MODULI SPACES, INDECOMPOSABLE OBJECTS AND POTENTIALS OVER A FINITE FIELD

GALYNA DOBROVOLSKA, VICTOR GINZBURG, AND ROMAN TRAVKIN

ABSTRACT. Given an $\mathbb{F}_q$-linear category such that the moduli space of its objects is a smooth Artin stack (and some additional conditions) we give formulas for an exponential sum over the set of absolutely indecomposable objects and a stacky sum over the set of all objects of the category, respectively, in terms of the geometry of the cotangent bundle on the moduli stack. The first formula was inspired by the work of Hausel, Letellier, and Rodriguez-Villegas. It provides a new approach for counting absolutely indecomposable quiver representations, vector bundles with parabolic structure on a projective curve, and irreducible étale local systems (via a result of Deligne). Our second formula resembles formulas appearing in the theory of Donaldson-Thomas invariants.

CONTENTS

1. MAIN RESULTS

1.1. **Exponential sums.** Fix a finite field k. Let $\mathscr{C}$ be a k-linear (not necessarily abelian) category that has a well-behaved moduli space of objects in the sense that there exists a *smooth* Artin moduli stack $\mathfrak{X}(\mathscr{C})$ parametrizing objects of $\mathscr{C}$. We assume that $\mathscr{C}$ has direct sums, moreover, it is Karoubian and the direct sum operation gives a morphism $\oplus : \mathfrak{X}(\mathscr{C}) \times \mathfrak{X}(\mathscr{C}) \to \mathfrak{X}(\mathscr{C})$, of stacks.

We fix a data (F, ϕ) where F is a k-linear functor from $\mathscr{C}$ to the category of finite dimensional vector spaces and $\phi : \mathfrak{X}(\mathscr{C}) \to \mathbb{A}^1$ a morphism, usually referred to as a 'potential'. We assume that F does not kill nonzero objects and the vector spaces $F(x)$, $x \in \mathscr{C}$, form a locally free sheaf $\mathfrak{F}$ on $\mathfrak{X}(\mathscr{C})$, see Definition 6.1.5 for more details. Furthermore, we assume that both ϕ and F are additive in the sense that one has $\phi(x \oplus y) = \phi(x) + \phi(y)$, resp. $F(x \oplus y) = F(x) \oplus F(y)$, for all objects x, y of $\mathscr{C}$. More precisely, we require that $\oplus^* \phi = pr_1^* \phi + pr_2^* \phi$, where $pr_j : \mathfrak{X}(\mathscr{C}) \times \mathfrak{X}(\mathscr{C}) \to \mathfrak{X}(\mathscr{C})$, $j = 1, 2$, denotes the projection to the j-th factor. The additivity of F is, by definition, the data of an isomorphism $\oplus^* \mathfrak{F} \cong pr_1^* \mathfrak{F} \oplus pr_2^* \mathfrak{F}$ that satisfies appropriate commutativity and associativity constraints.

1

One has a decomposition $\mathfrak{X}(\mathscr{C}) = \sqcup_{v \geq 0} \mathfrak{X}_v(\mathscr{C})$, where $\mathfrak{X}_v(\mathscr{C})$ is a substack such that the restriction of $\mathfrak{F}$ to $\mathfrak{X}_v(\mathscr{C})$ is a vector bundle of a fixed rank v. Thus, a closed k-point of $\mathfrak{X}_v(\mathscr{C})$ is an object of $\mathscr{C}$ such that $\dim F(x) = v$. It is often convenient to consider a slightly more general setting where $F = \oplus_{i \in I} F_i$ is a functor to the category of vector spaces graded by a fixed finite set I. The I-tuple $\mathbf{v} := (\dim F_i(x))_{i \in I} \in \mathbb{Z}_{\geq 0}^I$ is called the dimension (or dimension vector) of an object x of $\mathscr{C}$. Then, one has a decomposition $\mathfrak{X}(\mathscr{C}) = \sqcup_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^I} \mathfrak{X}_{\mathbf{v}}(\mathscr{C})$ where $\mathfrak{X}_{\mathbf{v}}(\mathscr{C})$ is the substack of objects of dimension $\mathbf{v}$. We will assume that for each $\mathbf{v}$ the stack $\mathfrak{X}_{\mathbf{v}}(\mathscr{C})$ has finite type. It follows that the set $[\mathrm{Ob}_{\mathbf{v}}\,\mathscr{C}] := [\mathfrak{X}_{\mathbf{v}}(\mathscr{C})(\mathsf{k})]$, of isomorphism classes of objects of our category of dimension $\mathbf{v}$, is finite.

Throughout the paper we fix an additive character $\psi : \mathsf{k} \to \bar{\mathbb{Q}}_\ell^\times$. We are interested in the following exponential sums:

$$E_{\mathbf{v}}(\mathscr{C}, F, \phi, \psi) = \sum_{x \in [\mathrm{Ob}_{\mathbf{v}}(\mathscr{C})]} \psi(\phi(x)), \qquad {}^{stacky}E_{\mathbf{v}}(\mathscr{C}, F, \phi, \psi) = \sum_{x \in [\mathrm{Ob}_{\mathbf{v}}(\mathscr{C})]} \frac{1}{\#\,\mathrm{Aut}(x)(\mathsf{k})} \cdot \psi(\phi(x)), \quad (1.1.1) \qquad \boxed{\text{E0}}$$

where $\mathrm{Aut}(x)(\mathsf{k})$ denotes the group of k-rational points of the algebraic group $\mathrm{Aut}(x)$ of automorphisms of x.

There are two classes of examples where our result is applicable.

Examples of the first class, to be discussed in more detail in §1.4, are categories of parabolic bundles on a smooth geometrically connected projective curve C over k with a fixed set of k-rational marked points. The functor F assigns to a parabolic bundle the direct sum of its fibers over the marked points. In the most basic special case of the category of vector bundles (without parabolic structure), one chooses an arbitrary k-rational point $c \in C$ and let the functor F send a vector bundle to its fiber over c. There are various modifications of the above setting. For example, one can consider categories of framed coherent sheaves on a projective variety, as in [MR] and [Me, Example 3.7]. It is also interesting to consider the category of locally projective modules over a fixed hereditary order on C, cf. [CI]. In all these cases, there are no nonconstant potentials ϕ.

Examples of categories of the second class are the categories A-mod of finite dimensional modules over a smooth associative k-algebra A. Recall that A is called *smooth* if the kernel of the map $A \otimes A \to A$, given by multiplication, is finitely generated and projective as an A-*bi*module. It is known that this condition insures that the stack $\mathfrak{X}(A\text{-mod})$ is smooth. Path algebras of quivers and coordinate rings of smooth affine curves are examples of smooth algebras. Any localization of a smooth algebra, eg. a multiplicative version of a path algebra, is smooth. We let F be the forgetful functor that assigns to an A-module M the underlying k-vector space. Given an element $a \in A/[A, A]$, there is an associated potential $\phi_a : \mathfrak{X}(A\text{-mod}) \to \mathbb{A}^1$, $M \mapsto \phi_a(M) := \mathrm{tr}(a, M)$, the trace of the a-action in M. In the case where A is the path algebra of a quiver Q, we get the category of finite dimensional representations of Q; here, the finite set I is the vertex set of Q and an element of $A/[A, A]$ is a linear combination of oriented cycles, i.e. cyclic paths, in Q.

$\boxed{\text{stab-rem}}$ *Remark* 1.1.2. The category A-mod is abelian and the forgetful functor is faithful. On the contrary, categories from the first class, such as the category of vector or parabolic bundles, are only quasi-abelian (usually not abelian) and the functor that sends a vector bundle to a direct some of finitely many fibers is, typically, not faithful. ◇

An important additional source of categories where our results are applicable comes from considering stability conditions on quasi-abelian categories. In more detail, let $\mathscr{C}$ be a quasi-abelian category equipped with a stability condition, cf. [An]. Any object $x \in \mathscr{C}$ has a canonical Harder-Narasimhan filtration $x = x^0 \supset x^1 \supset \ldots$, of finite length. Given a segment $S \subset \mathbb{R}$ (or a sector in the upper half plane, in the framework of Bridgeland stability) let $\mathscr{C}_S$ be a full subcategory of $\mathscr{C}$ whose objects are the objects $x \in \mathscr{C}$ such the slope of x^j/x^{j+1} is contained in S for all j. The moduli stack $\mathfrak{X}(\mathscr{C}_S)$ is easily seen, see e.g. [KS2], to be an open substack of $\mathfrak{X}(\mathscr{C})$. Hence, $\mathfrak{X}(\mathscr{C}_S)$ is

a smooth Artin stack whenever so is $\mathfrak{X}(\mathscr{C})$. Thus, starting with a data $(\mathscr{C}, F, \phi)$, where $\mathscr{C}$ is either the category of modules over a smooth algebra or the category of parabolic bundles, the results of this paper apply to the triple $(\mathscr{C}_S, F, \phi)$, for any choice of stability condition on $\mathscr{C}$ and a segment S.

Our main result provides formulas for the exponential sums (1.1.1) in terms of the geometry of a stack $\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)$ of 'framed objects'. A closed k-point of this stack is a pair (x, b), where x is an object of $\mathscr{C}$ of dimension $\mathbf{v}$ and b, the 'framing', is a collection $(b_i)_{i \in I}$ where b_i is a k-basis of the vector space $F_i(x)$, cf. §6.2 for a complete definition. The group $\mathrm{GL}_{\mathbf{v}} = \prod_i \mathrm{GL}_{v_i}$ acts on $\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)$ by changing the framing. One can show that the 1-dimensional torus $\mathbb{G}_m \subset \mathrm{GL}_{\mathbf{v}}$, of scalar matrices, acts trivially on $\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)$, so the $\mathrm{GL}_{\mathbf{v}}$-action factors through the group $\mathrm{PGL}_{\mathbf{v}} = \mathrm{GL}_{\mathbf{v}}/\mathbb{G}_m$. Let $q : \mathrm{T}^*\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F) \to \mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)$ be the cotangent stack. There is an induced $\mathrm{PGL}_{\mathbf{v}}$-action on $\mathrm{T}^*\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)$, the cotangent stack of $\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)$. Let $\mathfrak{pgl}_{\mathbf{v}} = \mathrm{Lie}\,\mathrm{PGL}_{\mathbf{v}}$ and $\mu : \mathrm{T}^*\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F) \to \mathfrak{pgl}_{\mathbf{v}}^*$ the moment map for the natural $\mathrm{PGL}_{\mathbf{v}}$-action on the cotangent stack. This map is $\mathrm{PGL}_{\mathbf{v}}$-equivariant, hence descends to a map $\underline{\mu} : \mathrm{T}^*\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)/\mathrm{PGL}_{\mathbf{v}} \to \mathfrak{pgl}_{\mathbf{v}}^*/\!\!/\mathrm{PGL}_{\mathbf{v}}$, where $\mathrm{T}^*\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)/\mathrm{PGL}_{\mathbf{v}}$, resp. $\mathfrak{pgl}_{\mathbf{v}}^*/\!\!/\mathrm{PGL}_{\mathbf{v}}$, stands for a stacky, resp. categorical, quotient by $\mathrm{PGL}_{\mathbf{v}}$. We obtain the following diagram

$$\mathfrak{pgl}_{\mathbf{v}}^*/\!\!/\mathrm{PGL}_{\mathbf{v}} \xleftarrow{\underline{\mu}} \mathrm{T}^*\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)/\mathrm{PGL}_{\mathbf{v}} \xrightarrow{p_1} \mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)/\mathrm{PGL}_{\mathbf{v}} \xrightarrow{p_2} \mathfrak{X}_{\mathbf{v}}(\mathscr{C}) \xrightarrow{\phi} \mathbb{A}^1. \qquad (1.1.3) \quad \boxed{\texttt{f}\star}$$

where p_1 is induced by the vector bundle projection $\mathrm{T}^*\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F) \to \mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)$ and the map p_2 forgets the framing. Let $\underline{\phi} = \phi \circ p_2 \circ p_1$ be the composite map.

Write $\mathfrak{S}_v$ for the Symmetric group on v letters, so $\mathfrak{S}_{\mathbf{v}} = \prod_i \mathfrak{S}_{v_i}$ is the Weyl group of $\mathrm{PGL}_{\mathbf{v}}$. Let $\mathrm{Irr}\,\mathfrak{S}_{\mathbf{v}}$ be the set of isomorphism classes of irreducible $\mathfrak{S}_{\mathbf{v}}$-representations and let sign, resp. triv, denote the sign, resp. trivial, character. Let $\mathfrak{t}_{\mathbf{v}}$ be the Cartan subalgebra of $\mathfrak{pgl}_{\mathbf{v}}$ formed by diagonal matrices. We will define a $\mathfrak{S}_{\mathbf{v}}$-stable Zariski open subset $\mathfrak{t}_{\mathbf{v}}^\diamond \subset \mathfrak{t}_{\mathbf{v}}^*$, see Definition 4.3.1, which is contained in the complement of (co)root hyperplanes. We have a canonical surjection $\pi_1^{et}(\mathfrak{t}_{\mathbf{v}}^\diamond/\!\!/\mathfrak{S}_{\mathbf{v}}) \twoheadrightarrow \mathfrak{S}_{\mathbf{v}}$. We also consider a $\mathrm{PGL}_{\mathbf{v}}$-stable open subset $\mathfrak{pgl}_{\mathbf{v}}^\diamond \subset \mathfrak{pgl}_{\mathbf{v}}^*$ such that $\mathfrak{pgl}_{\mathbf{v}}^\diamond/\!\!/\mathrm{PGL}_{\mathbf{v}} = \mathfrak{t}_{\mathbf{v}}^\diamond/\!\!/\mathfrak{S}_{\mathbf{v}}$ under the Chevalley isomorphism.

Proposition 1.1.4. *Let $(\mathscr{C}, F, \phi)$ be a triple satisfying all the conditions above and let $\mathcal{AS}_\psi$ be the Artin-Schreier local system on $\mathbb{A}^1$ associated with the additive character ψ. Then, for any dimension vector $\mathbf{v}$, the constructible complex $\underline{\mu}_! \underline{\phi}^* \mathcal{AS}_\psi\big|_{\mathfrak{pgl}_{\mathbf{v}}^\diamond/\!\!/\mathrm{PGL}_{\mathbf{v}}}$ is geometrically locally constant. Moreover, the restriction of the corresponding momodromy respresentation of $\pi_1^{et}(\mathfrak{pgl}_{\mathbf{v}}^\diamond/\!\!/\mathrm{PGL}_{\mathbf{v}})$ to the kernel of the composite homomorphism $\pi_1^{et}(\mathfrak{pgl}_{\mathbf{v}}^\diamond/\!\!/\mathrm{PGL}_{\mathbf{v}}) \to \pi_1^{et}(\mathfrak{t}_{\mathbf{v}}^\diamond/\!\!/\mathfrak{S}_{\mathbf{v}}) \twoheadrightarrow \mathfrak{S}_{\mathbf{v}}$ is unipotent.*

By construction, any element of $\mathfrak{pgl}_{\mathbf{v}}^\diamond$, viewed as an element of $\mathfrak{pgl}_{\mathbf{v}} \cong \mathfrak{pgl}_{\mathbf{v}}^*$, is regular semisimple. Thus, a coadjoint orbit $O \subset \mathfrak{pgl}_{\mathbf{v}}^\diamond$ gives a point in $\mathfrak{pgl}_{\mathbf{v}}^\diamond/\!\!/\mathrm{PGL}_{\mathbf{v}}$. The fiber of the map $\underline{\mu}$ over that point is $\mathcal{M}_O := \mu^{-1}(O)/\mathrm{PGL}_{\mathbf{v}}$, a stacky Hamiltonian reduction. Thanks to the above proposition, for any $\rho \in \mathrm{Irr}\,\mathfrak{S}_{\mathbf{v}}$ one has a well-defined space $H_c^\bullet(\mathcal{M}_O, \underline{\phi}^* \mathcal{AS}_\psi)^{\langle \rho \rangle}$, the generalized isotypic component of ρ under the monodromy action. To simplify notation, write $\wp^\phi = \underline{\phi}^* \mathcal{AS}_\psi$ and let $\mathsf{Tr}_{\mathsf{Fr}} | H_c^\bullet(\mathcal{M}_O, \wp^\phi)^{\langle \rho \rangle}$ denote the alternating sum of traces of the Frobenius.

To state our main result about exponential sums it is convenient to package the sums in (1.1.1) into generating series. To this end, let $(z_i)_{i \in I}$ be an I-tuple of indeterminates and write $z^{\mathbf{v}} := \prod_{i \in I} z_i^{v_i}$. Put $\mathbf{v} \cdot \mathbf{v} = \sum_{i \in I} v_i^2$ and $|\mathbf{v}| = \sum_{i \in I} v_i$. We form the following generating series

$$E(\mathscr{C}, F, \phi) = \sum_{\mathbf{v}} E_{\mathbf{v}}(\mathscr{C}, F, \phi), \qquad {}^{stacky}E(\mathscr{C}, F, \phi) = \sum_{\mathbf{v}} z^{\mathbf{v}} \cdot q^{-\frac{\mathbf{v} \cdot \mathbf{v}}{2}} \cdot {}^{stacky}E_{\mathbf{v}}(\mathscr{C}, F, \phi).$$

Let Exp denote the plethystic exponential and $d_{\mathbf{v}} = \dim \mathfrak{X}_{\mathbf{v}}$. Then, we have

Theorem 1.1.5. *Let* $\mathsf{k} = \mathbb{F}_q$, *let* $O \subset \mathfrak{pgl}_{\mathbf{v}}^{\diamond}$ *be a coadjoint orbit defined over* k, *and* $\mathcal{M}_O := \mu_{\mathbf{v}}^{-1}(O)/\mathrm{PGL}_{\mathbf{v}}$. *Then, one has the following equations:*

$$E(\mathscr{C}, F, \phi) \;=\; \mathrm{Exp}\left((1 - q^{-1}) \cdot \sum_{\mathbf{v} > 0} z^{\mathbf{v}} \cdot q^{-(d_{\mathbf{v}} + \frac{\mathbf{v} \cdot \mathbf{v} - |\mathbf{v}|}{2})} \cdot \mathsf{Tr}_{\mathsf{Fr}} | \, H_c^{\bullet}(\mathcal{M}_O, \wp^{\phi_{\mathbf{v}}})^{\langle \mathrm{sign} \rangle} \right) \qquad (1.1.6) \quad \boxed{\texttt{thm11}}$$

$${}^{stacky}E(\mathscr{C}, F, \phi) \;=\; \mathrm{Exp}\left(-q^{1/2} \cdot \sum_{\mathbf{v} > 0} (-1)^{|\mathbf{v}|} \cdot z^{\mathbf{v}} \cdot q^{-(d_{\mathbf{v}} + \frac{\mathbf{v} \cdot \mathbf{v} + |\mathbf{v}|}{2})} \cdot \mathsf{Tr}_{\mathsf{Fr}} | \, H_c^{\bullet}(\mathcal{M}_O, \wp^{\phi_{\mathbf{v}}})^{\langle \mathrm{triv} \rangle} \right). \quad (1.1.7) \quad \boxed{\texttt{thm1stack}}$$

Remarks 1.1.8. (i) Following Beilinson and Drinfeld, a stack X is called *good* if one has $\dim \mathrm{T}^* X = 2 \dim X$. One can show that if $\dim \mathfrak{X}_{\mathbf{v}}(\mathscr{C}) + n_{\mathbf{v}} = \frac{1}{2} \dim \mathcal{M}_O$ provided the stack $\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)$ is good. We note that in the case of vector bundles on a curve, the moduli stack $\mathfrak{X}_{\mathbf{v}}(\mathscr{C}) = Bun_{\mathbf{v}}$, of rank v vector bundles is *not* good. The corresponding stack $\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)/\mathrm{PGL}_{\mathbf{v}}$ is good, and it plays the role of an 'improved version' of the original stack $Bun_{\mathbf{v}} = \mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)/\mathrm{GL}_{\mathbf{v}}$. This improved version was introduced first by Beilinson and Drinfeld [BD] in a different way.

In general, the stacks $\mathfrak{X}_{\mathbf{v}}(\mathscr{C})$ and $\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)/\mathrm{PGL}_{\mathbf{v}}$ have the same objects but isomorphisms of objects are defined differently. The stack $\mathfrak{X}_{\mathbf{v}}(\mathscr{C})$ is a $\mathbb{G}_m$-gerbe over $\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)/\mathrm{PGL}_{\mathbf{v}}$. This construction can be applied in other situations; in particular, in §1.4 it will be used it in the case of parabolic Higgs bundles.

