NOTES ON GEOMETRIC LANGLANDS: GENERALITIES ON DG CATEGORIES

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Introduction

The purpose of this paper is to set up a language that will be used in the rest of the notes pertaining to DG categories and functors between them. A more complete treatment can be found in Lurie's book and the DAGs.

1. DG categories

In this section we'll be ignoring set-theoretical difficulties. Once the proper set-theoretic context is reinstated, the lemmas stated without proofs here can be found in Chapters 4 and 5 of [Lu1].

Unless specified otherwise, we'll be working with DG categories over the ground field k that are co-complete. Any such category is automatically closed under limits. No harm will be done if the reader perceives the word "DG category" as a stable ∞-category enriched over the category of complexes of k-vector spaces that contains all colimits.

In what follows we'll denote by Vect the unit DG category, i.e., one of complexes of k-vector spaces.

1.1. **Continuous vs. all functors.** For $\mathbf{C}_1$ and $\mathbf{C}_2$ as we consider the appropriately defined DG-category $\text{Funct}(\mathbf{C}_1, \mathbf{C}_2)$ of k-linear functors $\mathbf{C}_1 \to \mathbf{C}_2$, see e.g. [Dr], Sect. 16.8.

Remark. We note that $\text{Funct}(\mathbf{C}_1, \mathbf{C}_2)$ is *not* the category of naive DG-functors $\mathbf{C}_1 \to \mathbf{C}_2$, but rather one whose homotopy category is that of quasi-functors in the terminology of [Dr], Sect. 16.1 or homotopy functors in the terminology of [FG], Sects. 15.2 and 15.3.

We let $\text{Funct}_{\text{cont}}(\mathbf{C}_1, \mathbf{C}_2)$ be the full subcategory of $\text{Funct}(\mathbf{C}_1, \mathbf{C}_2)$ spanned by functors that commute infinite with direct sums (at the homotopy level). Equivalently, these are functors that commute with all colimits (here and elsewhere, by a "limit" and "colimit" we understand homotopy limit and colimit, respectively).

Both $\text{Funct}_{\text{cont}}(\mathbf{C}_1, \mathbf{C}_2)$ and $\text{Funct}(\mathbf{C}_1, \mathbf{C}_2)$ are co-complete. Note that limits and colimits in $\text{Funct}(\mathbf{C}_1, \mathbf{C}_2)$ are computed object-wise. This implies, in particular, that

$$\text{Funct}_{\text{cont}}(\mathbf{C}_1, \mathbf{C}_2) \subset \text{Funct}(\mathbf{C}_1, \mathbf{C}_2)$$

is stable under colimits, and that this embedding admits a right adjoint (see Lemma 1.1.1 below). However, this embedding does not usually commute with limits.

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Lemma 1.1.1. *For* $\mathbf{C}_1, \mathbf{C}_2$ *as above, any* $F \in \text{Funct}_{\text{cont}}(\mathbf{C}_1, \mathbf{C}_2)$ *admits a right adjoint in* $\text{Funct}(\mathbf{C}_2, \mathbf{C}_1)$*. The full subcategory of* $(\text{Funct}(\mathbf{C}_2, \mathbf{C}_1))^{\text{op}}$ *obtained as right adjoints of objects from* $\text{Funct}_{\text{cont}}(\mathbf{C}_1, \mathbf{C}_2)$ *is*

$$\text{Funct}_{cocont}(\mathbf{C}_2, \mathbf{C}_1) \subset \text{Funct}(\mathbf{C}_2, \mathbf{C}_1),$$

which consists of functors that commute with limits.

1.2. **The 2-category of DG categories.** We'd like to view the totality of DG categories as an $(\infty, 2)$-category in 2 ways, denoted $\text{DGCat}_{\text{cont}}$ and DGCat, respectively, where in both cases the objects are DG categories, and the 1-morphisms are

$$\text{Funct}_{\text{cont}}(\mathbf{C}_1, \mathbf{C}_2) \text{ and } \text{Funct}(\mathbf{C}_1, \mathbf{C}_2),$$

respectively.

For the most part, however, we'll be working with $\text{DGCat}_{\text{cont}}$. In this case, we'll also use the notation

$$\text{Hom}(\mathbf{C}_1, \mathbf{C}_2) := \text{Funct}_{\text{cont}}(\mathbf{C}_1, \mathbf{C}_2).$$

1.2.1. The trouble is, however, that the theory of $(\infty, 2)$-categories hasn't been adequately documented at the time of writing. The paper [Lu3] develops the notion of $(\infty, 2)$-category, but doesn't show that DG categories and its variants considered in the sequel form a $(\infty, 2)$-category.

A possible way out is as follows: for most applications (such as computation of limits and colimits), it would be sufficient to view $\text{DGCat}_{\text{cont}}$ and DGCat as just ∞-categories (i.e., $(\infty, 1)$-categories), by discrading non-invertible 2-morphisms, i.e., by considering as 1-morphisms the corresponding maximal sub-groupoids

$$\text{Funct}^{\circ}_{\text{cont}}(\mathbf{C}_1, \mathbf{C}_2) \subset \text{Funct}_{\text{cont}}(\mathbf{C}_1, \mathbf{C}_2) \text{ and } \text{Funct}^{\circ}(\mathbf{C}_1, \mathbf{C}_2) \subset \text{Funct}(\mathbf{C}_1, \mathbf{C}_2).$$

1.2.2. By definition, when considering $\text{DGCat}_{\text{cont}}$ or DGCat as an $(\infty, 1)$-category, we only consider natural transformations that are isomorphisms. However, one can recover all natural transformations as follows.

Namely, let $(0 \to 1)$ be the DG-category generated by two objects 0 and 1 with a unique arrow $0 \to 1$. Then for $\mathbf{C}_1$ and $\mathbf{C}_2$ as above, we can consider $\text{Funct}^{\circ}_{\text{cont}}(\mathbf{C}_1 \otimes (0 \to 1), \mathbf{C}_2)$. The fibers of the map

$$\text{Funct}^{\circ}_{\text{cont}}(\mathbf{C}_1 \otimes (0 \to 1), \mathbf{C}_2) \to \text{Funct}^{\circ}_{\text{cont}}(\mathbf{C}_1, \mathbf{C}_2) \times \text{Funct}^{\circ}_{\text{cont}}(\mathbf{C}_1, \mathbf{C}_2)$$

corresponding to the two functors $\text{Vect} \rightrightarrows (0 \to 1)$, over a given pair $F', F'' \in \text{Funct}^{\circ}_{\text{cont}}(\mathbf{C}_1, \mathbf{C}_2)$ is the groupoid of all natural transformations $F' \Rightarrow F''$.

A similar remark applies to $\text{DGCat}_{\text{cont}}$ replaced by DGCat.

1.3. **Limits and colimits in** $\text{DGCat}_{\text{cont}}$ **and** DGCat**.**

Lemma 1.3.1.

(1) *The* ∞*-categories* $\text{DGCat}_{\text{cont}}$ *and* DGCat *admit limits and colimits.*

(2) *The forgetful functor* $\text{DGCat}_{\text{cont}} \to \text{DGCat}$ *commutes with limits.*

This is done in [Lu1, Sect. 5.5.3].

Note, however, that the forgetful functor $\text{DGCat}_{\text{cont}} \to \text{DGCat}$ does not commute with colimits.

1.3.2. *A lemma on colimits.* Let I be an ∞-category. Let $i \mapsto \mathbf{C}_i$ be a functor $I \to \mathrm{DGCat}_{\mathrm{cont}}$. For $i, j \in I$ and $\alpha : i \to j$ let us denote the corresponding functor $F_\alpha : \mathbf{C}_i \to \mathbf{C}_j$.

By Lemma 1.1.1, each of the the functors F_α admits a right adjoint in $\mathrm{Funct}(\mathbf{C}_j, \mathbf{C}_i)$, denoted G_α. Thus, we obtain a functor $I^{\mathrm{op}} \to \mathrm{DGCat}$.

The following assertion is, to be the best of our knowledge, due to J. Lurie:

Lemma 1.3.3.
(a) *For every $j \in I$, the tautological evaluation functor*

$$\mathrm{ev}_j : \underset{i \in I^{\mathrm{op}}, G}{lim} \mathbf{C}_i \to \mathbf{C}_j$$

admits a left adjoint, to be denoted $'\mathrm{ins}_j$.
(b) *The functor*

$$\underset{j \in I, F}{colim} \mathbf{C}_j \to \underset{i \in I^{\mathrm{op}}, G}{lim} \mathbf{C}_i,$$

corresponding to the system of functors

$$'\mathrm{ins}_j : \mathbf{C}_j \to \underset{i \in I^{\mathrm{op}}, G}{lim} \mathbf{C}_i,$$

is an equivalence of categories.

