)$ but not $O_{\mathbb{C}}$.

The local field at the top divisor is denoted by B_{dR} . As usual, "local field at divisor" means "completion of the field of fractions w.r.t. the valuation corresponding to the divisor". Why "inf"? inf = infinitesimal. More precisely:

Lemma. The map $A_{inf} \twoheadrightarrow R$ is a ~~via~~ universal (pro)nilpotent thickening of R : given an epimorphism $T \rightarrow R$ with nilpotent kernel, there is a unique homomorphism $A_{inf} \rightarrow T$ such that the diagram commutes.

The ring $A_{crys} = A_{crys}(R)$.

$I := \text{Ker}(A_{inf} \twoheadrightarrow R)$. Inside $A_{inf}[\frac{1}{p}]$ consider the ~~ring~~ ^{ring generated by the} divided powers of I , then take the p -adic completion. Equivalently, for each $x \in I$ we add $f(x), f^2(x), f^3(x), \dots$ where $f(x) := \frac{x^p}{p}$. ~~(So the~~

(In other words, we force the Frobenius to be trivial modulo p).

Note that ~~if I~~ it is enough to take x to be ~~the~~ ^{elements of some system of} generators of I ; ~~moreover,~~ (or even generators modulo p).

Explanation of the notation "crys": A_{crys} is the universal p -adic divided power thickening of R . So $A_{crys} = \Gamma(\text{Spec } R/w(R), \mathcal{O}_{crys})$.