(ii) There are analogues of the equations of the theorem where the stack $\mathcal{M}_O$ is replaced by a similar stack involving Hamiltonian reduction by $\mathrm{GL}_{\mathbf{v}}$ rather than $\mathrm{PGL}_{\mathbf{v}}$. The resulting formulas have extra factors $q - 1 = \#\mathbb{G}_m(\mathbb{F}_q)$ at certain places.

(iii) In most examples the stack $\mathcal{M}_O$ in Theorem 1.1.5 is, in fact, a smooth scheme, cf. §10.

(iv) The equations of Theorem 1.1.5 have motivic nature. In particular, there are counterparts of these equations in the case where $\mathsf{k} = \mathbb{C}$, the field of complex numbers. In that case, the natural analogue of the function $\psi \circ \phi$ is the exponential function e^{ϕ}. The analogue of the exponential sum ${}^{stacky}E(\mathscr{C}, F, \phi)$ is an element $\frac{[\![\mathfrak{X}_{\mathbf{v}}(\mathscr{C}), e^{\phi}]\!]}{[\![\mathrm{PGL}_{\mathbf{v}}]\!]}$, where $[\![\mathrm{PGL}_{\mathbf{v}}]\!]$ is the motive of $\mathrm{PGL}_{\mathbf{v}}$, $[\![\mathfrak{X}_{\mathbf{v}}(\mathscr{C}), e^{\phi}]\!]$ is a class in the Grothendieck group of the category of *exponential monodromic mixed Hodge structures* defined by Kontsevich and Soibelman [KS2], and the fraction is an element of an appropriate localization of the Grothendieck group. It will become clear from section 5 that the correct analogue of the exponential sum $E(\mathscr{C}, F, \phi)$ is the element $\frac{[\![I(\mathfrak{X}_{\mathbf{v}}(\mathscr{C})), e^{\phi}]\!]}{[\![\mathrm{PGL}_{\mathbf{v}}]\!]}$, where $I(\mathfrak{X}_{\mathbf{v}})$ is the inertia stack of $\mathfrak{X}_{\mathbf{v}}(\mathscr{C})$. Finally, one can define a Hodge theoretic counterpart, $[\![\mathcal{M}_O, e^{\phi}]\!]^{\langle \rho \rangle}$, of the quantity $\mathsf{Tr}_{\mathsf{Fr}} | \, H_c^{\bullet}(\mathcal{M}_O, \wp^{\phi})^{\langle \rho \rangle}$.

(v) The motivic counterpart of formula (1.1.7) with $\phi = 0$ implies, in a special case where $\mathscr{C}$ is the category of representations of a symmetric quiver and zero potential, a special case of the 'positivity and integrality conjecture' for DT-invariants due to Kontsevich and Soibelman [KS2]. This special case of the conjecture has been proved in [HLV1] (cf. also [DM] in the general case). We believe that formula (1.1.7) might be relevant for understanding the conjecture in full generality. The essential point is that, in the case of quiver representations, the stack $\mathcal{M}_O$ is a scheme. In such a case, formula (1.1.7) indicates that every coefficient in the motivic counterpart of the formal power series $\mathrm{Log}({}^{stacky}E(\mathscr{C}, F, \phi))$ is an exponential motive of a *scheme* rather than a general stack.

1.2. Counting absolutely indecomposable objects.

For any finitely generated commutative k-algebra K one defines $K \otimes_{\mathsf{k}} \mathscr{C}$, the base change category, as a K-linear category whose objects are pairs (x, α), where x is an object of $\mathscr{C}$ and $\alpha : K \to \mathrm{End}(x)$ is an algebra homomorphism, to be thought of as a 'K-action on x'. A morphism $(x, \alpha) \to (x', \alpha')$, in $K \otimes_{\mathsf{k}} \mathscr{C}$ is, by definition, a morphism $u : x \to x'$ in $\mathscr{C}$ such that $\alpha'(a) \circ u = u \circ \alpha(a)$ for all $a \in K$. By construction, the algebra K maps to the center of the category $K \otimes_{\mathsf{k}} \mathscr{C}$. The natural forgetful functor $K \otimes_{\mathsf{k}} \mathscr{C} \to \mathscr{C}$ has a

left adjoint functor $\mathscr{C} \to K \otimes_k \mathscr{C}$, $x \to K \otimes_k x$, see [So, §2] for details. It is easy to show that if the category $\mathscr{C}$ is Karoubian, resp. exact, then so is $K \otimes_k \mathscr{C}$.

Definition 1.2.1. An object x of $\mathscr{C}$ is called *absolutely indecomposable* if $K \otimes_k x$ is an indecomposable object of $K \otimes_k \mathscr{C}$, for any *finite* field extension $K \supset k$.

It turns out that the generating series $E(\mathscr{C}, F, \phi)$ can be expressed in terms of absolutely indecomposable objects. Specifically, write

Let $\mathrm{AI}_\mathbf{v}(\mathscr{C})$, or $\mathrm{AI}_\mathbf{v}(\mathscr{C}, k)$, be the subset of $[\mathrm{Ob}_\mathbf{v}(\mathscr{C})]$ formed by the isomorphism classes of absolutely indecomposable objects. Let $E_\mathbf{v}^{\mathrm{AI}}(\mathscr{C}, F, \phi)$ be an exponential sum similar to $E_\mathbf{v}(\mathscr{C}, F, \phi)$, see (1.1.1), where summation over $[\mathrm{Ob}_\mathbf{v}(\mathscr{C})]$ is replaced by summation over the set $\mathrm{AI}_\mathbf{v}(\mathscr{C})$. It follows from a generalization of Hua's lemma [H] that one has $E(\mathscr{C}, F, \phi) = \mathrm{Exp}\left(E_\mathbf{v}^{\mathrm{AI}}(\mathscr{C}, F, \phi)\right)$ Therefore, formula (1.1.6) yields the following result.

Theorem 1.2.2. *Let* $k = \mathbb{F}_q$. *Then, we have*

$$E^{\mathrm{AI}}(\mathscr{C}, F, \phi) = \tfrac{q-1}{q} \cdot q^{-(\dim \mathfrak{X}_\mathbf{v} + \frac{\mathbf{v}\cdot\mathbf{v}-|\mathbf{v}|}{2})} \cdot \mathsf{Tr}_{\mathsf{Fr}}| \, H_c^\bullet(\mathcal{M}_O, \wp^{\phi_\mathbf{v}})^{\langle \mathrm{sign} \rangle}.$$

In the special case $\phi = 0$, the local system $\wp^\phi$ reduces to a constant sheaf and Theorem 1.2.2 yields a formula for the number of isomorphism classes of absolutely indecomposable objects:

$$\# \, \mathrm{AI}_\mathbf{v}(\mathscr{C}, \mathbb{F}_q) \;=\; \tfrac{q-1}{q} \cdot q^{-(\dim \mathfrak{X}_\mathbf{v} + \frac{\mathbf{v}\cdot\mathbf{v}-|\mathbf{v}|}{2})} \cdot \mathsf{Tr}_{\mathsf{Fr}}| \, H_c^\bullet(\mathcal{M}_O, \bar{\mathbb{Q}}_\ell)^{\langle \mathrm{sign} \rangle}. \tag{1.2.3}$$

Fix a finite set I and write kI for the algebra of functions $I \to k$, with pointwise operations. Let A be a kI-algebra, i.e. an associative k-algebra equipped with a k-algebra homomorphism $kI \to A$ (with a not necessarily central image). Let $\mathscr{C} = A$-mod and let F be the forgetful functor. Then, the stack $\mathfrak{X}_\mathbf{v}(\mathscr{C}, F)$ is nothing but the representation scheme $\mathrm{Rep}_\mathbf{v}(A)$, so we have $\mathfrak{X}_\mathbf{v}(\mathscr{C}) = \mathrm{Rep}_\mathbf{v}(A)/\mathrm{GL}_\mathbf{v}$, the moduli stack of $\mathbf{v}$-dimensional A-modules. The assumption that A be smooth implies that $\mathrm{Rep}_\mathbf{v}(A)$ is a smooth scheme. Hence, $\mathfrak{X}_\mathbf{v}(\mathscr{C})$ is a smooth Artin stack of dimension $\dim\left(\mathrm{Rep}_\mathbf{v}(A)/\mathrm{GL}_\mathbf{v}\right) = \dim \mathrm{Ext}^1(x, x) - \dim \mathrm{End}(x)$, for any $x \in \mathfrak{X}_\mathbf{v}(\mathscr{C})$.

To a large extent, our paper grew out of our attempts to better understand a remarkable proof of Kac's positivity conjecture by T. Hausel, E. Letellier, and F. Rodriguez-Villegas, [HLV1]. The present paper provides, in particular, a new proof of the main result of [HLV1]. Our approach is quite different from the one used in [HLV1].

In more detail, let $\mathscr{C}$ be the category of finite dimensional representations a quiver Q, equivalently, the category A-mod where A is the path algebra of Q. In that case the variety $\mathcal{M}_O$ becomes in this case a certain quiver variety. Our formula (1.2.3) is then equivalent to the main result of [HLV1], modulo the assertion, proved in [HLV2], that the cohomology of quiver varieties is pure and Tate, i.e. $H_c^{\mathrm{odd}}(\mathcal{M}_O, \bar{\mathbb{Q}}_\ell) = 0$ and the Frobenius acts on $H_c^{2j}(\mathcal{M}_O, \bar{\mathbb{Q}}_\ell)$ as a multiplication by q^j. In particular, we have that $\mathsf{Tr}_{\mathsf{Fr}}| \, H_c^\bullet(\mathcal{M}_O, \bar{\mathbb{Q}}_\ell) = \sum_{j \geq 0} q^j \cdot \dim H_c^{2j}(\mathcal{M}_O, \bar{\mathbb{Q}}_\ell)$ is a polynomial in q with non-negative integer coefficients, proving the Kac's positivity conjecture.

Remark 1.2.4. The fact that the cohomology of the quiver variety is pure and Tate also follows, using a deformation argument, cf. §11, from a much more general result of Kaledin [Ka] that says that rational cohomology groups of an arbitrary symplectic resolution are generated by algebraic cycles.

1.3. Deformation to the nilpotent cone. The RHS of formulas (1.1.6)-(1.1.7) involve an arbitrarily chosen coadjoint orbit $O \subset \mathfrak{pgl}_\mathbf{v}^\Diamond$ while the LHS of the formulas are independent of that choice. Therefore, it is tempting to find a replacement of the cohomology of the stack $\mathcal{M}_O$ by a more canonical object. Below, we explain how to do this in the case of quiver representations (and $\phi = 0$) or the case of parabolic bundles. In those cases, one can choose a stability condition θ

5

and consider of θ-semistable objects of the stack $\mathrm{T}^*\mathfrak{X}_\mathbf{v}(\mathscr{C}, F)/\mathrm{PGL}_\mathbf{v}$. The dimension vector $\mathbf{v}$ will be fixed throughout this subsection, so we drop it from the notation and write $G = \mathrm{PGL}_\mathbf{v}$, resp. $\mathfrak{g} = \mathfrak{pgl}_\mathbf{v}$. Let $\mathcal{M}^\theta$ be the coarse moduli space of θ-semistable objects of $\mathrm{T}^*\mathfrak{X}_\mathbf{v}(\mathscr{C}, F)/\mathrm{PGL}_\mathbf{v}$. The coarse moduli space is a quasi-projective variety whose closed $\bar{\mathsf{k}}$-points are polystable objects. In the case where $\mathscr{C}$ is the category of representations of a quiver Q, the coarse moduli space is a GIT quotient $\mathcal{M}^\theta = \mathrm{Rep}_\mathbf{v} \bar{Q} /\!\!/ G$, where $\mathrm{Rep}_\mathbf{v} \bar{Q}$ is the representation scheme of the double of the quiver Q. In the case of parabolic bundles the corresponding coarse moduli space is a well studied variety $\mathcal{M}^\theta$ of semistable parabolic Higgs bundles, cf. [Y].

In the above setting, the GIT counterpart of the moment $\mathrm{T}^* X_\mathbf{v}(\mathscr{C}, F) \to \mathfrak{pgl}_\mathbf{v}^*$ is a morphism $f^\theta : \mathcal{M}^\theta \to \mathfrak{g}^* /\!\!/ G$, where $\mathfrak{g}^* /\!\!/ G$ is a categorical quotient. An orbit $O \subset \mathfrak{g}^*$ maps to a point $\eta_O \in \mathfrak{g}^* /\!\!/ G$. The result below expresses the quantity in the RHS of (1.2.3) in terms of a 'limit' of the cohomology of the generic fiber $(f^\theta)^{-1}(\eta)$ as $\eta \to 0$, that is, the regular semisimple orbit O degenerates to the nilpotent cone $\mathrm{Nil} \subset \mathfrak{g}^*$.

To explain the result, let $\mathfrak{t}$ be the Cartan subalgebra of $\mathfrak{g}$ and $W = \mathfrak{S}_\mathbf{v}$ the Weyl group, so one has the Chevalley isomorphism $\mathfrak{g}^* /\!\!/ G = \mathfrak{t}^* /\!\!/ W$. Let $\mathfrak{t}^\diamond \subset \mathfrak{t}^*$ be the preimage of $\mathfrak{g}^\diamond/G$ and let $\mathcal{M}^\diamond := (f^\theta)^{-1}(\mathfrak{g}^\diamond /\!\!/ G)$. This is a Zariski open and dense subset contained in the stable locus of $\mathcal{M}^\theta$. Further, let $\mathcal{M}^\theta_\mathrm{nil} = (f^\theta)^{-1}(0)$, a closed subvariety of $\mathcal{M}^\theta$. We also consider another variety, $\mathcal{M}^\theta_0$, a closed subset of $\mathcal{M}^\theta_\mathrm{nil}$. In the case of quiver representations, $\mathcal{M}^\theta_0$ is defined as the image in $\mathrm{Rep}_\mathbf{v} \bar{Q} /\!\!/_\theta \mathrm{GL}_\mathbf{v}$ of the θ-semistable locus of the zero fiber of the moment map $\mathrm{Rep}_\mathbf{v} \bar{Q} \to \mathfrak{g}^*$. In the case of parabolic bundles, the variety $\mathcal{M}^\theta_0$ is formed by the Higgs bundles such that the Higgs field has no poles, equivalently, its residue at every marked point vanishes. Thus, one has a commutative diagram:

$$\mathcal{M}^\theta_0 \hookrightarrow \mathcal{M}^\theta_\mathrm{nil} \overset{\imath}{\hookrightarrow} \mathcal{M}^\theta \overset{\jmath}{\hookleftarrow} \mathcal{M}^\diamond \overset{pr}{\twoheadleftarrow} \mathcal{M}^\diamond \times_{\mathfrak{t}^\diamond /\!\!/ W} \mathfrak{t}^\diamond \qquad (1.3.1) \quad \boxed{\texttt{bs1}}$$

$$\begin{array}{ccccc} \downarrow f^\theta \quad \square & \downarrow f^\theta \quad \square & \downarrow f^\theta \quad \square & \downarrow \\ \{0\} \hookrightarrow \mathfrak{g}^* /\!\!/ G \hookleftarrow \mathfrak{t}^\diamond /\!\!/ W \twoheadleftarrow \mathfrak{t}^\diamond \end{array}$$

Here, $\imath$, resp. $\jmath$, is a closed, resp. open, imbedding, and all squares in the diagram are cartesian.

The map pr, the first projection, in the above diagram is an unramified covering, so one has a direct sum decomposition $pr_!\bar{\mathbb{Q}}_\ell = \oplus_{\rho \in \mathrm{Irr}(W)} \rho \otimes \mathcal{L}_\rho$, where $\mathrm{Irr}(W)$ is the set of isomorphism classes of irreducible representations of the group W, $\mathcal{L}_\rho$ is an irreducible local system on $\mathcal{M}^\diamond$ and $\bar{\mathbb{Q}}_\ell$ stands for a constant sheaf on $\mathcal{M}^\diamond \times_{\mathfrak{t}^\diamond /\!\!/ W} \mathfrak{t}^\diamond$. Let $\mathrm{IC}(\mathcal{L}_\rho)$ be the IC-extension of $\mathcal{L}_\rho$ (here, we use an unconventional normalization such that $\mathrm{IC}(\mathcal{L}_\rho)|_{\mathcal{M}^\theta} = \mathcal{L}_\rho$ is placed in homological degree zero rather than $-\dim \mathcal{M}^\theta$). The following result will be proved in section §10.

$\boxed{\texttt{la-thm}}$ **Theorem 1.3.2.** *Let $\mathscr{C}$ be either the category A-mod for a smooth algebra A or the category of parabolic bundles. Then, we have*

(i) *For any $\rho \in \mathrm{Irr}(W)$, the sheaf $\imath^* \mathrm{IC}(\mathcal{L}_\rho)$ is a semisimple perverse sheaf on $\mathcal{M}^\theta_\mathrm{nil}$, moreover, the cohomology $H^\bullet_c(\mathcal{M}^\theta_\mathrm{nil}, \mathrm{IC}(\mathcal{L}_\rho))$ is pure.*

(ii) *The support of the sheaf $\imath^* \mathrm{IC}(\mathcal{L}_\mathrm{sign})$ is contained in $\mathcal{M}^\theta_0$. Furthermore, if θ is sufficiently general then, in the 'coprime case', the map f^θ is a smooth morphism and $\imath^* \mathrm{IC}(\mathcal{L}_\mathrm{sign})$ is a constant sheaf on $\mathcal{M}^\theta_0$, up to shift.*

(iii) *For any $O \subset \mathfrak{g}^\diamond$, the scheme $(f^\theta)^{-1}(\eta_O)$ is smooth and one has an isomorphism $\mathcal{M}_O \cong (f^\theta)^{-1}(\eta_O)$, of stacks.*

(iv) *If $\mathscr{C}$ is either the category of quiver representations or the category of parabolic bundles then the monodromy action on $H^\bullet_c(\mathcal{M}_O, \bar{\mathbb{Q}}_\ell)$ factors through a W-action and one has a canonical isomorphism*

$$H^\bullet_c(\mathcal{M}_O, \bar{\mathbb{Q}}_\ell)^\rho = H^\bullet_c\big(\mathcal{M}^\theta_\mathrm{nil}, \imath^* \mathrm{IC}(\mathcal{L}_\rho)\big), \qquad \forall \rho \in \mathrm{Irr}\, W.$$

In particular, for θ sufficiently general in the 'coprime' case we have

$$\mathrm{Tr}_{\mathsf{Fr}}|\,H_c^{\bullet}(\mathcal{M}_O, \bar{\mathbb{Q}}_\ell)^{\mathrm{sign}} \;=\; q^{-\frac{1}{2}\dim\mathcal{M}_O}\cdot \#\mathcal{M}_0^{\theta}(\mathsf{k}). \tag{1.3.3}$$ `nopole`

Remark 1.3.4. We expect that parts (i)-(iii) of the theorem hold for more general quasi-abelian categories whenever one can construct coarse moduli spaces $\mathcal{M}^{\theta}$ with reasonable properties. Quite differently, the statement of part (iv) crucially relies on the fact that $\mathcal{M}^{\theta}$ is a *semi-projective* variety in the sense of [HV2]. We do not know examples beyond quiver representations and parabolic bundles where $\mathcal{M}^{\theta}$ is semi-projective. This is the reason we had to require in (iv) that the algebra A is a path algebra. ◇

The proof of Theorem 1.3.2 exploits an idea borrowed from [HLV2]. We use a deformation argument and the following general result that seems to be new and may be of independent interest.

`virt` **Proposition 1.3.5.** *Let $\widetilde{X} \xrightarrow{\pi} X \xrightarrow{p} S$ be a diagram of morphisms of schemes such that the scheme S is smooth, π is a proper, resp. $p\circ\pi$ is a smooth, morphism and, moreover, there is a Zariski open and dense subset $U \subset S$ such that the morphism $(p\circ\pi)^{-1}(U) \to p^{-1}(U)$, induced by from π by restriction, is smooth.*

Then, we have $\pi_*\bar{\mathbb{Q}}_\ell = \mathrm{IC}(\pi_*\bar{\mathbb{Q}}_\ell|_{p^{-1}(U)})$, i.e., the map π is virtually small in the sense of Meinhardt and Reineke [MR].