Proof. Let $\mathbf{D}$ be a DG category. By Lemma 1.1.1, maps in $\mathrm{DGCat}_{\mathrm{cont}}$

$$\Phi : \underset{i \in I^{\mathrm{op}}, G}{lim} \mathbf{C}_i \to \mathbf{D}$$

are in bijection with maps in DGCat

$$\Psi : \mathbf{D} \to \underset{i \in I^{\mathrm{op}}, G}{lim} \mathbf{C}_i$$

that commute with limits. The latter are the same as systems of maps in DGCat,

$$\Psi_i : \mathbf{D} \to \mathbf{C}_i,$$

each commuting with limits, and such that $G_\alpha \circ \Psi_j \simeq \Psi_i$ for $\alpha : i \to j$. The latter are the the same as systems of maps in $\mathrm{DGCat}_{\mathrm{cont}}$

$$\Phi_i : \mathbf{C}_i \to \mathbf{D}$$

satisfying

$$\Phi_j \circ F_\alpha \simeq \Phi_i,$$

which is the same as maps in $\mathrm{DGCat}_{\mathrm{cont}}$

$$\underset{i \in I, F}{colim} \mathbf{C}_i \to \mathbf{D}.$$

□

1.3.4. One can right down the functor

$$\underset{i \in I^{\mathrm{op}}, G}{lim} \mathbf{C}_i \to \underset{j \in I, F}{colim} \mathbf{C}_j$$

in Lemma 1.3.3 more explicitly.

Namely, it equals

$$\underset{i \in I}{colim}\ \mathrm{ins}_i \circ \mathrm{ev}_i,$$

where ins_i denotes the tautological functor

$$\mathbf{C}_i \to \underset{j \in I, F}{colim} \mathbf{C}_j,$$

1.3.5. Assume that in the above setting the functors G_α also belong to $\text{Funct}_{\text{cont}}(\mathbf{C}_j, \mathbf{C}_i)$. Assume also that for every diagram $A = i_1 \leftarrow j \rightarrow i_2$, the category $I_{A/}$ is contractible. (This happens, e.g., when I is filtered.)

Lemma 1.3.6. *Under the above circustamstances, the functor*

$$\mathbf{C}_{j_0} \overset{\text{ins}_{j_0}}{\longrightarrow} \underset{j\in I, G}{colim}\, \mathbf{C}_j \simeq \underset{i \in I^{\text{op}}, G}{lim}\, \mathbf{C}_i \overset{\text{ev}_{i_0}}{\longrightarrow} \mathbf{C}_{i_0}$$

is canonically isomorphic to

$$\underset{k\in I, \alpha: j_0 \to k, \beta: i_0 \to k}{colim} G_\beta \circ F_\alpha,$$

where the colimit is taken in $\text{Funct}_{\text{cont}}(\mathbf{C}_{j_0}, \mathbf{C}_{i_0})$.

Proof. Consider the category

$$'\mathbf{C} := \underset{\overleftarrow{i\in I^{\text{op}}, G}}{lax.lim}\, \mathbf{C}_i$$

of *all* assignments

$$i \mapsto (\mathbf{c}_i \in \mathbf{C}_i), \quad (\gamma : i \to i') \mapsto (G_\gamma(\mathbf{c}_{i'}) \overset{\phi_\gamma}{\longrightarrow} \mathbf{c}_i \in \mathbf{C}_i), \tag{1}$$

equipped with the data of making the maps ϕ_γ coherently associative. However, the maps ϕ_γ are *not* required to be isomorphisms.

We have a fully faithful embedding

$$\mathbf{C} := \underset{\overleftarrow{i\in I^{\text{op}}, G}}{lim}\, \mathbf{C}_j \hookrightarrow \underset{\overleftarrow{i\in I^{\text{op}}, G}}{lax.lim}\, \mathbf{C}_i =: {}'\mathbf{C}$$

whose essential image consists of those objects (1), for which the maps ϕ_γ are isomorphisms.

For a given index j and $\mathbf{c}_j \in \mathbf{C}_j$, the assignment

$$i \mapsto \underset{k\in I, \alpha: j \to k, \beta: i \to k}{colim} G_\beta \circ F_\alpha \tag{2}$$

naturally upgrades to an object of $'\mathbf{C}$. Indeed, for $\gamma : i \to i'$, the corresponding map is

$$G_\gamma \left(\underset{k\in I, \alpha: j\to k, \beta': i' \to k}{colim} G_{\beta'} \circ F_\alpha(\mathbf{c}_j) \right) \simeq \underset{\overrightarrow{k\in I, \alpha: j\to k, \beta': i' \to k}}{colim} G_\gamma \circ G_{\beta'} \circ F_\alpha(\mathbf{c}_j) \simeq$$
$$\simeq \underset{\overrightarrow{k\in I, \alpha: j\to k, \beta': i' \to k}}{colim} G_{\beta' \circ \gamma} \circ F_\alpha(\mathbf{c}_j) \to \underset{\overrightarrow{k\in I, \alpha: j\to k, \beta: i \to k}}{colim} G_\beta \circ F_\alpha(\mathbf{c}_j), \tag{3}$$

where the first isomorphism results from the continuity of the functor G_γ, and the last arrow corresponds to the functor of index categories

$$\{k \in I, \alpha : j \to k, \beta' : i' \to k\} \to \{k \in I, \alpha : j \to k, \beta : i \to k\}, \tag{4}$$

given by pre-composition with γ.

It is clear that the assignment (2) defines a functor $\mathbf{C}_j \to {}'\mathbf{C}$. Denote this functor by $'\text{ins}_j$.

It follows from the construction that for $\mathbf{c}_j \in \mathbf{C}_j$ and $\mathbf{c} \in \mathbf{C} \subset {}'\mathbf{C}$, we have a canonical isomorphism

$$\text{Maps}_{'\mathbf{C}}({}'\text{ins}_j(\mathbf{c}_j), \mathbf{c}) \simeq \text{Maps}_{\mathbf{C}_j}(\mathbf{c}_j, \text{ev}_j(\mathbf{c})).$$

Hence, by the $(\text{ins}_j, \text{ev}_j)$-adjunction, it remains to show that when I satisfies the assumtion of the lemma, the essential image of the functor $'\text{ins}_j$ belongs to $\mathbf{C} \subset {}'\mathbf{C}$.

I.e., we have to show that for an arrow $\gamma : i \to i'$ in I and $\mathbf{c}_j \in \mathbf{C}_j$, the map (3) is an isomorphism. For that it is sufficient to show that the functor (4) is cofinal. I.e., we have to show that for a given

$$k \in I, \alpha : j \to k, \beta : i \to k,$$

the category

$$\{k' \in I, \delta : k \to k', \beta' : i' \to k', \delta \circ \beta \sim \beta' \circ \gamma\}$$

is contractible. However, the latter is exactly the assumption on I.

□

1.4. **Tensor products.** One of the main advantages of $\mathrm{DGCat}_{\mathrm{cont}}$ as opposed to DGCat is the monoidal structure. Namely, $\mathrm{DGCat}_{\mathrm{cont}}$ carries a canonical symmetric monoidal structure given by tensor product:

For $\mathbf{C}_1, \mathbf{C}_2 \in \mathrm{DGCat}_{\mathrm{cont}}$ is characterized by the property that $\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_1 \otimes \mathbf{C}_2, \mathbf{D})$ consists of functors

$$\mathbf{C}_1 \times \mathbf{C}_2 \to \mathbf{D}$$

that are k-linear and commute with colimits in each variable.

The basic reference for this is [Lu2, Sect. 6.3.2]. Namely, Propositions 6.3.2.18 and 6.3.2.7 define on the the category of presentable stable ∞-categories a structure of symmetric monoidal ∞-category. One then makes $\mathrm{DGCat}_{\mathrm{cont}}$ into a symmetric monoidal category by realizing it as modules over the symmetric algebra object Vect in the symmetric monoidal ∞-category of presentable stable ∞-categories.

Lemma 1.4.1. *The formation of tensor products commutes with formation of colimits in each variable.*

The proof follows, e.g., from [Lu2, Prop. 6.3.1.16] using Lemma 1.3.3.

1.5. **Fully faithful functors.** Let $\Phi : \mathbf{C}_1 \to \mathbf{C}_2$ be a continuous functor between DG categories. Assume that Φ is fully faithfil.

Question 1.5.1. *Is it true that the functor $\Phi \otimes \mathrm{Id}_{\mathbf{D}} : \mathbf{C}_1 \otimes \mathbf{D} \to \mathbf{C}_2 \otimes \mathbf{D}$ is fully faithful for any $\mathbf{D} \in \mathrm{DGCat}$?*

The answer to the above question is "yes" at least in the following cases:

(a) If Φ admits a left adjoint, or a continuous right adjoint.