Assume in addition that π is generically finite. Then π is small. Furthermore, for any closed point $s \in S$ the map $(p\circ\pi)^{-1}(s) \to p^{-1}(s)$, induced by π, is semi-small.

A basic example of the setting of the proposition is provided by the Grothendieck-Springer resolution $\mu : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ for a semisimple Lie algebra $\mathfrak{g}$. In that case, we let $S = \mathfrak{t}$ be the Cartan subalgebra, resp. $\widetilde{X} = \widetilde{\mathfrak{g}}$ and $X = \mathfrak{g} \times_{\mathfrak{t}/W} \mathfrak{t}$, where W denotes the Weyl group. There is a natural smooth morphism $\widetilde{\mathfrak{g}} \to \mathfrak{t}$ that factors as a composition of a proper birational morphism $\pi : \widetilde{\mathfrak{g}} \to \mathfrak{g} \times_{\mathfrak{t}/W} \mathfrak{t}$ and the second projection $p : \mathfrak{g} \times_{\mathfrak{t}/W} \mathfrak{t} \to \mathfrak{t}$.

For applications to the proof of Theorem 1.3.2 one takes $S = \mathfrak{t}^*$ and $X = \mathcal{M}^{\theta} \times_{\mathfrak{t}^*/\!/W} \mathfrak{t}^*$. The corresponding variety $\widetilde{X}$ is defined as a certain GIT quotient for a stability condition θ' which is a sufficiently general deformation of the given stability condition θ, see §10. Proposition 1.3.5 insures that the resulting variety is a small resolution of $\mathcal{M}^{\theta} \times_{\mathfrak{t}^*/\!/W} \mathfrak{t}^*$. There is a strong similarity with the construction of M. Reineke [Rei], although the setting in *loc cit* involves quiver moduli spaces while our setting corresponds to the cotangent bundle of those moduli spaces.

`par-sec` **1.4. Parabolic bundles.** Let C be a smooth projective geometrically connected curve over $\mathbb{F}_q$, and $D = \{c_i \in C(\mathbb{F}_q),\ i \in I\}$ a fixed collection of pairwise distinct marked points sometimes thought of as a divisor $D = \sum_{i\in I} c_i$. Associated with a dimension vector $\mathbf{m} = (m_i)_{i\in I}$ with positive coordinates, there is a category $\mathscr{C} = \mathcal{V}ect(C, D, \mathbf{m})$ of parabolic bundles. An object of $\mathcal{V}ect(C, D, \mathbf{m})$ is a vector bundle, i.e. a locally free sheaf $\mathcal{V}$ on C equipped, for each $i \in I$, with an m_i-step partial flag $V_i^{\bullet} = (\mathcal{V}|_{c_i} = V_i^{(0)} \supseteq V_i^{(1)} \supseteq \ldots \supseteq V_i^{(m_i)} = 0)$ in the fiber of $\mathcal{V}$ at c_i. Put $r_i^{(j)} = \dim(V_i^{(j)}/V_i^{(j+1)})$ and call the array $\mathbf{r} = \{r_i^{(j)} \in \mathbb{Z}_{\geq 0},\ (i,j)\ |\ i \in I,\ j = 1,\ldots,m_i\}$ the *type* of the parabolic bundle. The integer $|\mathbf{r}| = \sum_{j=1}^{m_i} r_i^{(j)}$ is the same for all $i \in I$ and it is the rank of the vector bundle $\mathcal{V}$. Let $\Xi(\mathbf{m}) = \{(i,j)\ |\ i \in I,\ j = 1,\ldots,m_i\}$. It will be convenient to view various types $\mathbf{r}$ as dimension vectors $\mathbf{r} = (r_i^{(j)}) \in \mathbb{Z}_{\geq 0}^{\Xi(\mathbf{m})}$. A parabolic type such that $m_i = 1\ \forall i$, will be called 'trivial'. For any such type the corresponding category $\mathcal{V}ect(C, D, \mathbf{m})$ is the category of vector bundles without any flag data.

A morphism in $\mathcal{V}ect(C, S, \mathbf{m})$ is, by definition, a morphism $f : \mathcal{V} \to \mathcal{W}$ of locally free sheaves that respects the partial flags, i.e. such that $f(V_i^{(j)}) \subset W_i^{(j)}$, for all i,j. A direct sum of parabolic bundles $\mathcal{V}$ and $\mathcal{W}$ is the vector bundle $\mathcal{V} \oplus \mathcal{W}$ equipped with the partial flags defined by $(V \oplus$

7

$W)_i^{(j)} := V_i^{(j)} \oplus W_i^{(j)}$. This makes $\mathcal{V}ect(C, D, \mathbf{m})$ a k-linear, not necessarily abelian, Karoubian category. The assignment

$$\mathcal{V} \mapsto F(\mathcal{V}) = \bigoplus_{(i,j)\in\Xi(\mathbf{m})} V_i^{(j)}/V_i^{(j+1)}. \tag{1.4.1}$$

gives a k-linear functor from category $\mathcal{V}ect(C, D, \mathbf{m})$ to the category of finite dimensional $\Xi(\mathbf{m})$-graded vector spaces. One has a decomposition $\mathfrak{X}(\mathcal{V}ect(C, D, \mathbf{m})) = \sqcup_{\mathbf{r}\in\Xi(\mathbf{m})} \mathfrak{X}_{\mathbf{r}}(\mathcal{V}ect(C, D, \mathbf{m}))$, where $\mathfrak{X}_{\mathbf{r}}(\mathcal{V}ect(C, D, \mathbf{m}))$ is the stack parametrizing parabolic bundles of type $\mathbf{r}$. The corresponding framed stack $\mathfrak{X}_{\mathbf{r}}(\mathcal{V}ect(C, D, \mathbf{m}), F)$ parametrizes the data $(\mathcal{V}, (\mathcal{V}_i^{\bullet})_{i\in I}, b = (b_i^{(j)})_{(i,j)\in\Xi(\mathbf{m})})$, where $(\mathcal{V}, (\mathcal{V}_i^{\bullet})_{i\in I})$ is a parabolic bundle of type $\mathbf{r}$ and $b_i^{(j)}$ is a k-basis of the vector space $V_i^{(j)}/V_i^{(j+1)}$. The group $\mathrm{GL}_{\mathbf{r}} := \prod_{(i,j)\in\Xi(\mathbf{m})} \mathrm{GL}_{r_i^{(j)}}$ acts on $\mathfrak{X}(\mathcal{V}ect(C, D, \mathbf{m}), F)$ by changing the bases. One can show that this action factors through an action of $G_{\mathbf{r}} = G_{\mathbf{r}}/\mathbb{G}_m$.

Remark 1.4.2. Note that in order to have a nonzero functor F the set I of marked points must always be nonempty, even if one is only interested in vector bundles without flag data, i.e., parabolic bundles of trivial type. $\diamond$

In this paper, a Higgs field on a parabolic bundle $\mathcal{V}$ is, by definition, a morphism $\mathcal{V} \to \mathcal{V}\otimes\Omega_C^1(D)$ such that

$$res_i(u)(V_i^{(j)}) \subset V_i^{(j)} \otimes \Omega_C^1(D)|_{c_i}, \quad \forall\,(i,j) \in \Xi(\mathbf{m}). \tag{1.4.3}$$

Here, u is viewed as a section of $\mathscr{H}om(\mathcal{V}, \mathcal{V}) \otimes \Omega_C^1$ with simple poles at the marked points and $res_i(u) \in \mathrm{End}(\mathcal{V}|_{c_i})$ denotes the residue of u at the point c_i.

Remark 1.4.4. Our terminology is slightly different from the standard one. The objects that we call 'parabolic bundles' are usually referred to as 'quasi-parabolic bundlles', while the name 'parabolic bundle' is reserved to objects equipped with additional stability parameters. Since we work with stacks, these parameters will play no role in our considerations except for the section 10. Also, Higgs fields are often defined as morphisms $u : \mathcal{V} \to \mathcal{V}\otimes\Omega_C^1(D)$ satisfying a stronger requirement: $res_i(u)(V_i^{(j)}) \subset V_i^{(j+1)} \otimes \Omega_C^1(D)|_{c_i}, \forall\,(i,j) \in \Xi(\mathbf{m})$. $\diamond$

The stack $\mathrm{T}^*\mathfrak{X}_{\mathbf{r}}(\mathcal{V}ect(C, D, \mathbf{m}), F)$ may be identified with the stack of framed parabolic Higgs bundles. A k-point of this stack is a triple $(\mathcal{V}, b, u)$ where $\mathcal{V}$ is a parabolic bundle, b is a framing, and u is a Higgs field on $\mathcal{V}$. We let $\mathfrak{g}_{\mathbf{r}} = \mathrm{Lie}\,G_{\mathbf{r}}$ and use the natural identification $\mathfrak{g}_{\mathbf{r}}^* = \mathfrak{g}_{\mathbf{r}}$. The moment map $\mathrm{T}^*\mathfrak{X}_{\mathbf{r},d}(\mathcal{V}ect(C, D, \mathbf{m}), F) \to \mathfrak{g}_{\mathbf{r}}^*$ associated with the $G_{\mathbf{r}}$-action on the cotangent stack sends $(\mathcal{V}, u, b)$ to the collection $(\mathrm{gr}\,res_i(u))_{i\in I}$. Here, $\mathrm{gr}\,res_i(u)) \in \oplus_j \mathrm{End}(V_i^{(j)}/V_i^{(j+1)})$ is the map induced by $res_i(u)$ and the framing b provides an identification $\mathrm{End}(V_i^{(j)}/V_i^{(j+1)}) = \mathfrak{gl}_{r_i^j}$; thus, the collection $(\mathrm{gr}\,res_i(u))_{i\in I}$ may be identified with an element of $\mathfrak{g}_{\mathbf{r}}$. Note that the $G_{\mathbf{r}}$-conjugacy class of that element is independent of the choice of the framing b.

Let $\#\,\mathrm{AI}(\mathcal{P}\mathcal{B}un_{\mathbf{r},d}, \mathbb{F}_q)$ be the number of isomorphism classes of $\mathbb{F}_q$-rational absolutely indecomposable parabolic bundles of type $\mathbf{r}$ and degree d. Given a conjugacy class $O \subset \mathfrak{g}_{\mathbf{r}}$, let $Higgs_{\mathbf{r},d}(O)$ be (the Beilinson-Drinfeld modification, cf. Remark 1.1.8(i), of) the stack of parabolic Higgs bundles whose objects are pairs $(\mathcal{V}, u)$ such that $\mathrm{gr}\,res(u) \in O$. Using formula (1.2.3) we prove

Theorem 1.4.5. *For a very general $\mathbb{F}_q$-rational semisimple conjugacy class O in $\mathfrak{g}_{\mathbf{r}}$, we have an equation*

$$\#\,\mathrm{AI}(\mathcal{P}\mathcal{B}un_{\mathbf{r},d}, \mathbb{F}_q) = q^{-\frac{1}{2}\dim Higgs_{\mathbf{r},d}(O)} \cdot \mathsf{Tr}_{\mathsf{Fr}}|\,H_c^\bullet(Higgs_{\mathbf{r},d}(O), \bar{\mathbb{Q}}_\ell)^{\langle\mathrm{sign}\rangle}. \tag{1.4.6}$$

We remark that $\mathcal{P}\mathcal{B}un_{\mathbf{r},d}$ is an Artin stack of *infinite type* and the corresponding set $[\mathcal{P}\mathcal{B}un_{\mathbf{r},d}(\mathbb{F}_q)]$ is not finite. Therefore, Theorem 1.2.3 can not be applied to category $\mathcal{V}ect(C, D, \mathbf{m})$ directly. We will prove formula (1.4.6) by approximating category $\mathcal{V}ect(C, D, \mathbf{m})$ by an increasing family of subcategories such that the corresponding moduli stacks have finite type. We show that

the sets of absolutely indecomposable objects of these categories stabilize. This allows to find $\# \operatorname{AI}(\mathcal{PB}un_{\mathbf{r},d}, \mathbb{F}_q)$, in particular, this number is finite.

In the case of a trivial parabolic type, the LHS of the above formula reduces to $\# \operatorname{AI}_{r,d}(Bun_{r,d}, \mathbb{F}_q)$, the number of isomorphism classes of $\mathbb{F}_q$-rational absolutely indecomposable vector bundles of rank r and degree d. If, in addition, r and d are *coprime* the RHS of formula (1.4.6) can be expressed in terms of the coarse moduli space, $Higgs^{ss}_{r,d}$, of semistable Higgs bundles of rank r, degree d, and such that the Higgs field is regular on the whole of C. Specifically, formula (1.3.3) yields

$$\# \operatorname{AI}(Bun_{r,d}, \mathbb{F}_q) = q^{-\frac{1}{2}\dim Higgs^{ss}_{r,d}} \cdot \# Higgs^{ss}_{r,d}(\mathbb{F}_q) \quad \text{if } (r,d) = 1. \tag{1.4.7} \quad \boxed{\text{sch}}$$

Another, much more explicit, formula for $\# \operatorname{AI}(Bun_{r,d}, \mathbb{F}_q)$ has been obtained earlier by O. Schiffmann [Sch], cf. also [MS]. The approach in *loc cit* is totally different from ours and it is close, in spirit, to the strategy used in [CBvdB] in the case of quiver representations with an indivisible dimension vector. It seems unlikely that such an approach can be generalized to the case of vector bundles with $(r,d) > 1$, resp. the case of quiver representations with divisible dimension vector. Note that the more general formula (1.4.6) applies regardless of whether $(r,d) = 1$ or not. The relation between (1.4.6) and the approach used in [Sch] and [CBvdB] is discussed in more detail in §10.

1.5. **Higgs bundles and local systems.** Remarkably, indecomposable parabolic bundles provide a bridge between the geometry of Higgs bundles, on one hand, and counting *irreducible* ℓ-adic local systems on the punctured curve with prescribed monodromies at the punctures, on the other hand. In more detail, let $C_i^* = C_i \smallsetminus \{c_i\}$ where C_i is the henselization of C at c_i. Fix an I-tuple $(\mathscr{L}_i)_{i \in I}$, where $\mathscr{L}_i$ is a rank r tame semisimple $\bar{\mathbb{Q}}_\ell$-local system on C_i^* such that $\operatorname{Fr}^* \mathscr{L}_i \cong \mathscr{L}_i$.

In [De], Deligne addressed the problem of counting elements of the set $Loc_{\text{et}}((\mathscr{L}_i)_{i \in I}, q)$, of isomorphism classes of irreducible $\bar{\mathbb{Q}}_\ell$-local systems $\mathscr{L}$ on $C^* := C \smallsetminus (\bigcup_i c_i)$, such that $\operatorname{Fr}^* \mathscr{L} \cong \mathscr{L}$ and $\mathscr{L}|_{C_i^*} \cong \mathscr{L}_i$. Motivated by an old result of Drinfeld [Dr] in the case $r = 2$, one expects that the function $n \mapsto \# Loc_{\text{et}}((\mathscr{L}_i)_{i \in I}, q^n)$ behaves as if it counted $\mathbb{F}_{q^n}$-rational points of an algebraic variety over $\mathbb{F}_q$. The main result of Deligne confirms this expectation under the following additional assumptions on the collection $(\mathscr{L}_i)_{i \in I}$:

(1) Each of the local systems $\mathscr{L}_i$ is semisimple and tame.
(2) One has $\prod_{i \in I} \det \mathscr{L}_i = 1$, see [De], equation (2.7.2).
(3) *Genericity condition:* for any $0 < r' < r$ and any I-tuple $(\mathscr{L}'_i)_{i \in I}$ where $\mathscr{L}'_i$ is a direct summand of $\mathscr{L}_i$ of rank r', one has $\prod_{i \in I} \det \mathscr{L}'_i \neq 1$, see [De], equations (2.7.3) and (3.2.1).

Recall that the tame quotient of the geometric fundamental group of C_i^* is isomorphic to $\widehat{\mathbb{Z}}(1)$, cf. [De, §2.6]. Therefore, the local system $\mathscr{L}_i$ being tame by (1), it gives a homomorphism $\rho_i : \widehat{\mathbb{Z}}(1) \to \operatorname{GL}_r(\bar{\mathbb{Q}}_\ell)$. Condition (2) says that, writing $\prod_i \det \rho_i(z) = 1$ for all $z \in \widehat{\mathbb{Z}}(1)$. This equation is neccesary for the existence of a local system $\mathscr{L}$ on $C \smallsetminus D$ such that $\mathscr{L}|_{C_i^*} \cong \mathscr{L}_i, \; \forall i \in I$, see [De, §2.7]. Furthermore, condition (3) guarantees that any such local system $\mathscr{L}$ is automatically irreducible. Condition (1) implies also that each local system $\mathscr{L}_i$ is isomorphic to a direct sum of rank one local systems. Thus, one can write

$$\mathscr{L}_i = (\mathscr{L}_{i,1})^{\oplus r_i^{(1)}} \bigoplus \ldots \bigoplus (\mathscr{L}_{i,m_i})^{\oplus r_i^{(m_i)}}, \qquad \forall \, i \in I,$$

where the tuple $(r_i^{(1)}, \ldots, r_i^{(m_i)})$ is a partition of r with m_i parts and $\mathscr{L}_{i,j}, \; j \in [1, m_i]$, is an unordered collection of pairwise nonisomorphic rank one tame local systems. We say that the I-tuple $(\mathscr{L}_i)_{i \in I}$ has type $(r_i^{(j)})$.

9

According to Deligne [De, Théorème 3.5], for an I-tuple $(\mathscr{L}_i)_{i\in I}$ of type $\mathbf{r} = (r_i^{(j)})$ one has

$$\# Loc_{\mathrm{et}}((\mathscr{L}_i)_{i\in I},\, q) \;=\; \# \mathrm{AI}(\mathcal{P}\mathcal{B}un_{\mathbf{r},d}, \mathbb{F}_q), \tag{1.5.1}$$ de

for any fixed degree d. The proof of this equation, which is the main result of [De], rests on the validity of the Langlands conjecture for GL_n proved by Lafforgue. Comparing (1.5.1) and (1.4.6), we obtain

deligne **Corollary 1.5.2.** *Let $(\mathscr{L}_i)_{i\in I}$ be an I-tuple such that conditions (1)-(3) hold and let $\mathbf{r}$ be the corresponding type. Then, for any degree d and a very general conjugacy class O in $\mathfrak{g}_{\mathbf{r}}$ one has*

$$\# Loc_{et}((\mathscr{L}_i)_{i\in I},\, q) \;=\; q^{-\frac{1}{2}\dim Higgs_{\mathbf{r},d}(O)} \cdot \mathsf{Tr}_{\mathsf{Fr}}|\, H_c^\bullet(Higgs_{\mathbf{r},d}(O), \bar{\mathbb{Q}}_\ell)^{\langle \mathrm{sign}\rangle}. \tag{1.5.3}$$ DN

Remark 1.5.4. It would be interesting to find an analogue of (1.5.3) for a general reductive group G that relates the number of ℓ-adic G-local systems to the number of $\mathbb{F}_q$-rational points of a variety of Higgs bundles for the Langlands dual group of G. Note that the set of indecomposable parabolic bundles has no obvious counterpart for an arbitrary G. ◇

The RHS of (1.5.3) simplifies in the generic case, i.e. the case where one has $\mathscr{L}_i \not\cong \mathscr{L}_j$ for all $i \neq j$. This forces all $r_1^{(j)}$ be equal to 1, so $\mathbf{r}$ corresponds to the parabolic structure such that all partial flags are *complete* flags. The group $G_{\mathbf{r}}$ is in this case a torus. Therefore, the group $\mathfrak{S}_{\mathbf{r}}$ reduces to the identity element and the cohomology group in the RHS of (1.5.3) is equal to its sign-isotypic component. Hence, by the Lefschetz trace formula one finds that the RHS of (1.5.3) equals $q^{-\frac{1}{2}\dim Higgs_{r,d,\text{complete flags}}(O)} \cdot \# Higgs_{r,d,\text{complete flags}}(O)(\mathbb{F}_q)$.

Some very interesting phenomena also occur in the case where the set I consists of one element, so there is only one marked point $c \in C$. Let ζ be a primitive root of unity of order r and use simplified notation $Loc_{\mathrm{et}}^\zeta(q) = Loc_{\mathrm{et}}((\mathscr{L}_i)_{i\in I},\, q)$ for the set of isomorphism classes of tame rank r local systems $\mathscr{L}$ on $C \smallsetminus \{c\}$ such that $\mathrm{Fr}^*\mathscr{L} \cong \mathscr{L}$ and the monodromy of $\mathscr{L}$ at c equals $\zeta \cdot \mathrm{Id}$. Such a local system is automatically simple and the corresponding type $\mathbf{r}$ is *trivial*. Therefore, formula (1.4.7) is applicable and using Corollary 1.5.2, for any d coprime to r, we obtain

$$\# Loc_{\mathrm{et}}^\zeta(q) \;=\; q^{-\frac{1}{2}\dim Higgs_{r,d}^{\mathrm{ss}}} \cdot \mathsf{Tr}_{\mathsf{Fr}}|\, H_c^\bullet(Higgs_{r,d}^{\mathrm{ss}}, \bar{\mathbb{Q}}_\ell). \tag{1.5.5}$$ et

On the other hand, let C_{top} be a compact Riemann surface of the same genus as the algebraic curve C and $\mathrm{pt} \in C_{top}$ a marked point. For any field k that contains r-th roots of unity, one has a smooth affine algebraic variety Loc_{Betti}^ζ, called *character variety*, that parametrizes isomorphism classes of r-dimensional representations of the fundamental group of $C_{top} \smallsetminus \{\mathrm{pt}\}$ such that a small loop around the puncture pt goes to $\zeta \cdot \mathrm{Id}$. It is known that for $\mathsf{k} = \mathbb{C}$ there is a natural diffeomorphism $Loc_{Betti}^\zeta(\mathbb{C}) \cong Higgs_{r,d}^{\mathrm{ss}}(\mathbb{C})$, *of C^∞-manifolds*. In particular, these two C^∞-manifolds have isomorphic singular cohomology groups. However, these two manifolds are *not* isomorphic as complex algebraic varieties and the isomorphism of the cohomology induced by the diffeomorphism does not respect the Hodge structures. The number of $\mathbb{F}_q$-rational points of Loc_{Betti}^ζ is controlled by the E-polynomial associated with the Hodge structure. Specifically, it was shown in [HV] using a result of N. Katz that one has

$$\# Loc_{Betti}^\zeta(\mathbb{F}_q) \;=\; \sum_j q^j \cdot \left(\sum_k (-1)^k \dim \mathrm{gr}_{2j}^W H_c^k(Loc_{Betti}^\zeta(\mathbb{C}),\, \mathbb{C}) \right), \tag{1.5.6}$$ ltop

where $\mathrm{gr}_\bullet^W(-)$ denotes an associated graded space with respect to the weight filtration (the Hodge structure on $H^\bullet(Loc_{Betti}^\zeta(\mathbb{C}),\, \mathbb{C})$ is not pure).

It would be interesting to compare formulas (1.5.5) and (1.5.6) in the context of '$P = W$'-conjecture, see [CHM].

Remark 1.5.7. The above formulas may also be reformulated in terms of de Rham local systems. To do so, let R be a domain over $\mathbb{Z}$ that admits homomorphisms $R \to \mathbb{C}$ and $R \to \mathsf{k}$, where k is a finite field of characteristic p. Let C_R be a smooth projective curve over R with a good reduction to a smooth curve C_k over k, and let $C_\mathbb{C}$ be a complex curve obtained from C_R by specialization. Let $Loc_{\mathrm{DR}}^{\frac{1}{r}}(C_\mathbb{C})$, resp. $Loc_{\mathrm{DR}}^{\frac{1}{r}}(C_\mathsf{k})$, be the the moduli space of 'de Rham local systems', that is, of pairs $(\mathcal{V}, \nabla)$ where $\mathcal{V}$ is a rank r algebraic vector bundle on $C_\mathbb{C}$, resp. $C_\mathbb{F}$, and ∇ is a connection on $\mathcal{V}$ with regular singularity at pt such that $res_{\mathrm{pt}} \nabla = \frac{1}{r} \cdot \mathrm{Id}$. The Riemann-Hilbert correspondence provides an isomorphism $Loc_{\mathrm{Betti}}^{\zeta}(\mathbb{C}) \cong Loc_{\mathrm{DR}}^{\zeta}(\mathbb{C})$, of complex analytic manifolds (but not algebraic varieties) that respects the Hodge structures. Therefore, the RHS of (1.5.6) is not affected by replacing $Loc_{Betti}^{\zeta}(\mathbb{C})$ by $Loc_{DR}^{\zeta}(\mathbb{C})$.

On the other hand, recall that given a connection ∇ on a vector bundle $\mathcal{V}$ on C_k, there is an associated p-curvature, $\mathrm{curv}_p(\nabla)$, defined as the map $\xi \mapsto \nabla^p(\xi) - \nabla(\xi^p)$. Furthermore, it is known, cf. [BB], that the assignment $(\mathcal{V}, \nabla) \mapsto (\mathsf{Fr}^*\mathcal{V}, \mathrm{curv}_p(\nabla))$ provides an isomorphism between moduli of rank r local systems on C_k and rank r Higgs bundles on $C_\mathsf{k}^{(1)}$, the Frobenius twist of C_k, respectively. Assume that $p > r$ and fix an integer m prime to r. One can show that if ∇ has a simple pole at pt with residue $\frac{m}{r} \mathrm{Id}$ then $\mathrm{curv}_p(\nabla)$ has no singularity at pt. Moreover, this way one obtains an isomorphism of the stack $Loc_{\mathrm{DR}}^{\frac{1}{r}}(C_\mathsf{k})$ and the Frobenius twist of the stack $Higgs_{r,1}$. Thus, for the corresponding numbers of isomorphism classes of k-rational points one gets $\# Loc_{\mathrm{Betti}}^{\zeta}(C_\mathsf{k})(\mathsf{k}) = q^{-\frac{1}{2} \dim Higgs_{r,d}^{\mathrm{ss}}} \cdot \# Higgs_{r,d}^{\mathrm{ss}}(\mathsf{k})$ for any d.

Prime numbers as a twistor sphere.

1.6. Factorization sheaves.

Factorization sheaves have appeared in the literature in various contexts, see eg. [FG]. We will use the following settings.

Fix a smooth scheme C over k and write $\mathrm{Sym}^v C$ for the v-th symmetric power of C. For any pair $v_1, v_2 \in \mathbb{Z}_{\geq 0}$, let $C_{v_1, v_2}^{\mathrm{disj}}$ be an open subset of $C^{v_1 + v_2}$ formed by the tuples $(c_1, \ldots, c_{v_1}, c_1', \ldots, c_{v_2}') \in C^{v_1 + v_2}$ such that $c_j \neq c_k'$ for all j, k. Let $\mathrm{Sym}_{\mathrm{disj}}^{v_1 + v_2} C$ be the image of $C_{v_1, v_2}^{\mathrm{disj}}$ in $\mathrm{Sym}^{v_1 + v_2} C$. A factorization sheaf on $\mathrm{Sym}\, C$ is, by definition, a collection $\mathcal{F} = (\mathcal{F}_v)_{v \geq 0}$, where $\mathcal{F}_v \in D_{\mathrm{abs}}(\mathrm{Sym}^v C)$, equipped, for each pair v_1, v_2, with an isomorphism $\mathcal{F}_{v_1 + v_2}|_{\mathrm{Sym}_{\mathrm{disj}}^{v_1 + v_2} C} \xrightarrow{\sim} (\mathcal{F}_{v_1} \boxtimes \mathcal{F}_{v_2})|_{\mathrm{Sym}_{\mathrm{disj}}^{v_1 + v_2} C'}$ satisfying certain associativity and commutativity constraints, see §3. Here, the notation $D_{\mathrm{abs}}(X)$ stands for the the triangulated category of absolutely convergent Weil complexes on a stack X, as defined by Behrend [B1],[B2] and reminded in §2 below.

There are two other settings sheaves where the sheaf $\mathcal{F}_v$ on $\mathrm{Sym}^v C$ is replaced, in the above definition, by either a $\mathfrak{S}_v$-equivariant sheaf on C^v (equivalently, the categorical quotient $\mathrm{Sym}^v C = C^v /\!\!/ \mathfrak{S}_v$ is replaced by $C^v/\mathfrak{S}_v$, a stacky quotient of C^v by $\mathfrak{S}_v$) or by a sheaf on $\mathrm{Coh}_v(C)$, the stack of length v coherent sheaves on C. In the latter case, the subset $\mathrm{Sym}_{\mathrm{disj}}^{v_1 + v_2} C$ should be replaced by $\mathrm{Coh}_{v_1, v_2}^{\mathrm{disj}}$, an open substack of $\mathrm{Coh}_{v_1 + v_2}(C)$ whose objects are coherent sheaves of the form $M_1 \oplus M_2$ such that $\mathrm{supp}(M_1) \cap \mathrm{supp}(M_2) = \emptyset$ and $\mathrm{length}(M_j) = v_j$, $j = 1, 2$. In these two settings, the collections $\mathcal{F} = (\mathcal{F}_v)_{v \geq 0}$ where $\mathcal{F}_v \in D_{\mathrm{abs}}(C^v/\mathfrak{S}_v)$, resp. $\mathcal{F}_v \in D_{\mathrm{abs}}(\mathrm{Coh}_v(C))$, will be referred to as ' factorization sheaves on C', resp. ' factorization sheaves on $\mathrm{Coh}_v(C)$'.

Factorization sheaves on $\mathrm{Coh}\, C$ and $\mathrm{Sym}\, C$ are related via a pair of adjoint functors. Specifically, for each v one has an adjoint pair $D_{\mathrm{abs}}(\mathrm{Coh}_v C) \underset{\mathrm{ind}}{\overset{\mathrm{res}}{\rightleftarrows}} D_{\mathrm{abs}}(C^v/\mathfrak{S}_v)$, of functors of parabolic restriction and induction. One shows that these functors take factorization sheaves to factorization sheaves, cf. Remark 4.2.6.

11

Finally, there is also a version of the notion of factorization sheaf in the coherent world where one considers collections $M = (M_v)$ with $M_v \in D_{Coh}(C^v/\mathfrak{S}_v)$, an object of the derived category of coherent sheaves.

All the above can be extended to an I-graded setting by replacing a given scheme C by a disjoint union $\sqcup_{i \in I} C_i$, of several copies of C each 'colored' by an element of the set I. Given a dimension vector $\mathbf{v} = (v_i)$, let $C^{\mathbf{v}} = \prod_{i \in I} C_i^{v_i}$. Further, elements of $\text{Sym}^{\mathbf{v}} C$ are colored effective divisors on C with degree v_i for the i-th color. Similarly, the stack $\text{Coh}_{\mathbf{v}}(C)$ parametrizes finite length coherent sheaves whose support is a colored divisor of multi-degree $\mathbf{v}$. Thus, one may consider an object $\mathcal{F}_{\mathbf{v}}$ of $D_{\text{abs}}(\text{Sym}^{\mathbf{v}} C)$, resp. $D_{Coh}(\text{Sym}^{\mathbf{v}} C)$, $D_{\text{abs}}(C^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}})$, and $D_{\text{abs}}(\text{Coh}_{\mathbf{v}} C)$.

We will apply factorization sheaves to moduli problems using a construction that associates to a triple $(\mathscr{C}, F, \phi)$, as in §1.1, and an affine curve C a factorization sheaf on $\text{Coh} \, C$. To explain the construction, write $K = \mathsf{k}[C]$ for the coordinate ring of C and let $K \otimes \mathscr{C}$ be the corresponding base change category. For any object (x, α) of $K \otimes \mathscr{C}$, the composite homomorphism $K \xrightarrow{\alpha} \text{End}(x) \xrightarrow{F} \text{End} \, F(x)$ makes $F(x)$ a K-module. So the assignment $(x, \alpha) \mapsto F(x)$ yields a functor $K \otimes F : K \otimes \mathscr{C} \to K\text{-mod}$. Thus, one obtains a diagram of natural maps

$$\text{Coh}_{\mathbf{v}}(C) = \mathfrak{X}_{\mathbf{v}}(K\text{-mod}) \xleftarrow{\nu} \mathfrak{X}_{\mathbf{v}}(K \otimes \mathscr{C}) \xrightarrow{p} \mathfrak{X}_{\mathbf{v}}(\mathscr{C}) \xrightarrow{\phi_{\mathbf{v}}} \mathbb{A}^1 \qquad (1.6.1) \quad \boxed{\text{Cdiag}}$$

where the map ν is induced by the functor $K \otimes F$ and we have used an equivalence between the category of finite dimensional $\mathsf{k}[C]$-modules and of finite length coherent sheaves on C, respectively.

Remark 1.6.2. For $C = \mathbb{G}_m$, one has isomorphisms $\text{Coh}_{\mathbf{v}}(\mathbb{G}_m) \cong \text{GL}_{\mathbf{v}}/_{\text{Ad}} \text{GL}_{\mathbf{v}} \cong I(\text{pt}/\text{GL}_{\mathbf{v}})$, resp. $\mathfrak{X}(\mathsf{k}[\mathbb{G}_m] \otimes \mathscr{C}) \cong I(\mathfrak{X}(\mathscr{C}))$, where $I(\mathfrak{Y})$ denotes the *inertia stack* of a stack $\mathfrak{Y}$. Thus, in the special case $C = \mathbb{G}_m$, the map ν in (1.6.1) may be viewed as a map $I(\mathfrak{X}(\mathscr{C})) \to I(\text{pt}/\text{GL}_{\mathbf{v}})$. This map of inertia stacks is induced by the morphism $\mathfrak{X}_{\mathbf{v}}(\mathscr{C}) \to \text{pt}/\text{GL}_{\mathbf{v}}$ that comes from $\mathfrak{F}$, the rank $\mathbf{v}$ vector bundle on $\mathfrak{X}_{\mathbf{v}}(\mathscr{C})$, cf. §1.1. $\diamond$

The following result plays a crucial role in the proof of Theorem 1.1.5.

$\boxed{\text{cruc}}$ **Theorem 1.6.3.** *For any smooth affine curve C and a data $(\mathscr{C}, F, \phi)$, as in §1.1, the collection $\mathcal{R}^{F,\phi,C} = (\mathcal{R}_{\mathbf{v}}^{F,\phi,C})$ defined by the formula $\mathcal{R}_{\mathbf{v}}^{F,\phi,C} = \nu_! \wp^{\phi_{\mathbf{v}} \circ p}$ is a factorization sheaf on $\text{Coh} \, C$.*

$\boxed{\text{h-intro}}$

1.7. **Cohomology of factorization sheaves.** Fix a smooth scheme C with tangent, resp. cotangent, sheaf T_C, resp. T_C^*. Let $\mathbf{v}$ be a dimension vector, $C \subset C^{\mathbf{v}}$ the principal diagonal, and $\mathsf{N}_{\mathbf{v}} \to C$, resp. $\mathsf{N}_{\mathbf{v}}^* \to C$, the normal, resp. conormal bundle. The natural $\mathfrak{S}_{\mathbf{v}}$-action on $C^{\mathbf{v}}$ by permutation of factors induces a $\mathfrak{S}_{\mathbf{v}}$-action on $\mathsf{N}_{\mathbf{v}}$, resp. $\mathsf{N}_{\mathbf{v}}^*$. We will define a Zariski open $\mathfrak{S}_{\mathbf{v}}$-stable subset $\mathsf{N}_{\mathbf{v}}^\diamond \subset \mathsf{N}_{\mathbf{v}}^*$. To simplify the exposition we assume that there exists k-rational section $\eta_{\mathbf{v}} : C \to \mathsf{N}_{\mathbf{v}}^\diamond/\mathfrak{S}_{\mathbf{v}}$. We consider the following chain of functors

$$D_{\text{abs}}(C^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}) \xrightarrow{\mathsf{sp}_{\mathsf{N}}} D_{\text{abs}}(\mathsf{N}_{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}) \xrightarrow{\mathsf{F}_{\mathsf{N}}} D_{\text{abs}}(\mathsf{N}_{\mathbf{v}}^*/\mathfrak{S}_{\mathbf{v}}) \xrightarrow{\eta_{\mathbf{v}}^*} D_{\text{abs}}(C), \qquad (1.7.1) \quad \boxed{\text{Phi-fun}}$$

where F_{N} is the Fourier-Deligne transform along the fibers of the vector bundle N and sp_{N} is the Verdier specialization to the normal bundle. Let $\Phi : D_{\text{abs}}(C^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}) \to D_{\text{abs}}(C)$ be the composite functor. Also, write $\{n\}$ for the functor $[n](\frac{n}{2})$, of homological shift and a Tate twist. Then, our first important result concerning factorization sheaves reads, cf. Theorem 3.2.3.

Theorem 1.7.2. *Let $\mathcal{F} = (\mathcal{F}_{\mathbf{v}})_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^I \smallsetminus \{0\}}$ be a factorization sheaf on $\text{Sym} \, C$ such that for every $\mathbf{v}$ the sheaf $\Delta_{\mathbf{v}}^* \mathcal{F}_{\mathbf{v}}$ is a geometrically locally constant sheaf on C. Then, one has*

$$\sum_{\mathbf{v} \geq 0} z^{\mathbf{v}} \cdot [\text{R}\Gamma_c(C^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}, \mathcal{F}_{\mathbf{v}})] = \text{Sym}\left(\sum_{\mathbf{v} \neq 0} z^{\mathbf{v}} \cdot [\text{R}\Gamma_c\big(C, \, \Phi(\mathcal{F}_{\mathbf{v}})^{\langle\text{triv}\rangle}\big]\{\dim C(1 - |\mathbf{v}|)\}\big) \right). \qquad (1.7.3) \quad \boxed{\text{expC-intro}}$$

The above equation holds in $K_{\mathrm{abs}}(\mathrm{pt})[\![z]\!]$, where we use the notation $K_{\mathrm{abs}}(X)$ for a suitable Grothendieck group of the category $D_{\mathrm{abs}}(X)$. The symbol $H_c^\bullet(-)^{\langle\mathrm{triv}\rangle}$ in the RHS of the equation refers to the *generalized* isotypic component of the trivial representation of a certain monodromy action of the group $\mathfrak{S}_v$, cf. §2.5.

Observe that the LHS, resp. RHS, of the equation (1.7.3) depens only on the class of $\mathcal{F}_{\mathbf{v}}$ in $K_{\mathrm{abs}}(C^{\mathbf{v}})$), resp. of $\Phi(\mathcal{F}_v)^{\langle\mathrm{triv}\rangle}$ in $K_{\mathrm{abs}}(C)$. It will be explained in the main body of the paper that the map $K_{\mathrm{abs}}(\mathrm{Coh}_{\mathbf{v}}(C)) \to K_{\mathrm{abs}}(C)$ given the assignment $[\mathcal{F}_{\mathbf{v}}] \to [\Phi(\mathcal{F}_v)^{\langle\mathrm{triv}\rangle}]$ can be defined in a canonical way that does not involve a choice of $\eta_{\mathbf{v}}$. Furthermore, we will actually establish a version of (1.7.3) that holds without any additional assumptions and choices, see Theorem 3.2.3.