(b) If $\mathbf{D}$ is dualizable (see next section).

(c) Both $\mathbf{C}_1$ and $\mathbf{C}_2$ are dualizable, and $\mathbf{D}$ is isomorphic to the limit in $\mathrm{DGCat}_{\mathrm{cont}}$ of dualizable categories.

2. Dualizable categories

The notion of dualizable category is also, as far as we know, due to J. Lurie.

2.1. **Duality datum.** Let $\mathbf{C}$ be a DG-category. We say that $\mathbf{C}$ is dualizable, of there exists another DG-category $\mathbf{C}^\vee$ endowed with morphisms in $\mathrm{DGCat}_{\mathrm{cont}}$

$$\mu : \mathrm{Vect} \to \mathbf{C}^\vee \otimes \mathbf{C} \text{ and } \epsilon : \mathbf{C} \otimes \mathbf{C}^\vee \to \mathrm{Vect},$$

satisfying the usual duality axioms, i.e., the compositions

$$\mathbf{C} \overset{\mathrm{Id}_{\mathbf{C}} \otimes \mu}{\longrightarrow} \mathbf{C} \otimes \mathbf{C}^\vee \otimes \mathbf{C} \overset{\epsilon \otimes \mathrm{Id}_{\mathbf{C}}}{\longrightarrow} \mathbf{C}$$

and

$$\mathbf{C}^\vee \overset{\mu \otimes \mathrm{Id}_{\mathbf{C}}}{\longrightarrow} \mathbf{C}^\vee \otimes \mathbf{C} \otimes \mathbf{C}^\vee \overset{\mathrm{Id}_{\mathbf{C}} \otimes \epsilon}{\longrightarrow} \mathbf{C}^\vee$$

are isomorphic to the identity functor.

2.1.1. Tautologically, one can say that $\mathbf{C}$ is dualizable if it is such as a 0-object of the $(\infty, 1)$-category $\mathrm{DGCat}_{\mathrm{cont}}$, see e.g. Sect. 5 for a reminder what it means to be a dualizable object in a monoidal category.

2.1.2. Alternatively, $\mathbf{C}$ is dualizable if there exists a DG-category $\mathbf{C}^\vee$ endowed with a pairing $\mathbf{C} \otimes \mathbf{C}^\vee \to \mathrm{Vect}$ in $\mathrm{DGCat}_{\mathrm{cont}}$ which induces an equivalence

$$\mathbf{C}^\vee \otimes \mathbf{D} \to \mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}, \mathbf{D}) \tag{5}$$

for any $\mathbf{D} \in \mathrm{DGCat}_{\mathrm{cont}}$. In particular, $\mathbf{C}^\vee$ can be identified with $\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}, \mathrm{Vect})$.

2.1.3. Equivalently, there should exist a functor $\mathrm{Vect} \to \mathbf{C}^\vee \otimes \mathbf{C}$ in $\mathrm{DGCat}_{\mathrm{cont}}$ which for any $\mathbf{D}_1, \mathbf{D}_2$ induces an equivalence

$$\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C} \otimes \mathbf{D}_1, \mathbf{D}_2) \simeq \mathrm{Funct}_{\mathrm{cont}}(\mathbf{D}_1, \mathbf{C}^\vee \otimes \mathbf{D}_2). \tag{6}$$

2.1.4. Since the tensor product on $\mathrm{DGCat}_{\mathrm{cont}}$ is symmetric, we have that $\mathbf{C}^\vee$ is the dual of $\mathbf{C}$ if and only if $\mathbf{C}$ is the dual of $\mathbf{C}^\vee$.

2.1.5. Dualizable categories enjoy nice properties regarding limits and colimits:

Lemma 2.1.6. *Let $i \mapsto \mathbf{C}_i$ be a functor $I \to \mathrm{DGCat}_{\mathrm{cont}}$. Let $\mathbf{D}$ be dualizable.*
(1) *The natural functor*

$$\mathbf{D} \otimes \underset{I}{lim}\, \mathbf{C}_i \to \underset{I}{lim}\, (\mathbf{D} \otimes \mathbf{C}_i)$$

is an equivalence.
(2) *The natural functor*

$$\underset{I}{colim}\, \mathrm{Funct}_{\mathrm{cont}}(\mathbf{D}, \mathbf{C}_i) \to \mathrm{Funct}_{\mathrm{cont}}(\mathbf{D}, \underset{I}{colim}\, \mathbf{C}_i)$$

is an equivalence, i.e., $\mathbf{D}$ is compact as an object of $\mathrm{DGCat}_{\mathrm{cont}}$ (the colimit in LHS is taken within $\mathrm{DGCat}_{\mathrm{cont}}$).

Proof. For point (1), the LHS can be rewritten as

$$\mathrm{Funct}_{\mathrm{cont}}(\mathbf{D}^\vee, \underset{I}{lim}\, \mathbf{C}_i) \simeq \underset{I}{lim}\, \mathrm{Funct}_{\mathrm{cont}}(\mathbf{D}^\vee, \mathbf{C}_i),$$

which is equivalent to the RHS.

Point (2) follows from equation (5).

□

2.2. **Dual functors.** If $\mathbf{C}_1, \mathbf{C}_2$ are dualizable, there exists a canonical equivalence

$$\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_1, \mathbf{C}_2) \simeq \mathbf{C}_1^\vee \otimes \mathbf{C}_2 \simeq \mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_2^\vee, \mathbf{C}_1^\vee),$$

which we'll denote $F \mapsto F^\vee$.

2.2.1. *Limits and duals.* Let us return to the set-up of Sect. 1.3.5, and let us assume that the functors G_α belong to $\mathrm{DGCat}_{\mathrm{cont}}$. Assume that each $\mathbf{C}_i$ is dualizable. In this case, the data of $G_\alpha : \mathbf{C}_j \to \mathbf{C}_i$ gives rise to yet another functor $I \to \mathrm{DGCat}_{\mathrm{cont}}$, namely, $i \mapsto \mathbf{C}_i$ and $(\alpha : i \to j) \mapsto G_\alpha^\vee$.

Lemma 2.2.2. *Under the above circumstances,* $\underset{i\in I,F}{colim}\, \mathbf{C}_i$ *is dualizable, and its dual is canonically equivalent to* $\underset{i\in I,G^\vee}{colim}\, \mathbf{C}_i^\vee$. *This equivalence is uniquely characterized by the property that for* $i_0 \in I$, *we have*

$$(\mathrm{ins}_{i_0,G^\vee})^\vee \simeq \mathrm{ev}_{i_0,G}, \tag{7}$$

in a way compatible with arrows in I.

In formula (7), the notation $\mathrm{ins}_{i_0,G^\vee}$ means the functor

$$\mathrm{ins}_{i_0} : \mathbf{C}_{i_0}^\vee \to \underset{i\in I,G^\vee}{\underset{\longrightarrow}{colim}}\, \mathbf{C}_i^\vee,$$

and the notation $\mathrm{ev}_{i_0,G}$ means the functor

$$\mathrm{ev}_{i_0} : \underset{I^{\mathrm{op}},G}{\underset{\longleftarrow}{lim}}\, \mathbf{C}_i \to \mathbf{C}_{i_0}.$$

Proof. First, let construct a pairing

$$\left(\underset{i\in I,G^\vee}{\underset{\longrightarrow}{colim}}\, \mathbf{C}_i^\vee\right) \otimes \left(\underset{i\in I^{\mathrm{op}},G}{\underset{\longleftarrow}{lim}}\, \mathbf{C}_i\right) \to \mathrm{Vect}\,. \tag{8}$$

To do this, it is sufficient to specify the corresponding pairings

$$\mathbf{C}_{i_0}^\vee \otimes \left(\underset{i\in I^{\mathrm{op}},G}{\underset{\longleftarrow}{lim}}\, \mathbf{C}_i\right) \to \mathrm{Vect}, \quad i_0 \in I, \tag{9}$$

compatible with the functors $G_\alpha^\vee$. The pairing (9) is given by

$$\mathbf{C}_{i_0}^\vee \otimes \left(\underset{i\in I^{\mathrm{op}},G}{\underset{\longleftarrow}{lim}}\, \mathbf{C}_i\right) \overset{\mathrm{Id}\otimes \mathrm{ev}_{i_0}}{\longrightarrow} \mathbf{C}_{i_0}^\vee \otimes \mathbf{C}_{i_0} \to \mathrm{Vect}\,.$$

Now let us prove that the pairing (8) induces an equivalence

$$\left(\underset{i\in I,G^\vee}{\underset{\longrightarrow}{colim}}\, \mathbf{C}_i^\vee\right) \simeq \left(\underset{i\in I^{\mathrm{op}},G}{\underset{\longleftarrow}{lim}}\, \mathbf{C}_i\right)^\vee. \tag{10}$$