Applications to exponential sums involve factorization sheaves on $\mathrm{Coh}(C)$, where either $C = \mathbb{A}^1$ or $C = \mathbb{G}_m$. There is a well-known isomorphism of stacks $\mathrm{Coh}_{\mathbf{v}}(\mathbb{A}^1) \cong \mathfrak{gl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}$, resp. $\mathrm{Coh}_{\mathbf{v}}(\mathbb{G}_m) \cong \mathrm{GL}_{\mathbf{v}}/{}_{\mathrm{Ad}}\mathrm{GL}_{\mathbf{v}}$. The group $\mathrm{Aff} = \mathbb{G}_m \ltimes \mathbb{G}_a$ of affine-linear transformations acts on $\mathfrak{gl}_{\mathbf{v}}$ by $g \mapsto a \cdot g + b \cdot \mathrm{Id}$. For an *Aff*-equivariant factorization sheaf $\mathcal{F} = (\mathcal{F}_{\mathbf{v}})$, on $\mathfrak{gl} = \mathrm{Coh}\,\mathbb{A}^1$, formula (1.7.3) simplifies. Specifically, the sheaf $\mathcal{F}_{\mathbf{v}}$ descends to a $\mathbb{G}_m$-equivariant sheaf $\bar{\mathcal{F}}_{\mathbf{v}}$ on $\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}$. In such a case, taking Verdier's specialization becomes unnecessary. Furrther, we have a diagram $\mathrm{GL}_{\mathbf{v}} \overset{\varepsilon}{\hookrightarrow} \mathfrak{gl}_{\mathbf{v}} \overset{i}{\hookleftarrow} \{0\}$, where ε is an open imbedding of invertible matrices into all matrices. Restricting $\mathcal{F}$ via ε gives a $\mathbb{G}_m$-equivariant factorization sheaf $\varepsilon^*\mathcal{F} = (\varepsilon^*\mathcal{F}_{\mathbf{v}})$ on $\mathrm{GL} = \mathrm{Coh}\,\mathbb{G}_m$. Using that the operation of taking Sym corresponds, under the trace of Frobenius, to taking plethystic exponential, from formula (1.7.3) we deduce the following equations, cf. Theorem 4.3.2,

$$\sum_{\mathbf{v}} z^{\mathbf{v}} \cdot \mathsf{Tr}_{\mathsf{Fr}}|\, H_{c,\mathrm{GL}_{\mathbf{v}}}^\bullet(\mathrm{GL}_{\mathbf{v}},\, \varepsilon_{\mathbf{v}}^*\mathcal{F}_{\mathbf{v}}) \;=\; \mathrm{Exp}\left((q^{-\frac{1}{2}} - q^{\frac{1}{2}}) \cdot \sum_{\mathbf{v}\neq 0} (-1)^{|\mathbf{v}|} \cdot z^{\mathbf{v}} \cdot q^{\frac{|\mathbf{v}|}{2}} \cdot \mathsf{Tr}_{\mathsf{Fr}}|\, (\mathrm{F}_{\mathfrak{pgl}_{\mathbf{v}}}\bar{\mathcal{F}}_{\mathbf{v}}|_{\eta_{\mathbf{v}}})^{\langle\mathrm{sign}\rangle}\right),$$

$$\tag{1.7.4}\quad\boxed{\texttt{intro1}}$$

$$\sum_{\mathbf{v}} z^{\mathbf{v}} \cdot q^{-\frac{\mathbf{v}\cdot\mathbf{v}}{2}} \cdot \mathsf{Tr}_{\mathsf{Fr}}|\, H_{c,\mathrm{GL}_{\mathbf{v}}}^\bullet(i_{\mathbf{v}}^*\mathcal{F}_{\mathbf{v}}) \;=\; \mathrm{Exp}\left(\sum_{\mathbf{v}\neq 0} z^{\mathbf{v}} \cdot q^{-\frac{|\mathbf{v}|}{2}} \cdot \mathsf{Tr}_{\mathsf{Fr}}|\, (\mathrm{F}_{\mathfrak{pgl}_{\mathbf{v}}}\bar{\mathcal{F}}_{\mathbf{v}}|_{\eta_{\mathbf{v}}})^{\langle\mathrm{triv}\rangle}\right).$$

$$\tag{1.7.5}\quad\boxed{\texttt{intro2}}$$

where $\eta_{\mathbf{v}} \in \mathfrak{pgl}_{\mathbf{v}}^\diamond$ is a k-rational element.

Example 1.7.6. In the special case where the set I consists of one element and the factorization sheaf $\mathcal{F}$, on $\mathfrak{gl}$, is formed by the constant sheaves $\mathcal{F}_v = \bar{\mathbb{Q}}_\ell$, $\forall v$, one finds that $\mathrm{F}_{\mathfrak{pgl}_v}\bar{\mathcal{F}}_{\mathbf{v}}|_{\eta_v} = 0$ for all $v \neq 1$. terms in the sum in the RHS of formula (1.7.5) vanish except for term for $v = 1$. In that case, equation (1.7.5) is the standard identity for the *quantum dilogarithm*:

$$\sum_{v\geq 0} z^v \cdot \frac{(-q^{\frac{1}{2}})^{v^2}}{\#\mathrm{GL}_v(\mathbb{F}_q)} = \mathrm{Exp}\left(z \cdot \frac{q^{\frac{1}{2}}}{1-q}\right). \qquad\qquad \Diamond$$

The factorization sheaf $\mathcal{R}^{F,\phi,\mathbb{A}^1}$ is automatically Aff-equivariant, so the above equations are applicable for $\mathcal{F} = \mathcal{R}^{F,\phi,\mathbb{A}^1}$. Theorem 1.1.5 is an easy consequence of these equations.

Remark 1.7.7. There is a natural isomorphism of stacks $\mathrm{Coh}_{\mathbf{v}}\,\mathrm{pt} \cong \mathrm{pt}/\mathrm{GL}_{\mathbf{v}} = \mathbb{B}\mathrm{GL}_{\mathbf{v}}$. Hence, $H^\bullet(\mathrm{Coh}_{\mathbf{v}}\,\mathrm{pt}) = H^\bullet(\mathbb{B}\mathrm{GL}_{\mathbf{v}})$ is an algebra of symmetric polynomials, so one has $\mathrm{Spec}\,H^\bullet(\mathrm{Coh}_{\mathbf{v}}\,\mathrm{pt}) = \mathrm{Sym}^{\mathbf{v}}\,\mathbb{A}^1$. For any scheme C, we have a natural map $\mathrm{Coh}_{\mathbf{v}}\,C \to \mathrm{Coh}_{\mathbf{v}}\,\mathrm{pt} = \mathbb{B}\mathrm{GL}_{\mathbf{v}}$, induced by a constant map $C \to \mathrm{pt}$. This map takes a finite lentgh sheaf to its space of sections. The pull-back of the cohomology makes $H^\bullet(\mathrm{Coh}_{\mathbf{v}}\,C)$ a super-commutative $H^\bullet(\mathbb{B}\mathrm{GL}_{\mathbf{v}})$-algebra, equivalently, a coherent sheaf, $\mathcal{H}^\bullet(\mathrm{Coh}_{\mathbf{v}}\,C)$, of super-commutative algebras on $\mathrm{Sym}^{\mathbf{v}}\,\mathbb{A}^1$. By abstract nonsense, the collection $(\mathcal{H}^\bullet(\mathrm{Coh}_{\mathbf{v}}\,C))_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^I}$ gives a coherent factorization sheaf on $\mathrm{Sym}\,\mathbb{A}^1$ such that the factorization maps respect the algebra structures.

Now, given a sheaf $\mathcal{F}_{\mathbf{v}} \in D_{\mathrm{abs}}(\mathrm{Coh}_{\mathbf{v}}\,C)$, the cohomology $H_c^\bullet(\mathrm{Coh}_{\mathbf{v}}(C), \mathcal{F}_{\mathbf{v}})$ has the canonical structure of a graded module over the graded algebra $H^\bullet(\mathrm{Coh}_{\mathbf{v}}\,C)$. One can show that if a collection $\mathcal{F} = (\mathcal{F}_{\mathbf{v}})$ is a factorization sheaf on $\mathrm{Coh}\,C$ then the collection $H_c^\bullet(\mathcal{F}) = (H_c^\bullet(\mathrm{Coh}_{\mathbf{v}}\,C, \mathcal{F}_{\mathbf{v}}))$,

gives a coherent factorization sheaf on $\operatorname{Sym}\mathbb{A}^1$ and, moreover, the factorization maps respect the action of $(H^\bullet(\operatorname{Coh}_{\mathbf{v}} C))_{\mathbf{v}\in\mathbb{Z}^I_{\geq 0}}$, a coherent factorization sheaf of super-commutative algebras.

All the above applies, in particular, to the case where C is a smooth affine curve and $\mathcal{F} = \mathcal{R}^{F,\phi,C}$ is the factorization sheaf from Theorem 1.6.3. In that case we have

$$H^\bullet_c(\operatorname{Coh}_{\mathbf{v}}(C), \mathcal{R}^{F,\phi}_{\mathbf{v}}) \cong H^\bullet_c(\mathfrak{X}_{\mathbf{v}}(K\otimes\mathscr{C}), \wp^{\phi_{\mathbf{v}}\circ p}). \tag{1.7.8}$$

We conclude from that the collection of the cohomology spaces (1.7.8) has the natural structure of a coherent factorization sheaf on $\operatorname{Sym}\mathbb{A}^1$ equipped with a compatible action of the commutative algebras $(\mathcal{H}^\bullet_c(\operatorname{Coh}_{\mathbf{v}} C, \mathcal{F}_{\mathbf{v}}))$. This is a generalization of the construction from [KS2, §6.5]. $\diamond$

Remark 1.7.9. One can show that there is a natural graded algebra isomorphism

$$H^\bullet(\operatorname{Coh}_{\mathbf{v}}\mathbb{G}_m) \cong H^\bullet(\operatorname{GL}_{\mathbf{v}}/_{\operatorname{Ad}}\operatorname{GL}_{\mathbf{v}}) \cong \big(\operatorname{S}\mathfrak{h}\otimes\wedge\mathfrak{h}\big)^{\mathfrak{S}_{\mathbf{v}}},$$

where $\mathfrak{h} = \mathbb{A}^{\mathbf{v}}$ is the Cartan subalgebra of $\mathfrak{gl}_{\mathbf{v}}$. There is a generalization of this isomorphism to the case where $\mathbb{G}_m$ is replaced by an arbitrary smooth affine connected curve C. In the general case, the role of the super vector space $\mathfrak{h}\otimes(\mathbb{A}^1\oplus\mathbb{A}^1[1])$ is played by the super vector space $\mathfrak{h}\otimes(H^0(C)\oplus H^1(C))$. $\diamond$

We will use a version of the following well known construction. Let G be an algebraic group with Lie algebra $\mathfrak{g}$. Given a smooth G-variety Y, one defines $I(Y,\mathfrak{g}) := \{(a,y)\in\mathfrak{g}\times Y \mid \overrightarrow{a}(y) = 0\}$, where $\overrightarrow{a}$ stands for the vector field on Y induced by $a\in\mathfrak{g}$. Thus, $I(Y,\mathfrak{g})$ is a G-stable closed subvariety of $\mathfrak{g}\times Y$ and we let $\underline{\mathbf{C}}_{I(Y,\mathfrak{g})}$ be the constant sheaf on $I(Y,\mathfrak{g})$ extended by zero to a sheaf on $\mathfrak{g}\times Y$. Write $p : I(Y,\mathfrak{g})\to\mathfrak{g}$ for the first projection and $\mu : \operatorname{T}^*Y\to\mathfrak{g}^*$ for the moment map. There is a functor $\operatorname{F}_{\mathfrak{g}}$ of geometric Fourier transform (also called Fourier-Deligne transform) that sends constructible complexes on $\mathfrak{g}$ to constructible complexes on $\mathfrak{g}^*$. Then, basic properties of Fourier transform yield an isomorphism

$$\operatorname{F}_{\mathfrak{g}}(p_!\underline{\mathbf{C}}_{I(Y,\mathfrak{g})}) = \mu_!\underline{\mathbf{C}}_{\operatorname{T}^*Y}[?], \tag{1.7.10}$$

where $\underline{\mathbf{C}}_{I(Y,\mathfrak{g})}$ denotes the constant sheaf on $I(Y,\mathfrak{g})$ extended by zero to a sheaf on $\mathfrak{g}\times Y$ and $[?]$ stands for an appropriate shift in the derived category of constructible complexes on $\mathfrak{g}^*$.

We are interested in the case where $G = G_{\mathbf{v}} := \operatorname{GL}_{\mathbf{v}}$ and $Y = Y_{\mathbf{v}}$, a smooth $G_{\mathbf{v}}$-stack rather than a smooth G-variety. It turns out that the correct analogue of the quotient $I(Y,\mathfrak{g})/G$ is a stack $I^+_{\mathbf{v}}$, an 'infinitesimal inertia stack' Let $\varepsilon : G_{\mathbf{v}}/_{\operatorname{Ad}}G_{\mathbf{v}}\hookrightarrow\mathfrak{g}_{\mathbf{v}}/G_{\mathbf{v}}$ be the induced imbedding of quotient stacks. Similarly, the natural inclusion $\operatorname{Aut}(x)\subset\operatorname{End}(x)$, for any object x of $\mathscr{C}$, yields an open imbedding $\varepsilon_I : I_{\mathbf{v}}\subset I^+_{\mathbf{v}}$, where $I_{\mathbf{v}}$ is the usual inertia stack of $\mathfrak{X}_{\mathbf{v}}$, that is, the stack whose objects are pairs (x,f) such that $x\in\mathfrak{X}_{\mathbf{v}}$ and $f\in\operatorname{Aut}(x)$. Restricting the morphisms $p^+_{\mathfrak{g}}$ and p_X to $I_{\mathbf{v}}$ one obtains a diagram of cartesian squares

$$
\begin{array}{ccccc}
G_{\mathbf{v}}/_{\operatorname{Ad}}G_{\mathbf{v}} & \xleftarrow{\ p_{\mathfrak{g}}\ } & I_{\mathbf{v}} & \xrightarrow{\ p_X\ } & \mathfrak{X}_{\mathbf{v}} \\
{\scriptstyle\varepsilon}\big\uparrow & \square & {\scriptstyle\varepsilon_I}\big\uparrow & \square & \big\| \\
\mathfrak{g}_{\mathbf{v}}/G_{\mathbf{v}} & \xleftarrow{\ p^+_{\mathfrak{g}}\ } & I^+_{\mathbf{v}} & \xrightarrow{\ p^+_X\ } & \mathfrak{X}_{\mathbf{v}}
\end{array}
$$

We are going to apply the Grothendieck-Lefschetz trace formula for a local system on $\mathfrak{X}_{\mathbf{v}}$. Such a formula, proved by Behrend [B1]-[B2], involves a sum over the set of Frobenius fixed points, that is, over the set $[\mathfrak{X}_{\mathbf{v}}(\mathbb{F}_q)]$. It is essential that since $\mathfrak{X}_{\mathbf{v}}$ is a *stack* each isomorphism class x comes with a weight $\frac{1}{\#\operatorname{Aut}(x)}$. Note also that the number of points in the fiber over x of the map $p_X : [I_{\mathbf{v}}(\mathbb{F}_q)]\to[\mathfrak{X}_{\mathbf{v}}(\mathbb{F}_q)]$ equals $\#\operatorname{Aut}(x)$. Therefore, writing $(-)|_x$ for the stalk at x, for any local

14

system $\mathcal{F}$ on $\mathfrak{X}_{\mathbf{v}}$ we have $\mathsf{Tr}_{\mathsf{Fr}}|\left((p_X)_!p_X^*\mathcal{F}|_x\right) = \#\operatorname{Aut}(x)\cdot\mathsf{Tr}_{\mathsf{Fr}}|\left(\mathcal{F}|_x\right)$. Thus, we compute

$$
\begin{aligned}
E_{\mathbf{v}}([\mathfrak{X}(\mathbb{F}_q)]) &= \sum\nolimits_{x\in[\mathfrak{X}_{\mathbf{v}}(\mathbb{F}_q)]}\ \psi(\phi_{\mathbf{v}}(x)) = \sum\nolimits_{x\in[\mathfrak{X}_{\mathbf{v}}(\mathbb{F}_q)]}\ \mathsf{Tr}_{\mathsf{Fr}}|\left(\wp_\psi|_{\phi_{\mathbf{v}}(x)}\right)\\
&= \sum\nolimits_{x\in[\mathfrak{X}_{\mathbf{v}}(\mathbb{F}_q)]}\ \frac{1}{\#\operatorname{Aut}(x)}\cdot\#\operatorname{Aut}(x)\cdot\mathsf{Tr}_{\mathsf{Fr}}|\left(\wp_{\mathbf{v},\psi,\phi}|_x\right)\\
&= \sum\nolimits_{x\in[\mathfrak{X}_{\mathbf{v}}(\mathbb{F}_q)]}\ \frac{1}{\#\operatorname{Aut}(x)}\cdot\mathsf{Tr}_{\mathsf{Fr}}|\left((p_X)_!p_X^*\wp_{\mathbf{v},\psi,\phi}|_x\right)\qquad\text{(Grothendieck-Lefschetz)}\\
&= \mathsf{Tr}_{\mathsf{Fr}}|\,H_c^\bullet(\mathfrak{X}_{\mathbf{v}},\ (p_X)_!p_X^*\wp_{\mathbf{v},\psi,\phi}) \ = \ \mathsf{Tr}_{\mathsf{Fr}}|\,H_c^\bullet(I_{\mathbf{v}},\ p_X^*\wp_{\mathbf{v},\psi,\phi})\\
&= \mathsf{Tr}_{\mathsf{Fr}}|\,H_c^\bullet(I_{\mathbf{v}},\ \varepsilon_I^*(p_X^+)^*\wp_{\mathbf{v},\psi,\phi}) \ = \ \mathsf{Tr}_{\mathsf{Fr}}|\,H_c^\bullet\big(G_{\mathbf{v}}/_{\operatorname{Ad}}G_{\mathbf{v}},\ \varepsilon^*(p_{\mathfrak{g}}^+)_!(p_X^+)^*\wp_{\mathbf{v},\psi,\phi}\big)\qquad(1.7.11)\quad\boxed{\text{Ev}}
\end{aligned}
$$

1.8. **Relation to earlier works.** Formula (1.7.5) and Theorem 1.6.3 are related, in a way, to works of Davison, Meinhardt and Reineke, [MR], [Me], [DM]. In these papers, the authors study Donaldson-Thomas invariants of the moduli stack $\mathfrak{X}(\mathscr{C})^\theta$ of θ-semistable objects of a finitary abelian category $\mathscr{C}$. Let $\mathcal{M}^\theta$ be the corresponding coarse moduli space and $\mathcal{M}^{\theta-st}\subset\mathcal{M}^\theta$ the stable locus of $\mathcal{M}^\theta$. There is a well-defined morphism $p:\mathfrak{X}(\mathscr{C})^\theta\to\mathcal{M}^\theta$ that sends an object of $\mathscr{C}$ to an associated graded object with respect to a Jordan-Hölder filtration. Further, any cosed point of $\mathcal{M}^\theta$ is represented by a polystable object, i.e. a finite direct sum of stable objects. Hence, the direct sum operation yields an isomorphism $\operatorname{Sym}\mathcal{M}^{\theta-st}\overset{\sim}{\to}\mathcal{M}^\theta$, so the map p gives a morphism $\mathfrak{X}(\mathscr{C})^\theta\to\operatorname{Sym}\mathcal{M}^{\theta-st}$. Let $\underline{\mathbf{C}}$ be a constant sheaf on $\mathfrak{X}(\mathscr{C})^\theta$, with appropriated shifts and Tate twists. In this language, it was shown in [Me], [DM] that $p_!\underline{\mathbf{C}}$ is a factorization sheaf on $\operatorname{Sym}\mathcal{M}^{\theta-st}$. Furthermore, the main result of *loc cit* says, roughly speaking, that in $K_{\operatorname{abs}}(\mathcal{M}^{\theta-st})$ one has $[p_!\underline{\mathbf{C}}]=\operatorname{Sym}[\operatorname{IC}(\mathcal{M}^{\theta-st})]$. This result may be compared with equation (1.7.5).

We observe that in Donaldson-Thomas theory the category $\mathscr{C}$ in question is assumed to be *abelian*, furthermore, the framing functor F is assumed to be exact and faithful. Our setting is more general, e.g. it covers the case of parabolic bundles where the category is not abelian and the functor F is neither exact nor faithful. In such a case, there is no meaningful notion of Jordan-Hölder series, so there is no natural analogue of the morphism $\mathfrak{X}(\mathscr{C})^\theta\to\mathcal{M}^\theta$. On the other hand, one has the notion of an absolutely indecomposable object. The structure of the set of (isomorphism classes of) absolutely indecomposable objects is, typically, rather chaotic from an algebra-geometric viewpoint. There is no natural way, in general, to upgrade this set to a scheme or a stack. However, one can consider an *ind-constrictible scheme*, $X(\mathscr{C})$, that parametrizes isomorphism classes of objects of $\mathscr{C}$, and the set of isomorphism classes of absolutely indecomposable objects is the set of k-rational points of an ind-constrictible subscheme $X^{\operatorname{ind}}(\mathscr{C})$ of $X(\mathscr{C})$. Further, thanks to the Krull-Schmidt theorem there is a natural isomorphism $X(\mathscr{C})\cong\operatorname{Sym}X^{\operatorname{ind}}(\mathscr{C})$, of constructible schemes. This may be viewed as a substitute of the isomorphism $\operatorname{Sym}\mathcal{M}^{\theta-st}\overset{\sim}{\to}\mathcal{M}^\theta$, resp. the Krull-Schmidt theorem as a substitute of the Jordan-Hölder theorem. It seems that the formalism of constrictible schemes is not rich enough to do interesting geometry. This was one reason for us to consider the factorization sheaf $\mathcal{R}^{F,\phi,C}$, on $\operatorname{Coh}C$, rather than factorization sheaves on $\operatorname{Sym}X^{\operatorname{ind}}(\mathscr{C})$. More importantly, our approach is 'microlocal' in the sense that it involves the geometry of the cotangent bundle $\mathsf{T}\mathfrak{X}(\mathscr{C},F)$ and the monent map.

It is interesting to observe that the effect of replacing the zero fiber of the monent map by the *generic* fiber (the stack $\mathcal{M}_O$ in Theorem 1.1.5, cf. also the RHS of (1.7.3)) is similar, in a way, to replacing $[H_c^\bullet(\mathfrak{X}(\mathscr{C}))]$ by $[\operatorname{IH}(\mathcal{M}^{\theta-st})]$ (where $[\dots]$ stands for the class in an appropriate Grothendieck group), in the setting of [MR], [Me], [DM]. Understanding this phenomenon requires a better understanding of the geometry of the 'forgetting the legs' map from the *indecomposable* locus of $\mathfrak{X}(\widetilde{\mathscr{C}},\widetilde{F})^{\widetilde{\theta}}$ to $\mathfrak{X}(\mathscr{C},F)^\theta$, cf. §?? and Proposition 12.1.2. We plan to discuss this in the future.

1.9. **Layout of the paper.** In section 2 we recall basic notions concerning absolutely convergent constructible complexes and the Grothendieck-Lefschetz trace formula on stacks, as developed by Behrend [B1], [B2]. This section also contains the proof of a generalization of Hua's lemma and of Lemma 2.5.2 that allows to define the map Φ, see §1.6, in full generality. In section 3, we introduce factorization sheaves on a scheme and prove our main technical results involving co-homology of factorization sheaves. In section 4 we upgrade the results of section 3 to the setting of factorization sheaves on the stack $\operatorname{Coh} C$. Specifically we relate factorization sheaves on the stack $\operatorname{GL} = \operatorname{Coh} \mathbb{G}_m$ to factorization sheaves on symmetric powers of $\mathbb{G}_m$ via a radial part (aka 'parabolic restriction') functor. This is necessary for applications to moduli spaces. In section 5 we study equivariant inertia stacks. Section 6 begins with some generalities cocerning sheaves of categories and associated moduli stacks. This subject was studied by several authors in various contexts but we have been unable to find the setting that fits our purposes in the literature. We then study the equivariant inertia stack of a moduli stack and prove a factorization property which is equivalent to Theorem 1.6.3. In section 7 we explain a relation, via the Fourier transform, between the equivariant inertia stack of a G-stack and the moment map for the cotangent stack of that G-stack. Such a relation is well known in the case of G-schemes but its generalization to stacks is not straightforward. It requires a duality formalism for vector bundles on stacks which we develop in §7.1 and that may be of an independent interest. The proof of Theorem 1.1.5 is completed in §7.3. In section 8 we provide an alternative proof of Theorem 1.2.2 that bypasses the use of Hua's lemma. In section 9 we study indecomposable parabolic bundles. We use the Harder-Narasimhan filtration to define a family of certain truncated categories such that the corresponding moduli stacks have finite type and Theorem 1.1.5 is applicable. The counting of absolutely indecomposable parabolic bundles is then reduced to counting absolutely indecomposable objects of the truncated categories.

Sections 10, 11, and 12 do not depend on the rest of the paper and each of these sections may be of independent interest. Section 10 is devoted to the proof of Theorem 1.3.2. In §11.1 we prove Proposition 1.3.5. Section 11.2 contains a new proof of deformation invariance of the cohomology of a semi-projective variety. Although the result itself is not new, our proof is algebraic while the original proof in [HLV2] is based on a reduction to the case of the ground field $\mathbb{C}$ and a transcendental argument involving Riemannian metrics. In section 12 we show that the geometry of cells in the Calogero-Moser variety studied by G. Wilson [Wi] has a natural interpretation in terms of indecomposable representations of the Calogero-Moser quiver.

2. ABSOLUTELY CONVERGENT SHEAVES AND HUA'S FORMULA

2.1. We write $\bar{\mathsf{k}}$ for an algebraic closure of a field k. We choose a prime $\ell \neq \operatorname{char} \mathsf{k}$ and let $\underline{\mathbf{C}}_X$ denote the constant sheaf on a scheme (or stack) X with fiber $\bar{\mathbb{Q}}_\ell$.

Let k be a finite field and X a scheme over k. We write $\mathsf{Fr} = \mathsf{Fr}_{\bar{\mathsf{k}}/\mathsf{k}}$ for the Frobenius automorphism of k and $\pi_1^{et}(X)$ for the *geometric* étale fundamental group of X, that is, for the kernel of the canonical homomorphism of the full étale fundamental group of X to $\operatorname{Gal}(\bar{\mathsf{k}}/\mathsf{k})$.

Let G be a connected algebraic group over a finite field $\mathbb{F}$. Associated with a homomorphism $\psi : G(\mathbb{F}) \to \bar{\mathbb{Q}}_\ell^\times$ there is a rank one $\bar{\mathbb{Q}}_\ell$-local system $\wp_\psi$, on G, such that $\mathsf{Fr}^* \wp_\psi \cong \wp_\psi$ and one has $\psi(g) = \mathsf{Tr}_{\mathsf{Fr}}|\,(\wp_\psi)_g$, for any $g \in G(\mathbb{F})$. To define this local system one considers the Lang map $Lang : G \to G, \ g \mapsto g^{-1}\mathrm{Fr}(g)$. Thanks to Lang's theorem, the map $Lang$ is a Galois covering with Galois group $G(\mathbb{F})$. One defines $\wp_\psi$ to be the ψ-isotypic component of the local system $Lang_* \mathbf{C}_G$, on G. This construction is functorial in the sense that, for any morphism $\chi : G' \to G$ of algebraic groups over $\mathbb{F}$, there is a canonical isomorphism $\wp_{\psi \circ \chi} \cong \chi^* \wp_\psi$.

We write $\mathbb{A} := \mathbb{A}^1$ for an affine line and $\mathbb{G} := \mathbb{G}_m$ for the multiplicative group. Thus, $\mathbb{A}^n$ is an n-dimensional affine space, resp. $\mathbb{G}^n$ an n-dimensional torus. From now on, we reserve the notation ψ for an additive character, i.e. for the case $G = G_a$. The corresponding local system $\wp_\psi$ is the Artin-Schreier local system. Given a scheme (or a stack) X and a morphism $f : X \to \mathbb{A}$, we write $\wp^f := f^* \wp_\psi$, a rank 1 local system on X obtained from $\wp_\psi$ by pull-back via f.

2.2. We will systematically use the formalism of stacks. Throughtout the paper, *a stack* means *an Artin stack of finite type over* k. (We note that the stacks $\mathrm{Bun}_{r,d}$ are only *locally* of finite type but not of finite type for $d \geq 2$). Given a stack X over k and a field extension $K/$k we write $X(K) = X(\mathrm{Spec}\,K)$ for the groupoid (= category) of all 1-morphisms $x : \mathrm{Spec}\,K \to X$ in the 2-category of stacks. If the morphism x is defined over k, we let $[x]$ denote the isomorphism class of objects of the corresponding category over k. In this case we write $[x] \in [X(\mathsf{k})]$, where $[X(\mathsf{k})]$ stands for the set of isomorphism classes of all k-objects of X. Given a pair of objects $x, x' \in X(\mathsf{k})$ let $\mathrm{Mor}_{\mathsf{k}}(x, x')$ denote the set of morphisms $f : x \to x'$. We write $\mathrm{T}X$, resp. T^*X, for the tangent, resp. cotangent, stack on X.

Schemes will be identified with stacks via the functor of points, where sets are viewed as groupoids with trivial isomorphisms. For a scheme X over k, we will often identify the set $X(\mathsf{k})$ of k-rational points of X with the set $[X(\mathsf{k})]$ of isomorphism classes of objects of the corresponding stack, defined over k.

Given a stack X let $D(X)$ be the *bounded above* derived category of constructible complexes on X. Abusing the terminology, we will often refer to an arbitrary object of $D(X)$ as a "sheaf".

We refer the reader to [Ro] for a detailed exposition of the formalism of group actions on stacks. We will only consider G-actions where G is a linear algebraic group over k. A stack X equipped with a *strict G-action*, cf. [Ro, Definition 1.3], is called a G-stack. There is a quotient stack X/G that comes equipped with a canonical projection $X \to X/G$. The category $D(X/G)$ is equivalent to the G-equaivariant constructible derived category on X. Given a G-action on a scheme X we write $X /\!/ G$ for the corresponding categorical quotient. Then, there is a canonical morphism $\mathfrak{w} : X/G \to X /\!/ G$.

2.3. We will need to take the trace of Frobenius for sheaves on a stack X over a finite field k. The corresponding theory is developed in [B2], and [LO]. In particular, Behrend introduces [B2, §6] a triangulated category of *absolutely convergent ℓ-adic Weil complexes* on X. We will not reproduce a complete definition; an essential part being that for any point $x : \mathrm{pt} \to X$ defined over k, the trace of the Frobenius inverse on the (co)stalk $x^! \mathcal{F}$ of an absolutely convergent constructible complex $\mathcal{F}$ gives, for every field imbedding $\mathsf{j} : \bar{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$, an absolutely convergent series

$$\sum_{n \in \mathbb{Z}} \Big(\sum_{\lambda \in \mathrm{Spec}(\mathsf{Fr}^{-1}; H^n(x^! \mathcal{F}))} \mathrm{mult}(\lambda) \cdot |\mathsf{j}(\lambda)| \Big) < \infty,$$

where $\| - \|$ stands for the absolute value of a complex number, $\mathrm{Spec}(-)$ is the set of eigenvalues of Fr^{-1} in $H^n(x^! \mathcal{F})$, and $\mathrm{mult}(\lambda)$ is the corresponding multiplicity.

It will be more convenient for us to use a dual notion of an absolutely convergent constructible complex that involves the stalks $\mathcal{F}_x := x^*\mathcal{F}$ rather than the costalks. We let $D_{\mathrm{abs}}(X)$ denote the corresponding bounded above triangulated category, for which one additionally requires that the weights of Frobenius on all the stalks be uniformly bounded. The cohomology sheaves of an object of $D_{\mathrm{abs}}(X)$ vanish in degrees $\gg 0$ (whereas, in Behrend's definition the cohomology sheaves vanish in degrees $\ll 0$). The results of [B2] insure that for any morphism $f : X \to Y$, of stacks over $\Bbbk$, there are well defined functors $f_! : D_{\mathrm{abs}}(X) \to D_{\mathrm{abs}}(Y)$ and $f^* : D_{\mathrm{abs}}(Y) \to D_{\mathrm{abs}}(X)$. Objects of the category $D_{\mathrm{abs}}(\mathrm{pt})$ will be referred to as absolutely convergent complexes. Thus, for any $\mathcal{F} \in D_{\mathrm{abs}}(X)$ there is an absolutely convergent complex $\mathrm{R}\Gamma_c(X, \mathcal{F}) := f_!\mathcal{F}$, where $f : X \to \mathrm{pt}$ is a constant map.

Remark 2.3.1. Introducing a category of absolutely convergent constructible complexes is necessary already in the case where $X = \mathrm{pt}/\mathbb{G}$. Since $\mathrm{pt}/\mathbb{G}$ is homotopy equivalent to $\mathbb{P}^\infty$, the complex $\mathrm{R}\Gamma(\mathrm{pt}/\mathbb{G}, \mathbb{C})$ has nonzero cohomology in infinitely many *nonnegative* degrees: $0, 2, 4, \ldots$, and it is an absolutely convergent constructible complex on pt in the sense of [B2]. On the other hand, the complex $\mathrm{R}\Gamma_c(\mathrm{pt}/\mathbb{G}, \mathbb{C})$ is an object of $D_{\mathrm{abs}}(\mathrm{pt})$, in our definition. We have $\dim(\mathrm{pt}/\mathbb{G}) = -1$, and the complex $\mathrm{R}\Gamma_c(\mathrm{pt}/\mathbb{G}, \mathbb{C})$ has nonzero cohomology in infinitely many *negative* degrees: $-2 = -2\dim(\mathrm{pt}/\mathbb{G})$, $-4, -6, \ldots$. $\diamond$

Given an absolutely convergent complex $\mathcal{F} \in D_{\mathrm{abs}}(X)$, let $\mathscr{H}(\mathcal{F})$ denote the complex with zero differential and terms $\mathscr{H}^n(\mathcal{F})$, $n \in \mathbb{Z}$, the cohomology sheaves of $\mathcal{F}$. Let $K_{\mathrm{abs}}(X)$ be a quotient of the Grothendieck group of the category $D_{\mathrm{abs}}(X)$ by the relations $[\mathcal{F}] = [\mathscr{H}(\mathcal{F})]$ for all $[\mathcal{F}] \in D_{\mathrm{abs}}(X)$ (this relation is not, in general, a formal consequence of the relations in the Grothendieck group itself since $\mathcal{F}$ may have infinitely many nonzero cohomology sheaves). In the special case $X = \mathrm{pt}$ we use simplified notation $K_{\mathrm{abs}} = K_{\mathrm{abs}}(\mathrm{pt})$. For any field imbedding $\mathsf{j} : \bar{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$, the map $V \mapsto \sum_{n \in \mathbb{Z}} (-1)^n \cdot \mathsf{j}(\mathrm{tr}(\mathsf{Fr}; H^n(V)))$ induces a well-defined group homomorphism $K_{\mathrm{abs}} \to \mathbb{C}$. From now on, we fix an imbedding j and denote the corresponding homomorphism by $\mathsf{Tr}_{\mathsf{Fr}}$. In particular, from the Remark above we find $\mathsf{Tr}_{\mathsf{Fr}}|\, \mathrm{R}\Gamma_c(\mathrm{pt}/\mathbb{G}, \bar{\mathbb{Q}}_\ell) = \frac{1}{q} + \frac{1}{q^2} + \ldots = \frac{1}{q-1}$, where $q = \#\Bbbk$.

By definition, objects of $D_{\mathrm{abs}}(X)$ come equipped with a canonical weight filtration. The graded pieces of the weight filtration on $\mathcal{F}$ are pure, hence semisimple. We write $\mathcal{F}^{\mathrm{ss}}$ for the associated graded semisimple object and we call it the *semisimplification* of $\mathcal{F}$. If the stack X is an algebraic variety or, more generally, an orbifold then in $K_{\mathrm{abs}}(X)$ we have $[\mathcal{F}] = [\mathcal{F}^{\mathrm{ss}}]$. This can be proved using the equations $[\mathcal{E}] = [\mathscr{H}(\mathcal{E})]$ and the fact that the cohomology sheaves $\mathscr{H}^i(\mathcal{E})$ of any perverse sheaf $\mathcal{E}$ on X vanish in degrees $i > |\dim X|$.

We put $\mathsf{K}_{\mathrm{abs}}^I := \prod_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^I} K_{\mathrm{abs}} \cong K_{\mathrm{abs}}[\![z]\!]$, where $z = (z_i)_{i \in I}$. We write an element of $\mathsf{K}_{\mathrm{abs}}^I$ either as $V = \sum_{\mathbf{v}} z^{\mathbf{v}} V_{\mathbf{v}}$ or just $V = \sum_{\mathbf{v}} V_{\mathbf{v}}$, if no confusion is possible. We call $V_{\mathbf{v}}$ the *component of V of dimension* $\mathbf{v}$. The group $\mathsf{K}_{\mathrm{abs}}^I$ has the natural structure of a pre λ-ring, in particular, one has a product $[M] \cdot [M'] := [M \otimes M']$ and an operation $\mathrm{Sym} : \prod_{\mathbf{v} > 0} K_{\mathrm{abs}} \to \mathsf{K}_{\mathrm{abs}}^I$, $[M] \mapsto \mathrm{Sym}[M] = \prod_{\mathbf{v}} \mathrm{Sym}^{\mathbf{v}}[M]$, where $\mathrm{Sym}^{\mathbf{v}}[M] = \oplus_{m \geq 0} [((M^{\otimes m})_{\mathbf{v}})^{\mathfrak{S}_m}]$ and $(M^{\otimes m})_{\mathbf{v}}$ is the component of $M^{\otimes m}$ with respect to the decomposition induced by the decomposition M into components of various dimensions. We identify K_{abs} with the component of $\mathsf{K}_{\mathrm{abs}}^I$ corresponding to $\mathbf{v} = 0$. The class $1 = [\bar{\mathbb{Q}}_\ell] \in K_{\mathrm{abs}}$ is a unit of $\mathsf{K}_{\mathrm{abs}}^I$. Also, let $\mathbb{L}$ be the class $[\mathrm{R}\Gamma_c(\mathbb{A}, \bar{\mathbb{Q}}_\ell)] = \bar{\mathbb{Q}}_\ell[-2](-1) \in K_{\mathrm{abs}}(\mathrm{pt})$. It will be convenient to enlarge $\mathsf{K}_{\mathrm{abs}}^I$ by adjoining a formal square root $\mathbb{L}^{1/2}$ of $\mathbb{L}$. For $\mathcal{F} \in D_{\mathrm{abs}}(X)$ and an integer n, we often use the notation $\mathcal{F}\{n\} = \mathcal{F}[n](\frac{n}{2})$, so in $K_{\mathrm{abs}}(X)$ one has $[\mathcal{F}\{n\}] = [\mathcal{F}] \otimes \mathbb{L}^{\otimes -n/2}$.

2.4. **Generalized Hua's formula.** Let $f = \sum_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^I} z^{\mathbf{v}} \cdot f_{\mathbf{v}}$ be a formal power series where each coefficient $f_{\mathbf{v}}$ is a function $\mathbb{Z}_{\geq 1}^I \to \bar{\mathbb{Q}}_\ell$, $m \mapsto f_{\mathbf{v}}(m)$. Assuming that $f_0 = 0$, we define the plethystic exponential of f as a similar formal power series $\mathrm{Exp}\, f = \sum_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^I} z^{\mathbf{v}} \cdot \mathrm{Exp}_{\mathbf{v}} f$, where the functions

18

$\mathrm{Exp}_{\mathbf{v}}\, f$ are determined by the equation

$$\sum_{\mathbf{v}>0} z^{\mathbf{v}}\cdot(\mathrm{Exp}_{\mathbf{v}}\, f)(m) \;:=\; \exp\left(\sum_{n\geq 1}\sum_{\mathbf{v}>0}\frac{z^{n\cdot\mathbf{v}}}{n}\cdot f_{\mathbf{v}}(n\cdot m)\right),\qquad m\in\mathbb{Z}_{\geq 1}^{I}.$$

In a special case where the functions $f_{\mathbf{v}}$ have the form $f_{\mathbf{v}}(n)=\mathbf{f}_{\mathbf{v}}(q^n)$, for some $q\in\bar{\mathbb{Q}}_\ell$ and some Laurent polynomials $\mathbf{f}_{\mathbf{v}}\in\bar{\mathbb{Q}}_\ell[\mathbf{q}]$ in one variable, the above formula takes a more familiar form

$$\mathrm{Exp}\left(\sum_{\mathbf{v}>0} z^{\mathbf{v}}\cdot\mathbf{f}_{\mathbf{v}}(\mathbf{q})\right) \;:=\; \exp\left(\sum_{n\geq 1}\sum_{\mathbf{v}>0}\frac{z^{n\cdot\mathbf{v}}}{n}\cdot\mathbf{f}_{\mathbf{v}}(\mathbf{q}^n)\right). \qquad (2.4.1)\quad \boxed{\texttt{exp-pol}}$$

Let k be a finite field and K/k a finite field extention. Let $(\mathscr{C},F)$ be a data as in §$\overset{\text{exp-sec}}{1.1}$, so $\mathscr{C}$ is a k-linear category and F is an additive k-linear functor from $\mathscr{C}$ to the category of finite dimensional I-graded k-vector spaces that does not kill nonzero objects. Let $\mathfrak{X}(\mathscr{C})=\sqcup_{\mathbf{v}}\mathfrak{X}_{\mathbf{v}}(\mathscr{C})$ be the corresponding decomposition of the moduli stack of $\mathscr{C}$. In §$\overset{\text{Ob}}{6.1}$, we will define, by 'extension of scalars', a K-linear category $\mathscr{C}_K=\mathscr{C}_K$ equipped with a K-linear functor $F_K=K\otimes F$ such that the pair $(\mathscr{C}_K,F_K)$ satisfies the assumptions of §$\overset{\text{exp-sec}}{1.1}$ over the the ground field K and such that for the corresponding moduli stacks there are canonical isomorphisms $\mathfrak{X}_{\mathbf{v}}(\mathscr{C}_K)\cong K\otimes_{\mathsf{k}}\mathfrak{X}_{\mathbf{v}}(\mathscr{C})$. In particular, for the sets of isomorphism classes of objects one has $[\mathrm{Ob}_{\mathbf{v}}(\mathscr{C}_K)]=[\mathfrak{X}_{\mathbf{v}}(\mathscr{C})(K)]$. Further, for each $\mathbf{v}\neq 0$, let $\mathrm{AI}_{\mathbf{v}}(K)$ let be a subset of $[\mathrm{Ob}_{\mathbf{v}}(\mathscr{C}_K)]$ formed by the absolutely indecomposable objects of the category $\mathscr{C}_K$.