By [De, Prop. 2.3], it suffices to show that for any $\mathbf{D} \in \mathrm{DGCat}_{\mathrm{cont}}$ the functor

$$(\underset{i\in I^{\mathrm{op}},G}{\underset{\longleftarrow}{lim}}\, \mathbf{C}_i) \otimes \mathbf{D} \to \mathrm{Funct}_{\mathrm{cont}}\left((\underset{i\in I,G^\vee}{\underset{\longrightarrow}{colim}}\, \mathbf{C}_i^\vee), \mathbf{D}\right)$$

induced by (8) is an equivalence. To this end, one checks that the above functor is isomorphic to the composition of the equivalences

$$(\underset{i\in I^{\mathrm{op}},G}{\underleftarrow{lim}}\, \mathbf{C}_i)\otimes\mathbf{D}\simeq(\underset{i\in I,F}{\underrightarrow{colim}}\,\mathbf{C}_i)\otimes\mathbf{D}\simeq\underset{i\in I,(F\otimes\mathrm{Id})}{\underrightarrow{colim}}(\mathbf{C}_i\otimes\mathbf{D})\simeq\underset{i\in I^{\mathrm{op}},(G\otimes\mathrm{Id})}{\underleftarrow{lim}}(\mathbf{C}_i\otimes\mathbf{D})\simeq$$

$$\simeq\underset{i\in I^{\mathrm{op}}}{\underleftarrow{lim}}\,\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_i^\vee,\mathbf{D})\simeq\mathrm{Funct}_{\mathrm{cont}}\left((\underset{i\in I,G^\vee}{\underrightarrow{colim}}\,\mathbf{C}_i^\vee),\mathbf{D}\right),$$

where the first and third equivalence are obtained by applying Lemma 1.3.3, and the limit of $\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_i^\vee,\mathbf{D})$ is taken with respect to the functors

$$\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_j^\vee,\mathbf{D})\longrightarrow\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_i^\vee,\mathbf{D}),\quad \mathsf{F}\mapsto\mathsf{F}\circ G_\alpha^\vee,\quad \alpha:i\to j.$$

The characterization of the equivalence (10) given by (7) follows from the definition of the pairing (8). □

2.2.3. It's also easy to see that if, under the above circumstances, each of the categories $\mathbf{C}_i$ is compactly generated, then so is $\mathbf{C}:=\underset{I,F}{colim}\,\mathbf{C}_i$. Indeed, the functors $\Phi_i:\mathbf{C}_i\to\mathbf{C}$ send compact objects to compact ones.

2.3. **Compactly generated categories.** Assume now that $\mathbf{C}$ is compactly generated. I.e., we can write $\mathbf{C}\simeq\mathrm{Ind}(\mathbf{C}^c)$, where $\mathbf{C}^c$ is a small (non-cocomplete) DG category consisting of compact objects of $\mathbf{C}$.

For any DG-category $\mathbf{D}$ we have that $\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C},\mathbf{D})$ is equivalent to the category

$$\mathrm{Funct}(\mathbf{C}^c,\mathbf{D})$$

of just k-linear functors $\mathbf{C}^c\to\mathbf{D}$.

Note that we have a canonical pairing in $\mathrm{DGCat}_{\mathrm{cont}}$:

$$\mathrm{Ind}(\mathbf{C}^c)\otimes\mathrm{Ind}((\mathbf{C}^c)^{\mathrm{op}})\to\mathrm{Vect},$$

given by the Yoneda pairing $\mathbf{C}^c\times(\mathbf{C}^c)^{\mathrm{op}}\to\mathrm{Vect}$.

Proposition 2.3.1. *The above pairing makes* $\mathrm{Ind}((\mathbf{C}^c)^{\mathrm{op}})$ *into a dual of* $\mathbf{C}$. *In particular, any compactly generated DG-category is dualizable.*

Proof. We'll check that for any $\mathbf{D}$ the above pairing defines an equivalence

$$\mathrm{Ind}((\mathbf{C}^c)^{\mathrm{op}})\otimes\mathbf{D}\to\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C},\mathbf{D})\simeq\mathrm{Funct}(\mathbf{C}^c,\mathbf{D}).$$

It will be convenient to use the following characterization of the the tensor product operation on $\mathrm{DGCat}_{\mathrm{cont}}$:

$$\mathbf{D}_1\otimes\mathbf{D}_2\simeq\left(\mathrm{Funct}_{\mathrm{cont}}(\mathbf{D}_1,\mathbf{D}_2^{\mathrm{op}})\right)^{\mathrm{op}}.$$

Hence, we obtain that

$$\mathrm{Ind}((\mathbf{C}^c)^{\mathrm{op}})\otimes\mathbf{D}\simeq\left(\mathrm{Funct}((\mathbf{C}^c)^{\mathrm{op}},\mathbf{D}^{\mathrm{op}})\right)^{\mathrm{op}}\simeq\mathrm{Funct}(\mathbf{C}^c,\mathbf{D}),$$

as required.

□

2.3.2. *Duality and adjunction.* Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be two compactly generated categories, and $F : \mathbf{C}_1 \to \mathbf{C}_2$ a functor that sends $\mathbf{C}_1^c$ to $\mathbf{C}_2^c$. This is a necessary and sufficient for the adjoint functor

$$F^R : \mathbf{C}_2 \to \mathbf{C}_1$$

to belong to $\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_2, \mathbf{C}_1)$.

Let $F^{c,op}$ denote the functor

$$(\mathbf{C}_1^c)^{\mathrm{op}} \to (\mathbf{C}_2^c)^{\mathrm{op}},$$

obtained from $F^c : \mathbf{C}_1^c \to \mathbf{C}_2^c$ by reversing the arrows. Let $F^{\mathrm{op}} := \mathrm{Ind}(F^{c,op})$ be its ind-extension

$$\mathbf{C}_1^\vee \simeq \mathrm{Ind}(\mathbf{C}_1^{\mathrm{op}}) \to \mathrm{Ind}(\mathbf{C}_2^{\mathrm{op}}) \simeq \mathbf{C}_2^\vee.$$

Lemma 2.3.3. *Under the above circumstances, the functor F^{op} is the left adjoint of $F^\vee$, and F is the left adjoint of $(F^{\mathrm{op}})^\vee$.*

3. The Barr-Beck-Lurie theorem

3.1. **The General set-up.** Let us first review the general Barr-Beck-Lurie theorem. Let $\mathbf{C}$ and $\mathbf{D}$ be ∞-categories that admit limits and colimits. Let

$$F : \mathbf{C} \rightleftarrows \mathbf{D} : G$$

be a pair of mutually adjoint functors. Consider the monad $A = G \circ F : \mathbf{C} \to \mathbf{C}$. The functor G naturally factors through

$$\mathbf{C} \leftarrow A\text{-mod}_{\mathbf{C}} \overset{G'}{\leftarrow} \mathbf{D}.$$

Note that the functor G' itself admits a left adjoint F', defined as follows: for $X \in A\text{-mod}_{\mathbf{C}}$ the assignment

$$n \mapsto F(A^{\times n}(X))$$

is naturally a simplicial object of $\mathbf{D}$, which we denote by $F(A_\bullet(X))$ and we define

$$F'(X) := |F(A_\bullet(X))|.$$

Proposition 3.1.1. *The functor G' and F' define mutually inverse equivalences if and only if the following two conditions hold:*

- *The functor G is conservative.*
- *If $X_\bullet \in \mathbf{D}$ is a simplicial object, such that $G(X_\bullet)$ is split, then the natural map $|G(X_\bullet)| \to G(|X_\bullet|)$ is an isomorphism.*

Proof. We'll prove the "if" direction. Since G' is conservative, it is enough to show that F' is fully faithful, i.e., that the adjunction map

$$X \mapsto G' \circ F'(X)$$

is an isomorphism. It is enough to show that it is an isomorphism after applying the forgetful functor $A\text{-mod}_{\mathbf{C}} \to \mathbf{C}$.

Note that the simplicial object $F(A_\bullet(X)) \in \mathbf{D}$ is such that $G(F(A_\bullet(X)))$ is split with the -1-st term being X. Hence, by assumption

$$G(|F(A_\bullet(X))|) \simeq |G(F(A_\bullet(X)))| \simeq X,$$

as required. □

3.1.2. *Easy Barr-Beck.* A particularly easy case of the above proposition is when the functor G commutes with colimits. In this case we see that the functors F' and G' are mutually inverse if and only if G is conservative.