Now, for each $n\geq 1$ let K_n be a degree n extension of k and $\mathrm{tr}_{K_n/\mathsf{k}}:\ K_n\to\mathsf{k}$ the trace map. Thus, $\psi_n:=\psi\circ\mathrm{tr}_{K_n/\mathsf{k}}$ is an additive character of K_n associated with the additive character $\psi:\mathsf{k}\to\bar{\mathbb{Q}}_\ell^{\times}$. Given a potential $\phi:\mathfrak{X}(\mathscr{C})\to\mathbb{A}$, we define a function $\mathsf{E}_{\mathbf{v}}(\mathscr{C},F,\phi):\mathbb{Z}_{\geq 0}\to\bar{\mathbb{Q}}_\ell$, resp. $\mathsf{E}_{\mathbf{v}}^{\mathrm{AI}}(\mathscr{C},F,\phi):\mathbb{Z}_{>0}\to\bar{\mathbb{Q}}_\ell$, by the formula

$$\mathsf{E}_{\mathbf{v}}(\mathscr{C},F,\phi)(n)=\sum_{x\in[\mathrm{Ob}_{\mathbf{v}}(\mathbb{F}_{q^n}\otimes\mathscr{C})]}\psi_n(\phi(x)),\quad\text{resp.}\quad \mathsf{E}_{\mathbf{v}}^{\mathrm{AI}}(\mathscr{C},F,\phi)(n)=\sum_{x\in[\mathrm{AI}_{\mathbf{v}}(\mathbb{F}_{q^n})]}\psi_n(\phi(x)).$$

Then, our generalization of Hua's formula reads

$\boxed{\texttt{eai}}$ **Proposition 2.4.2.** *For any data $(\mathscr{C},F,\phi)$ as in section $\overset{\text{exp-sec}}{1.1}$, we have*

$$\sum_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^{I}} z^{\mathbf{v}}\cdot\mathsf{E}_{\mathbf{v}}(\mathscr{C},F,\phi) \;=\; \mathrm{Exp}\left(\sum_{\mathbf{v}>0} z^{\mathbf{v}}\cdot\mathsf{E}_{\mathbf{v}}^{\mathrm{AI}}(\mathscr{C},F,\phi)\right).$$

In the special case where $\phi=0$ and $\mathscr{C}$ is the category of finite dimemsional quiver representations over $\mathsf{k}=\mathbb{F}_q$, it was shown by V. Kac that the functions $\#[\mathfrak{X}_{\mathbf{v}}(\mathscr{C})(\mathbb{F}_{q^n})]$ and $\#[\mathrm{AI}_{\mathbf{v}}(\mathbb{F}_{q^n})]$ are Laurent polynomials in q. In this case one can use $(\overset{\text{exp-pol}}{2.4.1})$ and the formula of the proposition reduces to the original formula due to Hua $\overset{\text{H}}{[\text{H}]}$.

The proof the proposition given below amounts, essentially, to a more conceptual reinterpretation of Hua's argument. In fact, the proof in *loc cit* can be carried over in our, more general, setting as well. Indeed, the only property of the function $\psi\circ\phi:\ X(\mathscr{C})(\bar{\mathsf{k}})\to\bar{\mathsf{k}}$ which is necessary for Hua's argument to go through is the equation $(\psi\circ\phi)(x\oplus x')=(\psi\circ\phi)(x)\cdot(\psi\circ\phi)(x'),\ x,x'\in X(\mathscr{C})(\bar{\mathsf{k}})$. The latter equation holds thanks to the additivity of the potential ϕ.

The proof of the Hua formula begins with the following

$\boxed{\texttt{krull-lem}}$ **Lemma 2.4.3.** *For any finite field extension K/k, all Hom-spaces in the the category $\mathscr{C}_K$ have finite dimension over K and, moreover, $\mathscr{C}_K$ is a Krull-Schmidt category, i.e. any object is isomorphic to a finite direct sum of indecomposable objects, defined uniquely up to isomorphism and permutation of direct summands.*

19

Proof. Recall that automorphism groups of objects of an Artin stack are algebraic groups. For $x \in \mathscr{C}_K$, the group $\mathrm{Aut}(x)$ is the group of invertible elements of the algebra $\mathrm{End}(x)$, hence it is Zariski open in $\mathrm{End}(x)$. Thus, we have $\dim \mathrm{End}(x) = \dim \mathrm{Aut}(x) < \infty$. Applying this to a direct sum $x \oplus y$ we deduce that $\mathrm{Hom}(x, y) < \infty$. The second statement holds by Theorem A.1 in [CYZ], which asserts that an additive Karoubian category has the Krull-Schmidt property if and only if $\mathrm{End}(x)$ is semi-perfect for every object x of this category. In particular this is true in our case since $\mathrm{End}(x)$ is finite-dimensional, hence semi-perfect. $\square$

Given a reduced scheme Y, let $\mathrm{Sym}^n Y = Y^n /\!\!/ \mathfrak{S}_n$ and write $p_n : Y^n \to \mathrm{Sym}^n Y$ for the quotient map. We put $\mathrm{Sym}\, Y = \sqcup_{n \geq 0} \mathrm{Sym}^n Y$ where $\mathrm{Sym}^0 Y = \{\mathrm{pt}\}$. Given a sheaf $\mathcal{F} \in D_{\mathrm{abs}}(Y)$, for each $n \geq 1$ put $\mathrm{Sym}^n \mathcal{F} = \left((p_n)_! \mathcal{F}^{\boxtimes n} \right)^{\mathfrak{S}_n}$. Also let $\mathrm{Sym}^0 \mathcal{F} = \underline{\mathbf{C}}_{\mathrm{pt}}$. Let $\mathrm{Sym}\, \mathcal{F}$ be a sheaf on $\mathrm{Sym}\, Y$ such that the restriction of $\mathrm{Sym}\, \mathcal{F}$ to $\mathrm{Sym}^n Y$ equals $\mathrm{Sym}^n \mathcal{F}$. Then, one has the following equations for generating functions in one variable z (these equations are well known from calculations involving the L-function associated with the sheaf $\mathcal{F}$):

$$\sum_{n \geq 0} z^n \cdot \left(\sum_{w \in (\mathrm{Sym}^n Y)^{\mathsf{Fr}}} \mathsf{Tr}_{\mathsf{Fr}} | \, H^\bullet(\mathrm{Sym}^n \mathcal{F}|_w) \right) = \sum_{n \geq 0} z^n \cdot \mathsf{Tr}_{\mathsf{Fr}} | \, H_c^\bullet(\mathrm{Sym}^n Y, \mathrm{Sym}^n \mathcal{F})$$

$$= \exp\left(\sum_{n \geq 1} \frac{z^n}{n} \cdot \mathrm{tr}_{\mathsf{Fr}^n} | H_c^\bullet(Y, \mathcal{F}) \right) = \exp\left(\sum_{n \geq 1} \frac{z^n}{n} \cdot \left(\sum_{y \in Y^{\mathsf{Fr}^n}} \mathrm{tr}_{\mathsf{Fr}^n} | H^\bullet(\mathcal{F}|_y) \right) \right). \quad (2.4.4) \quad \boxed{\mathrm{expn}}$$

Here, the first and third equality hold by the Grothendieck-Lefschetz trace formula, and the second equality is a consequence of the Künneth formula one

Observe that for any finite decomposition $Y = \sqcup_\alpha Y_\alpha$, where $i_\alpha : Y_\alpha \hookrightarrow Y$ is a locally closed subvariety, in K_{abs} one has $[H_c^\bullet(Y, \mathcal{F})] = \sum_\alpha [H_c^\bullet(Y_\alpha, i_\alpha^* \mathcal{F})]$. It follows that one can assign a well defined element $[H_c^\bullet(Y, \mathcal{F})] \in K_{\mathrm{abs}}$ and extend equations (2.4.4) to sheaves on constructible schemes. Recall that a constructible scheme is, roughly speaking, a reduced scheme Y defined up to an equivalence given by partitioning Y into smaller, locally closed subsets, cf. [KS1].

We will use a graded version of the above. Specifically, let $Y_{\mathbf{v}}$, $\mathbf{v} \in \mathbb{Z}_{\geq 0}^I$, be a collection of reduced schemes of finite type over k and $\mathcal{F}_{\mathbf{v}} \in D_{\mathrm{abs}}(Y_{\mathbf{v}})$, a collection of sheaves. We put $Y = \sqcup_{\mathbf{v}} Y_{\mathbf{v}}$, a disjoint union of the schemes $Y_{\mathbf{v}}$. For each $n \geq 1$ and $\mathbf{v} \in \mathbb{Z}_{\geq 0}^I$ let $\mathrm{Sym}^{\mathbf{v}} Y_n$ be the image of $\cup_{\{\mathbf{v}_1, \dots \mathbf{v}_n \in \mathbb{Z}_{\geq 0}^I | \mathbf{v}_1 + \dots + \mathbf{v}_n = \mathbf{v}\}} \mathrm{Sym}^{\mathbf{v}_1} Y \times \dots \mathrm{Sym}^{\mathbf{v}_n} Y$ under the quotient map $p_n : Y^n \to \mathrm{Sym}^n Y$. Let $\mathrm{Sym}^{\mathbf{v}} \mathcal{F}_n$ be the restriction of the sheaf $\mathrm{Sym}^n \mathcal{F}$ to $\mathrm{Sym}^{\mathbf{v}} Y_n$. It is clear that $\mathrm{Sym}^{\mathbf{v}} Y = \sqcup_n \mathrm{Sym}^{\mathbf{v}} Y_n$. This is a disjoint union of finitely many schemes of finite type and we have $\mathrm{Sym}^{\mathbf{v}} \mathcal{F} \in D_{\mathrm{abs}}(\mathrm{Sym}^{\mathbf{v}} Y)$, where $\mathcal{F}$ is a sheaf such that $\mathcal{F}|_{\mathrm{Sym}^{\mathbf{v}} Y_n} = \mathrm{Sym}^{\mathbf{v}} \mathcal{F}_n$. Further, it is clear that one has a decomposition $\mathrm{Sym}\, Y = \sqcup_{\mathbf{v}} \mathrm{Sym}^{\mathbf{v}} Y$.

For each $\mathbf{v}$, write $z^{\mathbf{v}} = \prod_{i \in I} z_i^{v_i}$ and let $\mathsf{E}(Y_{\mathbf{v}}, \mathcal{F}_{\mathbf{v}}) : \mathbb{Z}_{>0} \to \bar{\mathbb{Q}}_\ell$ be a function defined by the assignment $n \mapsto \sum_{y \in (\mathrm{Sym}^{\mathbf{v}} Y)^{\mathsf{Fr}^n}} \mathrm{tr}_{\mathsf{Fr}^n} | H^\bullet(\mathcal{F}_y)$. With this notation, a generalization of (2.4.4) to the $\mathbb{Z}_{\geq 0}^I$-graded setting reads

$$\sum_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^I} z^{\mathbf{v}} \cdot \left(\sum_{w \in (\mathrm{Sym}^{\mathbf{v}} Y)^{\mathsf{Fr}}} \mathsf{Tr}_{\mathsf{Fr}} | \, H^\bullet(\mathrm{Sym}^{\mathbf{v}} \mathcal{F}|_w) \right) = \mathrm{Exp}\left(\sum_{\mathbf{v} > 0} z^{\mathbf{v}} \cdot \mathsf{E}(Y_{\mathbf{v}}, \mathcal{F}_{\mathbf{v}}) \right). \quad (2.4.5) \quad \boxed{\mathrm{graded\text{-}exp}}$$

Sketch of proof of Proposition 2.4.2. The argument below is based on an observation, due to Kontsevich and Soibelman [KS1, §2], that for each $\mathbf{v}$ there is a canonically defined *constructible* scheme $X_{\mathbf{v}}$ of finite type over k such that one has a bijection of sets $[\mathfrak{X}_{\mathbf{v}}(\bar{\mathsf{k}})] \cong X_{\mathbf{v}}(\bar{\mathsf{k}})$. Here, $[\mathfrak{X}_{\mathbf{v}}(\bar{\mathsf{k}})]$ stands for the set of isomorphism classes of objects $\mathfrak{X}_{\mathbf{v}}(\mathscr{C})(\bar{\mathsf{k}})$, resp. $X_{\mathbf{v}}(\bar{\mathsf{k}})$ for the set of closed

$\bar{k}$-points of $X_\mathbf{v}$. Similarly, it was argued in [KS1] that for each $\mathbf{v} \neq 0$ there is a canonically de-fined constructible k-subscheme $X_\mathbf{v}^{\mathrm{ind}}$, of $X_\mathbf{v}$, such that one has a bijection between the subset $[\mathfrak{X}_\mathbf{v}^{\mathrm{ind}}(\mathscr{C})(\bar{k})] \subset [\mathfrak{X}_\mathbf{v}(\mathscr{C})(\bar{k})]$, of isomorphism classes of indecomposable objects, and the set $X_\mathbf{v}^{\mathrm{ind}}(\bar{k})$ of closed $\bar{k}$- points of the constructible scheme $X_\mathbf{v}^{\mathrm{ind}}$. It follows that for any finite field exten-sion K/k there is a bijection $X_\mathbf{v}^{\mathrm{ind}}(K) \cong [\mathrm{AI}_\mathbf{v}(\mathscr{C}_K)]$. Further, one has an ind-constructible scheme $X = \sqcup_\mathbf{v} X_\mathbf{v}$, resp. $X^{\mathrm{ind}} = \sqcup_{\mathbf{v}>0} X_\mathbf{v}^{\mathrm{ind}}$. Then, the Krull-Schmidt property implies that the di-rect sum map, $\oplus : \mathrm{Sym}^n X^{\mathrm{ind}} \to X$, $(x_1, \ldots, x_n) \mapsto x_1 \oplus \ldots \oplus x_n$, induces a bijection of sets $\oplus : (\mathrm{Sym}\, X^{\mathrm{ind}})(\bar{k}) \xrightarrow{\sim} X(\bar{k})$. This bijection sends $(\mathrm{Sym}^\mathbf{v} X^{\mathrm{ind}})(\bar{k})$ to $X_\mathbf{v}(\bar{k})$. Further, a potential $\phi : \mathfrak{X}(\mathscr{C}) \to \mathbb{A}$ induces morphisms $X_\mathbf{v} \to \mathbb{A}$, of constructible schemes. The additivity of the po-tential implies an isomorphism $\wp^{\phi \circ \oplus} \cong \wp^\phi \boxtimes \ldots \boxtimes \wp^\phi$, of sheaves on $\mathrm{Sym}\, X^{\mathrm{ind}}$. We conclude that equation (2.4.5) becomes, in the case where $Y_\mathbf{v} = X_\mathbf{v}^{\mathrm{ind}}$ and $\mathcal{F}_\mathbf{v} = \wp^\phi|_{X^{\mathrm{ind}}_\mathbf{v}}$, the required equation of Proposition 2.4.2. $\qquad\square$

2.5. Generalized isotypic decomposition. The construction of this subsection will (only) be used in §3.3.

Consider a diagram $\widetilde{X} \xrightarrow{a} X \xrightarrow{b} Y$ of morphisms of schemes and put $c := b \circ a$. We make the following assumptions:

(1) The morphism a is an unramified finite Galois covering;
(2) Each of the morphisms b and c is a Zariski locally trivial fibration;
(3) All fibers of the morphisms b and c are connected.

Let Γ be the Galois group of the Galois covering a and the group Γ acts naturally on $\widetilde{X}$. A local system on $\widetilde{X}$ is said to be *constant along c* if it is isomorphic to a pull-back via c of a local system on Y. Let $\widetilde{\mathcal{F}}$ be a Γ-equivariant local system on $\widetilde{X}$ which is constant along c. Therefore, on any local system $\mathcal{F}$ on X one obtains, thanks to assumption (3), an action of the local system of groups which is obtained by taking the associated bundle for the action of Γ on itself by conjugation. The group Γ being finite, this yields a canonical direct sum decomposition into isotypic components:

$$\mathcal{F} = \oplus_{\rho \in \mathrm{Irr}(\Gamma)} \mathcal{F}^\rho, \tag{2.5.1}$$

where $\mathrm{Irr}(\Gamma)$ denotes the set of isomorphism classes of irreducible Γ-representations in finite di-mensional $\bar{\mathbb{Q}}_\ell$-vector spaces.

Next, we say that a $\bar{\mathbb{Q}}_\ell$-local system $\tilde{\mathcal{F}}$ on $\widetilde{X}$ is *unipotent along c* if it admits a finite filtration by local sub-systems such that $\mathrm{gr}\,\tilde{\mathcal{F}}$, an associated graded local system, is constant along c. The restriction of such an $\tilde{\mathcal{F}}$ to any fiber of c is a local system with unipotent monodromy.

Lemma 2.5.2. *Let $\mathcal{F}$ be a $\bar{\mathbb{Q}}_\ell$-local system on X such that the local system $a^*\mathcal{F}$, on $\widetilde{X}$, is unipotent along c. Then, there is a canonical 'generalized isotypic decomposition':*

$$\mathcal{F} = \oplus_{\rho \in \mathrm{Irr}(\Gamma)} \mathcal{F}^{\langle\rho\rangle}. \tag{2.5.3}$$

The direct summand $\mathcal{F}^{\langle\rho\rangle}$ is uniquely determined by the requirement that it admits a filtration such that $a^* \mathrm{gr}(\mathcal{F}^{\langle\rho\rangle})$, a pull-back of an associated graded local system, is constant along c and using the notation of (2.5.1), we have $\mathrm{gr}(\mathcal{F}^{\langle\rho\rangle}) = (\mathrm{gr}(\mathcal{F}^{\langle\rho\rangle}))^\rho$, equivalently, $(\mathrm{gr}(\mathcal{F}^{\langle\rho\rangle}))^\sigma = 0$, $\forall \sigma \in \mathrm{Irr}(\Gamma)$, $\sigma \neq \rho$.

Proof. It will be convenient to identify $\bar{\mathbb{Q}}_\ell$-local systems on a scheme with representations of the geometric (étale) fundamental group of that scheme.

We have the following diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1^{et}(\widetilde{X}) & \longrightarrow & \pi_1^{et}(X) & \longrightarrow & \Gamma & \longrightarrow & 1 \\
& & \downarrow{\scriptstyle c_*} & & \downarrow{\scriptstyle b_*} & & & & \\
& & \pi_1^{et}(Y) & \overset{\mathrm{Id}}{=\!=\!=} & \pi_1^{et}(Y) & & & &
\end{array}
$$

where the maps b_* and c_* are induced by the morphisms b and c, respectively. It follows from assumption (1) that the horizontal row in the diagram is an exact sequence, and it follows from assumptions (2)-(3) that the vertical maps are surjective.