3.2. **Monads and tensor products.** Let $\mathbf{C}_1, \mathbf{C}_2$ be DG-categories, and let $\mathsf{A}_i : \mathbf{C}_i \to \mathbf{C}_i$, $i = 1, 2$ be monads that belong to $\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_i, \mathbf{C}_i)$. Let $\mathbf{C} := \mathbf{C}_1 \otimes \mathbf{C}_2$, and let $\mathsf{A} = \mathsf{A}_1 \otimes \mathsf{A}_2$.

Proposition 3.2.1. *The natural functor*

$$\mathsf{A}_1\text{-mod}_{\mathbf{C}_1} \otimes \mathsf{A}_2\text{-mod}_{\mathbf{C}_2} \to \mathsf{A}\text{-mod}_{\mathbf{C}}$$

is an equivalence.

Proof. Let $\mathsf{F}_i, \mathsf{G}_i$ denote the pair of adjoint functors

$$\mathsf{F}_i : \mathsf{A}_i\text{-mod}_{\mathbf{C}_i} \leftrightarrows \mathbf{C}_i : \mathsf{G}_i.$$

Consider the forgetful functor

$$\mathsf{G}_1 \otimes \mathsf{G}_2 : \mathsf{A}_1\text{-mod}_{\mathbf{C}_1} \otimes \mathsf{A}_2\text{-mod}_{\mathbf{C}_2} \to \mathbf{C}_1 \otimes \mathbf{C}_2 \simeq \mathbf{C}.$$

Its left adjoint is $\mathsf{F}_1 \otimes \mathsf{F}_2$, and the resulting monad on $\mathbf{C}$ is A. By Sect. 3.1.2, it suffices to show that $\mathsf{G}_1 \otimes \mathsf{G}_2$ is conservative. The latter is equivalent to the image of $\mathsf{F}_1 \otimes \mathsf{F}_2$ generating $\mathsf{A}_1\text{-mod}_{\mathbf{C}_1} \otimes \mathsf{A}_2\text{-mod}_{\mathbf{C}_2}$. However, this follows from the fact that the image of F_i generates $\mathsf{A}_i\text{-mod}_{\mathbf{C}_i}$, since G_i is conservative by assumption.

□

4. Module categories

4.1. **The set-up.** Let $\mathbf{O}$ be a monoidal category. We'll always be assuming that the monoidal operation $mult^*_{\mathbf{O}} : \mathbf{O} \otimes \mathbf{O} \to \mathbf{O}$ belongs to $\mathrm{Funct}_{\mathrm{cont}}(\mathbf{O} \otimes \mathbf{O}, \mathbf{O})$.

By an $\mathbf{O}$-module we'll mean a category $\mathbf{C}$ endowed with an associative action $act^*_{\mathbf{O},\mathbf{C}}$: $\mathbf{O} \otimes \mathbf{C} \to \mathbf{C}$, such that this functor belongs to $\mathrm{Funct}_{\mathrm{cont}}(\mathbf{O} \otimes \mathbf{C}, \mathbf{C})$.

For two $\mathbf{O}$-module categories $\mathbf{C}_1$ and $\mathbf{C}_2$, we shall denote by $\mathrm{Hom}_{\mathbf{O}\text{-}\mathbf{mod}}(\mathbf{C}_1, \mathbf{C}_2)$ the DG category of functors $\mathbf{C}_1 \to \mathbf{C}_2$ that are compatible with the $\mathbf{O}$-action, and belong to $\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_1, \mathbf{C}_2)$.

4.1.1. We make $\mathbf{O}$-module categories into an $(\infty, 2)$-category, denoted $\mathbf{O}\text{-}\mathbf{mod}$, by setting 1-morphisms to be $\mathrm{Hom}_{\mathbf{O}}(\mathbf{C}_1, \mathbf{C}_2)$. However, the same reservation pertaining to the notion of $(\infty, 2)$-category as in the case of $\mathrm{DGCat}_{\mathrm{cont}}$ (see Sect. 1.2.1) applies.

As in the case of DGCat, we can alternatively view $\mathbf{O}\text{-}\mathbf{mod}$ as an $(\infty, 1)$-category, by discarding the non-invertible 2-morphisms, i.e., by considering the maximal sub-groupoid

$$\mathrm{Hom}^{\circ}_{\mathbf{O}\text{-}\mathbf{mod}}(\mathbf{C}_1, \mathbf{C}_2) \subset \mathrm{Hom}_{\mathbf{O}\text{-}\mathbf{mod}}(\mathbf{C}_1, \mathbf{C}_2).$$

As for DGCat, the $(\infty, 2)$-category structure can be essentially recovered from the $(\infty, 1)$-category by considering the arrows category, using the fact that $\mathbf{O}\text{-}\mathbf{mod}$ is *tensored over* DGCat.

When considering a functor $\Phi : \mathbf{O}_1\text{-}\mathbf{mod} \to \mathbf{O}_2\text{-}\mathbf{mod}$ as $(\infty, 1)$-categories, we can recover it as a 2-functor between the corresponding $(\infty, 2)$-categories once Φ is endowed with a structure of being tensored over DGCat.

4.2. **Duality of module categories.** The following notion has also been explained to us by J. Lurie:

Let $\mathbf{O}$ be a monoidal category, and $\mathbf{C}_1$ and $\mathbf{C}_2$ be left and right $\mathbf{O}$-modules respectively. An $\mathbf{O}$-duality datum is a pair of functors

$$\mu : \text{Vect} \to \mathbf{C}_2 \underset{\mathbf{O}}{\otimes} \mathbf{C}_1 \text{ and } \epsilon : \mathbf{C}_1 \otimes \mathbf{C}_2 \to \mathbf{O},$$

where ϵ is $\mathbf{O} \otimes \mathbf{O}^{\text{op}}$-linear, such that the composition

$$\mathbf{C}_1 \overset{\mu}{\to} \mathbf{C}_1 \otimes (\mathbf{C}_2 \underset{\mathbf{O}}{\otimes} \mathbf{C}_1) \simeq (\mathbf{C}_1 \otimes \mathbf{C}_2) \underset{\mathbf{O}}{\otimes} \mathbf{C}_1 \overset{\epsilon}{\to} \mathbf{O} \underset{\mathbf{O}}{\otimes} \mathbf{C}_1 \simeq \mathbf{C}_1$$

is the identity, and so is the functor

$$\mathbf{C}_2 \overset{\mu}{\to} (\mathbf{C}_2 \underset{\mathbf{O}}{\otimes} \mathbf{C}_1) \otimes \mathbf{C}_2 \simeq \mathbf{C}_2 \underset{\mathbf{O}}{\otimes} (\mathbf{C}_1 \otimes \mathbf{C}_2) \overset{\epsilon}{\to} \mathbf{C}_2 \underset{\mathbf{O}}{\otimes} \mathbf{O} \simeq \mathbf{C}_2,$$

with the natural compatibility axioms holding.

4.3. Let $\mathbf{C}$ be a left $\mathbf{O}$-module category, and assume that it is dualizable in the above sense. Let $\mathbf{C}^\vee$ denote the dual category.

Lemma 4.3.1. *For any left $\mathbf{O}$-module category $\mathbf{C}'$ we have a natural equivalence*

$$\mathbf{C}^\vee \underset{\mathbf{O}}{\otimes} \mathbf{C}' \simeq \text{Hom}_{\mathbf{O}\text{-}\mathbf{mod}}(\mathbf{C}, \mathbf{C}').$$

Corollary 4.3.2. *If $\mathbf{C}$ is dualizable as a right module category, then the functor*

$$\mathbf{C}' \mapsto \mathbf{C} \underset{\mathbf{O}}{\otimes} \mathbf{C}' : \mathbf{O}\text{-}\mathbf{mod} \to \text{DGCat}_{\text{cont}}$$

commutes with limits.

4.4. Modules over an algebra. Let $\mathbf{O}$ be a monoidal category, and let $A \in \mathbf{O}$ be an associative algebra. Let $A\text{-mod}_{\mathbf{O}}$ denote the category of A-modules on $\mathbf{O}$. Tautologically, this category is the same as that of modules over the monad

$$\mathsf{A}' : \mathbf{O} \to \mathbf{O} : X \mapsto A \otimes X.$$

The category $A\text{-mod}_{\mathbf{O}}$ is naturally a right module over $\mathbf{O}$.

4.5. Let $\mathbf{O}_1$ and $\mathbf{O}_2$ be two monoidal categories and $A_i \in \mathbf{O}_1$ be algebras. By Proposition 3.2.1, we have:

Lemma 4.5.1. *The natural functor*

$$A_1\text{-mod}_{\mathbf{O}_1} \otimes A_2\text{-mod}_{\mathbf{O}_2} \to (A_1 \otimes A_2)\text{-mod}_{\mathbf{O}_1 \otimes \mathbf{O}_2}$$

is an equivalence.

4.6. Consider now the category of A-bimodules on $\mathbf{O}$, denoted $A\text{-bimod}_{\mathbf{O}}$. Consider also the category $A\text{-mod}_{\mathbf{O}} \underset{\mathbf{O}}{\otimes} A^{\text{op}}\text{-mod}_{\mathbf{O}^{\text{op}}}$.

Proposition 4.6.1. *The natural functor*

$$A\text{-mod}_{\mathbf{O}} \underset{\mathbf{O}}{\otimes} A^{\text{op}}\text{-mod}_{\mathbf{O}^{\text{op}}} \to A\text{-bimod}_{\mathbf{O}}$$

is an equivalence.

Proof. Let $\mathsf{G} : A\text{-mod}_{\mathbf{O}} \to \mathbf{O}$ and $\mathsf{G}^{\mathrm{op}} : A^{\mathrm{op}}\text{-mod}_{\mathbf{O}^{\mathrm{op}}} \to \mathbf{O}$ denote the forgetful functors, and let F and F^{op} denote their adjoints. We have a pair of mutually adjoint functors

$$\mathsf{F} \otimes \mathsf{F}^{\mathrm{op}} : \mathbf{O} \simeq \mathbf{O} \underset{\mathbf{O}}{\otimes} \mathbf{O} \leftrightarrows A\text{-mod}_{\mathbf{O}} \underset{\mathbf{O}}{\otimes} A^{\mathrm{op}}\text{-mod}_{\mathbf{O}^{\mathrm{op}}} : \mathsf{G} \otimes \mathsf{G}^{\mathrm{op}}.$$

The resulting monad on $\mathbf{O}$ corresponds to the category $A\text{-bimod}_{\mathbf{O}}$. Hence, by Proposition 3.1.2, it suffices to check that the functor $\mathsf{G} \otimes \mathsf{G}^{\mathrm{op}}$ is conservative. The latter is equivalent to the fact that the image of $\mathsf{F} \otimes \mathsf{F}^{\mathrm{op}}$ generates $A\text{-mod}_{\mathbf{O}} \underset{\mathbf{O}}{\otimes} A^{\mathrm{op}}\text{-mod}_{\mathbf{O}^{\mathrm{op}}}$. However, the latter follows since the image of F (resp., F^{op}) generates $A\text{-mod}_{\mathbf{O}}$ (resp., $A^{\mathrm{op}}\text{-mod}_{\mathbf{O}^{\mathrm{op}}}$), since the functors G and G^{op} are conservative.

□

4.7. Let $\mathbf{O}$ and A be as above.

Proposition 4.7.1. *The categories $A\text{-mod}_{\mathbf{O}}$ and $A^{\mathrm{op}}\text{-mod}_{\mathbf{O}^{\mathrm{op}}}$ are mutually $\mathbf{O}$-dual.*

Proof. The pairing $\epsilon : A^{\mathrm{op}}\text{-mod}_{\mathbf{O}^{\mathrm{op}}} \otimes A\text{-mod}_{\mathbf{O}} \to \mathbf{O}$ is the usual Hochschild homlogy functor. The functor

$$\mu : \mathrm{Vect} \to A\text{-mod}_{\mathbf{O}} \otimes A^{\mathrm{op}}\text{-mod}_{\mathbf{O}^{\mathrm{op}}}$$

corresponds to the object

$$A \in A\text{-bimod}_{\mathbf{O}} \simeq A\text{-mod}_{\mathbf{O}} \otimes A^{\mathrm{op}}\text{-mod}_{\mathbf{O}^{\mathrm{op}}}.$$

□

Corollary 4.7.2. *For a left $\mathbf{O}$-module category $\mathbf{C}$, we have a natural equivalence:*

$$\mathrm{Hom}_{\mathbf{O}\text{-}\mathbf{mod}}(A^{\mathrm{op}}\text{-mod}_{\mathbf{O}^{\mathrm{op}}}, \mathbf{C}) \simeq A\text{-mod}_{\mathbf{O}} \underset{\mathbf{O}}{\otimes} \mathbf{C}.$$

4.8. Let $\mathbf{C}$ be a left $\mathbf{O}$-module category. For A as above, we can consider the monad $\mathsf{A}_{\mathbf{C}}$ on $\mathbf{C}$ given by tensor product with A. Let $A\text{-mod}_{\mathbf{C}}$ denote the corresponding category of modules.

Proposition 4.8.1. *The natural functor*

$$A\text{-mod}_{\mathbf{O}} \underset{\mathbf{O}}{\otimes} \mathbf{C} \to A\text{-mod}_{\mathbf{C}}$$

is an equivalence.

Proof. The proof follows again from Proposition 3.1.2: it suffices to observe that the forgetful functor

$$A\text{-mod}_{\mathbf{O}} \underset{\mathbf{O}}{\otimes} \mathbf{C} \to \mathbf{O} \underset{\mathbf{O}}{\otimes} \mathbf{C} \simeq \mathbf{C}$$

is conservative.

□

4.9. Compact generation of tensor products. Let $\mathbf{O}$ be a monoidal DG category, and let $\mathbf{C}_1$ and $\mathbf{C}_2$ be left and right $\mathbf{O}$-module categories, respectively.

Assume now that $\mathbf{O}$ and that the monoidal operation $\mathbf{O} \otimes \mathbf{O} \to \mathbf{O}$ admits a continuous right adjoint, and that so do the action functors $\mathbf{O} \otimes \mathbf{C}_1 \to \mathbf{C}_1$ and $\mathbf{C}_2 \otimes \mathbf{O} \to \mathbf{C}_2$.

Proposition 4.9.1. *Under the above circumstances, the right adjoint to the tautological functor*

$$\mathbf{C}_2 \otimes \mathbf{C}_1 \to \mathbf{C}_2 \underset{\mathbf{O}}{\otimes} \mathbf{C}_1$$

is continuous.

Corollary 4.9.2. *Assume that* $\mathbf{O}$, $\mathbf{C}_1$, $\mathbf{C}_2$ *are compactly generated, and that the functors*

$$\mathbf{O}\otimes\mathbf{O}\to\mathbf{O},\ \mathbf{O}\otimes\mathbf{C}_1\to\mathbf{C}_1,\ \mathbf{C}_2\otimes\mathbf{O}\to\mathbf{C}_2$$

preserve compact objects. Then the functor

$$\mathbf{C}_2\otimes\mathbf{C}_1\to\mathbf{C}_2\underset{\mathbf{O}}{\otimes}\mathbf{C}_1$$

sends compact objects to compact ones. In particular, $\mathbf{C}_2\underset{\mathbf{O}}{\otimes}\mathbf{C}_1$ *is compactly generated.*

Proof. (of Proposition 4.9.1)

By definition, the category $\mathbf{C}_2\underset{\mathbf{O}}{\otimes}\mathbf{C}_1$ is given as the geometric realization of the simplicial category

$$i\mapsto\mathbf{C}_1\otimes\mathbf{O}^{\otimes i}\otimes\mathbf{C}_2.$$

By Lemma 1.3.3, the above geometric realization can be rewritten as the totalization of the corresponding cosimplicial category obtained by taking the right adjoints.

Moreover, the right adjoint to the tautological functor

$$\mathbf{C}_2\otimes\mathbf{C}_1\to|\mathbf{C}_1\otimes\mathbf{O}^{\otimes\bullet}\otimes\mathbf{C}_2|=\mathbf{C}_2\underset{\mathbf{O}}{\otimes}\mathbf{C}_1$$

is the evaluation functor

$$\mathrm{Tot}(\mathbf{C}_1\otimes\mathbf{O}^{\otimes\bullet}\otimes\mathbf{C}_2|)\to\mathbf{C}_2\otimes\mathbf{C}_1.$$

Now, the assumption of the proposition implies that the above cosimplicial category belongs to $\mathrm{DGCat}_{\mathrm{cont}}$. In particular, the above evaluation functor is continuous.

□

5. Dualizability in a monoidal category

5.1. Inner Hom. Let $\mathbf{O}$ be a monoidal DG category and $\mathbf{C}$ an $\mathbf{O}$-module category.

For $X,Y\in\mathbf{C}$ we set $\underline{\mathrm{Hom}}_{\mathbf{O}}(X,Y)\in\mathbf{O}$ to be the object such that

$$\mathrm{Hom}_{\mathbf{O}}(Z,\underline{\mathrm{Hom}}_{\mathbf{O}}(X,Y))\simeq\mathrm{Hom}_{\mathbf{C}}(Z\otimes X,Y),$$

functorially in $Z\in\mathbf{O}$.

An object $X\in\mathbf{C}$ is called relatively $\mathbf{O}$-left compact if the functor

$$Y\mapsto\underline{\mathrm{Hom}}_{\mathbf{O}}(X,Y):\mathbf{C}\to\mathbf{O}$$

commutes with colimits.

Lemma 5.1.1.

(1) *Suppose that* $\mathbf{O}$ *is compactly generated. Then every compact object in* $\mathbf{C}$ *is relatively compact.*

(2) *If* $1_{\mathbf{O}}$ *is compact, then the converse to (1) hold.*

5.2. Let us now consider $\mathbf{O}$ as a module over itself.

Recall that an object $X \in \mathbf{O}$ is said to be left-dualizable if there exists an object $X^\vee \in \mathbf{O}$ endowed with the

$$1_\mathbf{O} \to X \otimes X^\vee \text{ and } X^\vee \otimes X \to 1_\mathbf{O},$$

satisfying the usual axioms.

Recall also the following:

Lemma 5.2.1.
(1) *If X is dualizable, we have*

$$\mathrm{Hom}(Y, X \otimes Z) \simeq \mathrm{Hom}(X^\vee \otimes Y, Z) \textit{ and } \mathrm{Hom}(Z \otimes X, Y) \simeq \mathrm{Hom}(Z, Y \otimes X^\vee).$$

(2) *If either X or Y is left-dualizable, then*

$$\underline{\mathrm{Hom}}_\mathbf{O}(X, Y) \simeq Y \otimes \underline{\mathrm{Hom}}(X, 1_\mathbf{O}).$$

(3) *An object X is left-dualizable if and only if there exists an object $X^\vee$ endowed with a functorial isomorphism $\underline{\mathrm{Hom}}_\mathbf{O}(X, Y) \simeq Y \otimes X^\vee$.*

5.2.2. Evidently, if an object X is left-dualizable, then it's relatively compact.

Proposition 5.2.3. *Suppose that $\mathbf{O}$ is generated by left-dualizable objects. Then every relatively compact object is left-dualizable.*

Proof. Let X be relatively $\mathbf{O}$-left compact. We need to establish the isomorphism

$$Y \otimes \underline{\mathrm{Hom}}_\mathbf{O}(X, 1_\mathbf{O}) \simeq \underline{\mathrm{Hom}}_\mathbf{O}(X, Y).$$

By assumption, both sides commute with colimits in Y. Hence, it's enough to establish it for a generating set of Y's. However, the isomorphism does hold whenever Y is left-dualizable. □

6. Rigid monoidal categories

6.1. Let $\mathbf{O}$ be a monoidal category. Let $mult^*_\mathbf{O}$ denote the tensor product functor $\mathbf{O} \otimes \mathbf{O} \to \mathbf{O}$, and $unit^*_\mathbf{O} : \mathrm{Vect} \to \mathbf{O}$ the unit. We shall say that $\mathbf{O}$ is rigid if the following conditions hold:

- The right adjoint of $mult^*_\mathbf{O}$, denoted $mult_{\mathbf{O}*}$, belongs to $\mathrm{Funct}_{\mathrm{cont}}(\mathbf{O}, \mathbf{O} \otimes \mathbf{O})$.
- The functor $mult_{\mathbf{O}*} : \mathbf{O} \to \mathbf{O} \otimes \mathbf{O}$ is compatible with the left and right actions of $\mathbf{O}$. [1]
- The right adjoint of $unit^*_\mathbf{O}$, denoted $unit_{\mathbf{O}*}$, belongs to $\mathrm{Funct}_{\mathrm{cont}}(\mathbf{O}, \mathrm{Vect})$ (equivalently, the object $unit^*_\mathbf{O}(k) \in \mathbf{O}$ is compact).

If this happens, it's easy to see that the data of

$$\epsilon : \mathbf{O} \otimes \mathbf{O} \overset{mult^*_\mathbf{O}}{\longrightarrow} \mathbf{O} \overset{unit_{\mathbf{O}*}}{\longrightarrow} \mathrm{Vect}$$

and

$$\mu : \mathrm{Vect} \overset{unit^*_\mathbf{O}}{\longrightarrow} \mathbf{O} \overset{mult_{\mathbf{O}*}}{\longrightarrow} \mathbf{O} \otimes \mathbf{O}$$

define an isomorphism $\mathbf{O} \to \mathbf{O}^\vee$; let's denote this isomorphism $\phi^1_\mathbf{O}$.

6.1.1. Note that when $\mathbf{O}$ is compactly generated, the condition that $\mathbf{O}$ be rigid is equivalent to $\mathbf{O}^c$ being a rigid monoidal category in the usual sense (i.e., every object admits a left and a right dual).

[1] A priori, it's only lax compatible

6.1.2. Reversing the multiplication on $\mathbf{O}$ we obtain another identification $\mathbf{O} \to \mathbf{O}^\vee$, which we denote $\phi^2_{\mathbf{O}}$. We have $\phi^2_{\mathbf{O}} = \phi^1_{\mathbf{O}} \circ \varphi_{\mathbf{O}}$, where $\varphi_{\mathbf{O}}$ is an automorphism of $\mathbf{O}$.

It is easy to see, however, that $\mathbf{O}$ is naturally an automorphism of $\mathbf{O}$ as a monoidal category. [2] (When $\mathbf{O}$ is compactly generated, φ is the ind-exetension of the automorphism of $\mathbf{O}^c$ given by $X \mapsto (X^\vee)^\vee$, i.e., the discrepancy between left duals and right duals).

The category $\mathbf{O}^\vee$ acquires a natural bimodule structure over $\mathbf{O}$. It is easy to see that the functor $\phi^1_{\mathbf{O}}$ is compatible with the right $\mathbf{O}$-module structures, while $\phi^2_{\mathbf{O}}$ is compatible with the left $\mathbf{O}$-module structures.

6.2. Let us list some properties of rigid monoidal categories.

Lemma 6.2.1. *Let $\mathbf{O}$ be a rigid monoidal category. Then:*

(1) *The diagram*

$$\begin{array}{ccc} \mathbf{O}^\vee & \xrightarrow{(mult^*_{\mathbf{O}})^\vee} & \mathbf{O}^\vee \otimes \mathbf{O}^\vee \\ {\scriptstyle \phi^i}\uparrow & & \uparrow{\scriptstyle \phi^i \otimes \phi^i} \\ \mathbf{O} & \xrightarrow{(mult_{\mathbf{O}})_*} & \mathbf{O} \otimes \mathbf{O} \end{array}$$

commutes for $i = 1, 2$.

(2) *Let $F : \mathbf{O}_1 \to \mathbf{O}_2$ be a monoidal functor between rigid monoidal categories. Then its right adjoint F^R belongs to $\mathrm{Funct}_{\mathrm{cont}}(\mathbf{O}_2, \mathbf{O}_1)$, and the following diagram commutes:*

$$\begin{array}{ccc} \mathbf{O}_2 & \xrightarrow{F^R} & \mathbf{O}_1 \\ {\scriptstyle \phi^i_{\mathbf{O}_2}}\downarrow & & \downarrow{\scriptstyle \phi^i_{\mathbf{O}_1}} \\ \mathbf{O}_2^\vee & \xrightarrow{F^\vee} & \mathbf{O}_1^\vee \end{array}$$

for $i = 1, 2$.

The proof follows readily from the definitions.

6.2.2. *Modules over rigid categories.*

Proposition 6.2.3. *Let $\mathbf{O}$ be rigid, and let $\mathbf{C}$ be a left module category over $\mathbf{O}$. Then the right adjoint $(act_{\mathbf{O},\mathbf{C}})_*$ to $act^*_{\mathbf{O},\mathbf{C}}$ belongs to* $\mathrm{Funct}_{\mathrm{cont}}$*, and is given by*

$$\mathbf{C} \simeq \mathrm{Vect} \otimes \mathbf{C} \overset{\mu \otimes \mathrm{Id}_{\mathbf{C}}}{\longrightarrow} \mathbf{O} \otimes \mathbf{O} \otimes \mathbf{C} \overset{\mathrm{Id}_{\mathbf{O}} \otimes act^*_{\mathbf{O},\mathbf{C}}}{\to} \mathbf{O} \otimes \mathbf{C},$$

i.e.,

$$\mathbf{C} \to \mathbf{O}^\vee \otimes \mathbf{C} \overset{\phi^2_{\mathbf{O}} \otimes \mathrm{Id}_{\mathbf{C}}}{\simeq} \mathbf{O} \otimes \mathbf{C},$$

where $\mathbf{C} \to \mathbf{O}^\vee \otimes \mathbf{C}$ is obtained from $act^*_{\mathbf{O},\mathbf{C}}$ via the duality data for $(\mathbf{O}, \mathbf{O}^\vee)$. A similar statement holds for right modules over $\mathbf{O}$.

Proof. By tensoring up over $\mathbf{O}$, it is enough to consider the universal case, when $\mathbf{C} = \mathbf{O}$. In this case, we need to compare the following two functors. One is

$$\mathbf{O} \to \mathbf{O}^\vee \otimes \mathbf{O} \overset{\phi^2_{\mathbf{O}} \otimes \mathrm{Id}_{\mathbf{O}}}{\simeq} \mathbf{O} \otimes \mathbf{O},$$

[2] We are grateful to J. Lurie for pointing out this issue.

written above, and the other is $(mult_{\mathbf{O}})_*$. We consider both sides as endowed with an action of $\mathbf{O}$ on the right. The assumption on $\mathbf{O}$ says that both functors are compatible with this action. Hence, it is enough to identify the two compositions

$$\text{Vect} \overset{unit^*_{\mathbf{O}}}{\to} \mathbf{O} \rightrightarrows \mathbf{O} \otimes \mathbf{O}$$

coincide. But this is easy to see that both identify with μ.

□

As a corollary, we obtain:

Corollary 6.2.4. *Let $\mathbf{O}$ be rigid, and let $F : \mathbf{C}_1 \to \mathbf{C}_2$ be a functor between $\mathbf{O}$-module categories. Suppose that F, when viewed as a functor between just DG categories admits a left (resp., continuous right) adjoint G. Then the diagram*

$$\begin{array}{ccc} \mathbf{O} \otimes \mathbf{C}_1 & \xleftarrow{\text{Id}_{\mathbf{O}} \otimes F} & \mathbf{O} \otimes \mathbf{C}_2 \\ {\scriptstyle act^*_{\mathbf{O},\mathbf{C}_1}}\downarrow & & \downarrow{\scriptstyle act^*_{\mathbf{O},\mathbf{C}_2}} \\ \mathbf{C}_1 & \longleftarrow & \mathbf{C}_2, \end{array}$$

that a priori commutes up to a natural transformation, actually commutes. In particular, G has a natural structure of functor between $\mathbf{O}$-module categories.

6.3. Hohschild homology and cohomology. Let $\mathbf{O}$ be a monoidal category and $\mathbf{K}$ a bimodule category. Recall that in this case we can form the "Hochschild homology" category $\text{HH}_{\mathbf{O}}(\mathbf{K})$, defined as the geometric realization of the simplicial category

$$\mathbf{K} \leftleftarrows \mathbf{O} \otimes \mathbf{K}...$$

In particular, if $\mathbf{K} = \mathbf{C}_1 \otimes \mathbf{C}_2$ with $\mathbf{C}_1$ being a right module and $\mathbf{C}_2$ a left module, we have $\text{HH}_{\mathbf{O}}(\mathbf{C}_1 \otimes \mathbf{C}_2) =: \mathbf{C}_1 \underset{\mathbf{O}}{\otimes} \mathbf{C}_2$, the tensor product of $\mathbf{C}_1$ and $\mathbf{C}_2$ over $\mathbf{O}$.

Let $\mathbf{O}$ and $\mathbf{K}$ be as before. We can also define the "Hochschild cohomology" category $\text{CH}_{\mathbf{O}}(\mathbf{K})$, defined as the totalization of the co-simplicial category

$$\mathbf{K} \rightrightarrows \mathbf{O}^\vee \otimes \mathbf{K}...$$

In particular, for two left module categories $\mathbf{C}_1, \mathbf{C}_2$, by setting $\mathbf{K} = \text{Hom}(\mathbf{C}_1, \mathbf{C}_2)$, we have $\text{CH}_{\mathbf{O}}(\text{Hom}(\mathbf{C}_1, \mathbf{C}_2)) \simeq \text{Hom}_{\mathbf{O}\text{-}\mathbf{mod}}(\mathbf{C}_1, \mathbf{C}_2)$.

For $\mathbf{C}_1$ a left $\mathbf{O}$-module and $\mathbf{C}_2$ a right $\mathbf{O}$-module, we will use the notation

$$\mathbf{C}_1 \overset{\mathbf{O}}{\otimes} \mathbf{C}_2 := \text{CH}_{\mathbf{O}}(\mathbf{C}_1 \otimes \mathbf{C}_2).$$

6.3.1. Assume now that $\mathbf{O}$ is rigid. From Lemma 1.3.3 and Proposition 6.2.3, we obtain:

Proposition 6.3.2. $\text{HH}_{\mathbf{O}}(\mathbf{K}) \simeq \text{HC}_{\mathbf{O}}(\mathbf{K}')$, *where $\mathbf{K}'$ is the same as $\mathbf{K}$ as a right $\mathbf{O}$-module, and the left $\mathbf{O}$-module structure is twisted by φ.*

Corollary 6.3.3. *Let $\mathbf{O}$ be a rigid monoidal category.*
(1) *Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be two left $\mathbf{O}$-modules with $\mathbf{C}_1$ dualizable as a category. Then*

$$\text{Hom}_{\mathbf{O}\text{-}\mathbf{mod}}(\mathbf{C}_1, \mathbf{C}_2) \simeq \mathbf{C}_1^\vee \underset{\mathbf{O}}{\otimes} \mathbf{C}'_2,$$

where $\mathbf{C}'_2$ is obtained from $\mathbf{C}_2$ by twisting the action by φ.
(2) *Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be right and left $\mathbf{O}$-modules, both dualizable as categories. Then $\mathbf{C}_1 \underset{\mathbf{O}}{\otimes} \mathbf{C}_2$ is dualizable and its dual identifies with $\mathbf{C}'^\vee_2 \underset{\mathbf{O}}{\otimes} \mathbf{C}_1^\vee$.*

(3) *Let* $\mathbf{C}$ *be a left* $\mathbf{O}$*-module category. Then for another category* $\mathbf{D}$*,*

$$\mathrm{Hom}_{\mathbf{O}\text{-}\mathbf{mod}}(\mathbf{C}, \mathbf{O} \otimes \mathbf{D}) \simeq \mathrm{Hom}(\mathbf{C}, \mathbf{D}).$$

6.4. Let $\mathbf{O}$ be a monoidal category and $\mathbf{C}_l$ and $\mathbf{C}_r$ be left and right $\mathbf{O}$-modules, respectively.

Corollary 6.4.1. *Suppose that* $\mathbf{O}$ *is rigid, and let* $\mathbf{C}_l$ *and* $\mathbf{C}_r$ *be as above. The following data are equivalent:*

- *An* $\mathbf{O}$*-duality between* $\mathbf{C}_l$ *and* $\mathbf{C}_r$*;*
- *An isomorphism* $\mathbf{C}_r \to \mathbf{C}_l^\vee$ *as right* $\mathbf{O}$*-modules (in particular,* $\mathbf{C}_l$ *must be dualizable as a category).*

Corollary 6.4.2. *Let* $\mathbf{O}$ *be rigid. Then an* $\mathbf{O}$*-module category* $\mathbf{C}$ *is dualizable as a module category if and only if it is dualizable as a plain DG-category.*

6.5. Suppose now that $\mathbf{C}_1$ and $\mathbf{C}_2$ are bi-modules over $\mathbf{O}$. A duality between them is a datum of maps of $\mathbf{O}$-bimodules:

$$\epsilon_{\mathbf{O}} : \mathbf{C}_1 \underset{\mathbf{O}}{\otimes} \mathbf{C}_2 \to \mathbf{O}, \ \mu_{\mathbf{O}} : \mathbf{O} \to \mathbf{C}_2 \underset{\mathbf{O}}{\otimes} \mathbf{C}_1,$$

such that the usual axioms are satisfied.

We have:

Corollary 6.5.1. *Let* $\mathbf{O}$ *be rigid and let* $\mathbf{C}$ *be an* $\mathbf{O}$*-bimodule. Then for a bi-module* $\mathbf{C}'$ *a duality datum for the pair* $(\mathbf{C}, \mathbf{C}')$ *is equivalent to an isomorphism of bimodules* $\mathbf{C}' \simeq \mathbf{C}^\vee$ *(in particular,* $\mathbf{C}$ *must be dualizable as a category).*

References

[De] P. Deligne, *Catégories tannakiennes*, in: "The Grothendieck Festschrift", Vol. II, 111–195, Progr. Math. **87**, Birkhäuser Boston, Boston, MA, 1990.
[Dr] V. Drinfeld, *DG Quotients of DG Categories*, arXiv:math:0210114.
[FG] E. Frenkel and D. gGaitsgory, *D-modules on the affine flag variety and representations of affine Kac-Moody algebras*, arXiv: 0712.0788.
[Lu1] J. Lurie, *Higher topos theory.*
[Lu2] J. Lurie, *Higher Algebra.*
[Lu3] J. Lurie, $(\infty, 2)$*-categories and Goodwillie calculus-I.*