Let $K = \mathrm{Ker}(b_*)$, resp. $\widetilde{K} = \mathrm{Ker}(c_*)$. We may view $\pi_1^{et}(\widetilde{X})/\widetilde{K}$ and $K/\widetilde{K}$ as normal subgroups of $\pi_1^{et}(X)/\widetilde{K}$. Then, diagram chase yields $\pi_1^{et}(X)/\widetilde{K} = (\pi_1^{et}(\widetilde{X})/\widetilde{K}) \cdot (K/\widetilde{K})$ and $(\pi_1^{et}(\widetilde{X})/\widetilde{K}) \cap (K/\widetilde{K}) = \{1\}$. It follows that the group $\pi_1^{et}(X)/\widetilde{K}$ is a direct product of the subgroups $\pi_1^{et}(\widetilde{X})/\widetilde{K}$ and $K/\widetilde{K}$. Further, we have that $K/\widetilde{K} \cong \Gamma$ and $\pi_1^{et}(\widetilde{X})/\widetilde{K} = \pi_1^{et}(Y)$. We deduce an exact sequence

$$
1 \to \widetilde{K} \to \pi_1^{et}(X) \to \Gamma \times \pi_1^{et}(Y) \to 1,
$$

such that the group $\pi_1^{et}(\widetilde{X}) \subset \pi_1^{et}(X)$ gets identified with the preimage of $\{1\} \times \pi_1^{et}(Y) \subset \Gamma \times \pi_1^{et}(Y)$.

Now, let $f : \pi_1^{et}(X) \to \mathrm{GL}(V)$ be the representation that corresponds to a local system $\mathcal{F}$ on X. The assumption that $a^*\mathcal{F}$ be unipotent along c translates into the condition that $f|_{\widetilde{K}}$ is a unipotent representation of the group $\widetilde{K}$. In such a case, by elementary group theory, there is a canonical generalized isotypic decomposition $V|_K = \oplus_{\rho \in \mathrm{Irr}(\Gamma)} V^{\langle \rho \rangle}$ where each $V^{\langle \rho \rangle}$ is K-stable. The direct summand $V^{\langle \rho \rangle}$ is characterized by the property that it is the maximal K-subrepresentation of $V|_K$ such that any irreducible K-subquotient of $V^{\langle \rho \rangle}$ is isomorphic to a pull-back of ρ via the projection $K \twoheadrightarrow \Gamma$.

We claim that each space $V^{\langle \rho \rangle}$ is in fact $\pi_1^{et}(X)$-stable. To see this, choose a $\pi_1^{et}(X)$-stable increasing filtration $0 = V_0 \subset V_1 \subset \ldots \subset V_m = V$ such that the action of $\widetilde{K}$ on V_i/V_{i-1} is trivial. To prove the claim it suffices to show that for any $i \geq 1$ the space $V_i \cap V^{\langle \rho \rangle}$ is $\pi_1^{et}(X)$-stable and, moreover, one has

$$
V_i = \oplus_{\rho \in \mathrm{Irr}(\Gamma)} (V_i \cap V^{\langle \rho \rangle}).
$$

This is easily proved by induction on i using that the $\pi_1^{et}(X)$-action on V_i/V_{i-1} factors through an action of $\pi_1^{et}(X)/\widetilde{K} = \Gamma \times \pi_1^{et}(Y)$ and the characterizing property of the subspaces $V^{\langle \rho \rangle}$.

The claim insures that, for each $\rho \in \mathrm{Irr}(\Gamma)$, there is a local subsystem $\mathcal{F}^{\langle \rho \rangle}$, of $\mathcal{F}$, that corresponds to the $\pi_1^{et}(X)$-subrepresentation $V^{\langle \rho \rangle} \subset V$. We leave to the reader to check that the resulting decomposition $\mathcal{F} = \oplus_{\rho \in \mathrm{Irr}(\Gamma)} \mathcal{F}^{\langle \rho \rangle}$ satisfies all the required properties. $\qquad\square$

Similarly, we say that a local system $\mathcal{F}$ on X is *unipotent relative to c* if it admits a finite filtration by local sub-systems such that $a^* \mathrm{gr}\, \mathcal{F}$, a pull-back of an associated graded local system, is isomorphic to a local system of the form $c^*\mathcal{F}_Y$ for some local system $\mathcal{F}_Y$ on Y. Let $D_{\mathrm{abs}}^{\mathrm{unip}}(X, c)$, denote the full triangulated subcategory of $D_{\mathrm{abs}}(X)$ whose objects are complexes $\mathcal{F}$ such that each cohomology sheaf of $\mathcal{F}$ is a local system unipotent relative to c. Let $K_{\mathrm{abs}}^{\mathrm{unip}}(X, c)$, denote the corresponding Grothendieck group with the modification similar to the one explained in §2.3. Also, let $K_{\mathrm{abs}}^{\mathrm{loc}}(Y)$ be the Grothendieck group of the category of absolutely convergent local systems on C. The functor $\mathrm{R}\Gamma_c(Y, -)$ induces a homomorphism $K_{\mathrm{abs}}^{\mathrm{loc}}(Y) \to K_{\mathrm{abs}}(\mathrm{pt})$, $[\mathcal{F}_Y] \mapsto [\mathrm{R}\Gamma_c(Y, \mathcal{F}_Y)]$.

Let $\mathrm{Irr}(\Gamma)$ be the set of isomorphism classes of irreducible Γ-representations in finite dimensional $\bar{\mathbb{Q}}_\ell$-vector spaces For each $\rho \in \mathrm{Irr}(\Gamma)$, we define a group homomorphism

$$
K_{\mathrm{abs}}^{\mathrm{unip}}(X, c) \to K_{\mathrm{abs}}^{\mathrm{loc}}(Y), \quad [\mathcal{F}] \mapsto [\mathcal{F}]^{\langle \rho \rangle}, \tag{2.5.4} \quad \boxed{\texttt{hor}}
$$

as follows.

Let $\mathcal{F} \in D_{\mathrm{abs}}^{\mathrm{unip}}(X, c)$. One can assume, without changing the class of $\mathcal{F}$ in the Grothendieck group, that $\mathcal{F}$ is a local system and, morover, that $a^*\mathcal{F} \cong c^*\mathcal{F}_Y$ for some local system $\mathcal{F}_Y$ on Y The local system $a^*\mathcal{F}$ comes equipped with a canonical Γ-equivariant structure which induces, via the isomorphism $a^*\mathcal{F} \cong c^*\mathcal{F}_Y$, a Γ-equivariant structure on $c^*\mathcal{F}_Y$. The group Γ acts along the fibers of c and these fibers are connected. Therefore, we have $\mathcal{F}_Y = R^0 c_* a^*\mathcal{F}$ (an underived direct image) and giving the Γ-equivariant structure on $c^*\mathcal{F}_Y$ is equivalent to giving an action $\Gamma \to \mathrm{Aut}(\mathcal{F}_Y)$, of Γ on $\mathcal{F}_Y$. Hence, one has a canonical direct sum decomposition into Γ-isotypic components:

$$\mathcal{F}_Y = \oplus_{\rho \in \mathrm{Irr}(\Gamma)} \mathcal{F}_Y^\rho \qquad (2.5.5) \quad \boxed{\texttt{isotyp}}$$

By definition, the map ($\overset{\texttt{hor}}{2.5.4}$) sends the class $[\mathcal{F}] \in K_{\mathrm{abs}}^{\mathrm{unip}}(X, c)$ to the class $[\mathcal{F}]^{\langle \rho \rangle} := [(R^0 c_* a^*\mathcal{F})^\rho] \in K_{\mathrm{abs}}^{\mathrm{loc}}(Y)$. It is easy to check that this map is well-defined and the class thus defined does not depend on the choices of various filtrations involved in the construction. In particular, note that the uniform boundedness of weights in the definition of $K_{\mathrm{abs}}(X)$ implies that the K-group above is the same as the K-group of pure complexes, hence the K-group is the same as for sheaves with monodromy along the fibers of c. Therefore the above K-group decomposes as a direct sum over representations of Γ, and the Γ-isotypic components along the fibers and descend to the base.

Remark 2.5.6. Under these conditions we also have an isomorphism obtained by taking the direct sum of the generalized isotypic component morphisms over all irreducible representations of the finite group Γ

$$K_{\mathrm{abs}}^{\mathrm{unip}}(X, c) \to \oplus_{\mathrm{Irr}(\Gamma)} K_{\mathrm{abs}}^{\mathrm{loc}}(Y)$$

3. FACTORIZATION SHEAVES

<div style="margin-left:-2em">$\boxed{\texttt{sec3}}$</div>

In this section we introduce factorization sheaves and prove a result (Theorem $\overset{\texttt{expC}}{3.2.3}$) involving cohomology of such sheaves and the Fourier transform. This theorem, which has an independent interest, plays a crucial role in relating exponential sums over indecomposable objects with the geometry of moment maps.

<div style="margin-left:-2em">$\boxed{\texttt{cond-gen}}$</div>

3.1. Given a nonempty finite set J let $\mathfrak{S}_J$ denote the group of bijections $J \to J$ and $\mathrm{sign} : \mathfrak{S}_J \to \{\pm 1\}$ the sign-character. Given a positive integer v we write $[v] := \{1, 2, \ldots, v\}$ and $\mathfrak{S}_v := \mathfrak{S}_{[v]}$, the Symmetric group on v letters. For $v = 0$, we define $[v] = \emptyset$, resp. $\mathfrak{S}_v = \{1\}$. Given a dimension vector $\mathbf{v} = \{v_i\}_{i \in I}$, we put $[\mathbf{v}] = \sqcup_i [v_i]$, resp. $|\mathbf{v}| = \sum_i v_i$, and $\mathfrak{S}_\mathbf{v} = \prod_i \mathfrak{S}_{[v_i]} = \prod_i \mathfrak{S}_{v_i}$.

Let C be a connected scheme over a field k and C^J the scheme of maps $\gamma : J \to C$. This scheme is isomorphic to a cartesian power of C (for $J = \emptyset$ we declare that $C^J = \{\mathrm{pt}\}$) and it comes equipped with a natural $\mathfrak{S}_J$-action. For an integer $v \geq 0$, resp. dimension vector $\mathbf{v} = \{v_i\}_{i \in I}$, we use simplified notation $C^v := C^{[v]}$, resp. $C^\mathbf{v} := C^{[\mathbf{v}]}$.

Let $\mathbf{v}^1, \ldots, \mathbf{v}^m$ be an m-tuple of dimension vectors. One has the following canonical maps

$$\prod_\alpha C^{\mathbf{v}^\alpha}/\mathfrak{S}_{\mathbf{v}_\alpha} \cong C^{\sqcup_\alpha [\mathbf{v}_\alpha]}/(\prod_\alpha \mathfrak{S}_{[\mathbf{v}_\alpha]}) \to C^{\sqcup_\alpha [\mathbf{v}_\alpha]}/\mathfrak{S}_{\sqcup_\alpha [\mathbf{v}_\alpha]} \cong C^{\mathbf{v}^1 + \cdots + \mathbf{v}^m}/\mathfrak{S}_{\mathbf{v}^1 + \cdots + \mathbf{v}^m}. \quad (3.1.1) \quad \boxed{\texttt{imath}}$$

Here, the last isomorphism uses the fact that for any set J with v elements there is a canonical isomorphism $C^J/\mathfrak{S}_J \cong C^v/\mathfrak{S}_v$, of stacks. Let $\imath_{\mathbf{v}^1, \ldots, \mathbf{v}^m}$ denote the composite map in ($\overset{\texttt{imath}}{3.1.1}$).

Further, we put

$$C_{\mathbf{v}^1, \ldots, \mathbf{v}^m}^{\mathrm{disj}} := \{\gamma : \sqcup_\alpha [\mathbf{v}^\alpha] \to C \mid \gamma([\mathbf{v}^\alpha]) \cap \gamma([\mathbf{v}^\beta]) = \emptyset, \quad \forall \alpha \neq \beta\}. \quad (3.1.2) \quad \boxed{\texttt{hec}}$$

Thus, $C_{\mathbf{v}^1, \ldots, \mathbf{v}^m}^{\mathrm{disj}}$ is a Zariski open and dense, $(\prod_\alpha \mathfrak{S}_{\mathbf{v}^\alpha})$-stable subscheme of $\prod_\alpha C^{\mathbf{v}^\alpha}$. We obtain a diagram

$$C_{\mathbf{v}^1, \ldots, \mathbf{v}^m}^{\mathrm{disj}}/(\prod_\alpha \mathfrak{S}_{\mathbf{v}^\alpha}) \overset{\jmath_{\mathbf{v}^1, \ldots, \mathbf{v}^m}}{\hookrightarrow} \prod_\alpha(C^{\mathbf{v}^\alpha}/\mathfrak{S}_{\mathbf{v}^\alpha}) \overset{\imath_{\mathbf{v}^1, \ldots, \mathbf{v}^m}}{\twoheadrightarrow} C^{\mathbf{v}^1 + \cdots + \mathbf{v}^m}/\mathfrak{S}_{\mathbf{v}^1 + \cdots + \mathbf{v}^m}, \quad (3.1.3) \quad \boxed{\texttt{ij}}$$

23

where $\jmath_{\mathbf{v}^1,\ldots,\mathbf{v}^m}$ is the natural open imbedding.

Let $\tau_{\mathbf{v},\mathbf{w}} C^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}} \times C^{\mathbf{w}}/\mathfrak{S}_{\mathbf{w}} \xrightarrow{\sim} C^{\mathbf{w}}/\mathfrak{S}_{\mathbf{w}} \times C^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}$ be the flip morphism. It is clear that $\tau_{\mathbf{v},\mathbf{w}}$ maps the set $C^{\mathrm{disj}}_{\mathbf{v},\mathbf{w}}/(\mathfrak{S}_{\mathbf{v}} \times \mathfrak{S}_{\mathbf{w}})$ to $C^{\mathrm{disj}}_{\mathbf{w},\mathbf{v}}/(\mathfrak{S}_{\mathbf{w}} \times \mathfrak{S}_{\mathbf{v}})$. Furthermore, the following diagram commutes

$$
\begin{array}{ccc}
C^{\mathrm{disj}}_{\mathbf{v},\mathbf{w}}/(\mathfrak{S}_{\mathbf{v}} \times \mathfrak{S}_{\mathbf{w}}) & \xrightarrow{\quad\tau_{\mathbf{v},\mathbf{w}}\quad} & C^{\mathrm{disj}}_{\mathbf{w},\mathbf{v}}/(\mathfrak{S}_{\mathbf{w}} \times \mathfrak{S}_{\mathbf{v}}) \\
& \searrow^{\imath_{\mathbf{v},\mathbf{w}}} \quad \swarrow^{\imath_{\mathbf{w},\mathbf{v}}} & \\
& C^{\mathbf{v}+\mathbf{w}}/\mathfrak{S}_{\mathbf{v}+\mathbf{w}} &
\end{array}
$$

It follows that there is a isomorphism of functors $\psi_{\mathbf{v},\mathbf{w}} : \imath^*_{\mathbf{v},\mathbf{w}} \xrightarrow{\sim} \tau^*_{\mathbf{v},\mathbf{w}} \circ \imath^*_{\mathbf{w},\mathbf{v}}$.

fac-def **Definition 3.1.4.** A *weak factorization sheaf* $\mathcal{F}$ on $\mathrm{Sym}\, C$ is the data of a collection $(\mathcal{F}_{\mathbf{v}})_{\mathbf{v} \in \mathbb{Z}^I_{\geq 0}}$, $\mathcal{F}_{\mathbf{v}} \in D(C^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}})$, equipped, for each pair $\mathbf{v}_1, \mathbf{v}_2$ of dimension vectors, with an isomorphism

$$
\varphi_{\mathbf{v}_1,\mathbf{v}_2} : \jmath^*_{\mathbf{v}_1,\mathbf{v}_2} \imath^*_{\mathbf{v}_1,\mathbf{v}_2} \mathcal{F}_{\mathbf{v}_1+\mathbf{v}_2} \to \jmath^*_{\mathbf{v}_1,\mathbf{v}_2}(\mathcal{F}_{\mathbf{v}_1} \boxtimes \mathcal{F}_{\mathbf{v}_2}),
$$

of sheaves on $C^{\mathrm{disj}}_{\mathbf{v}_1,\mathbf{v}_2}/(\mathfrak{S}_{\mathbf{v}_1} \times \mathfrak{S}_{\mathbf{v}_2})$, such that:

Associativity constraint: For any triple $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, of dimension vectors (using slightly imprecise notation) one has

$$
\jmath^*_{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3}\big((\varphi_{\mathbf{v}_1,\mathbf{v}_2} \boxtimes \mathrm{Id}_{\mathbf{v}_3}) \circ \varphi_{\mathbf{v}_1+\mathbf{v}_2,\mathbf{v}_3}\big) = \jmath^*_{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3}\big(\varphi_{\mathbf{v}_1,\mathbf{v}_2+\mathbf{v}_3} \circ (\mathrm{Id}_{\mathbf{v}_1} \boxtimes \varphi_{\mathbf{v}_2,\mathbf{v}_3})\big). \qquad (3.1.5) \quad \text{fact2}
$$

Commutativity constraint: For any $\mathbf{v}$, the following diagram commutes

$$
\begin{array}{ccc}
\jmath^*_{\mathbf{v},\mathbf{w}} \imath^*_{\mathbf{v},\mathbf{w}} \mathcal{F}_{\mathbf{v}+\mathbf{w}} & \xrightarrow{\quad\varphi_{\mathbf{v},\mathbf{w}}\quad} & \jmath^*_{\mathbf{v},\mathbf{w}}(\mathcal{F}_{\mathbf{v}} \boxtimes \mathcal{F}_{\mathbf{w}}) \\
{\scriptstyle \jmath^*_{\mathbf{v},\mathbf{w}}(\psi_{\mathbf{v},\mathbf{w}})}\Big\downarrow & & \Big\downarrow{\scriptstyle \cong} \\
\tau^*_{\mathbf{v},\mathbf{w}} \jmath^*_{\mathbf{w},\mathbf{v}} \imath^*_{\mathbf{w},\mathbf{v}} \mathcal{F}_{\mathbf{v}+\mathbf{w}} & \xrightarrow{\tau_{\mathbf{v},\mathbf{w}}(\varphi_{\mathbf{w},\mathbf{v}})} & \tau^*_{\mathbf{v},\mathbf{w}} \jmath^*_{\mathbf{w},\mathbf{v}}(\mathcal{F}_{\mathbf{w}} \boxtimes \mathcal{F}_{\mathbf{v}})
\end{array} \qquad (3.1.6) \quad \text{fac-diag}
$$

Here the vertical map on the right comes from the natural isomorphism $\mathcal{F}_{\mathbf{v}} \boxtimes \mathcal{F}_{\mathbf{w}} \xrightarrow{\sim} \tau^*_{\mathbf{v},\mathbf{w}}(\mathcal{F}_{\mathbf{w}} \boxtimes \mathcal{F}_{\mathbf{v}})$.

Given an action on C of a group H, we get the diagonal H-action on $C^{\mathbf{v}}$ for any I-graded set $\mathbf{v}$. An H-*equivariant* weak factorization sheaf is a weak factorization sheaf $\mathcal{F} = (\mathcal{F}_{\mathbf{v}})$ equipped, for each $\mathbf{v}$, with an H-equivariant structure on $\mathcal{F}_{\mathbf{v}}$ and such that the morphisms $\varphi_{\mathbf{v}^1 \ldots,\mathbf{v}^m}$ respect the H-equivariant structures (here $\mathcal{F}_{\mathbf{v}^1} \boxtimes \ldots \boxtimes \mathcal{F}_{\mathbf{v}^m}$, an $H \times \ldots \times H$-equivariant sheaf, is viewed as an H-equivariant sheaf via the diagonal imbedding $H \hookrightarrow H \times \ldots \times H$).

Let $\mathcal{F} = (\mathcal{F}_{\mathbf{v}})$ be a factorization sheaf. Using the associativity constraint one constructs inductively, for any $m \geq 2$ and $\mathbf{v}_1, \ldots, \mathbf{v}_m$, an isomorphism

$$
\varphi_{\mathbf{v}^1,\ldots,\mathbf{v}^m} : \jmath^*_{\mathbf{v}^1,\ldots,\mathbf{v}^m} \imath^*_{\mathbf{v}^1,\ldots,\mathbf{v}^m} \mathcal{F}_{\mathbf{v}_1+\ldots+\mathbf{v}_m} \to \jmath^*_{\mathbf{v}^1,\ldots,\mathbf{v}^m}(\mathcal{F}_{\mathbf{v}^1} \boxtimes \ldots \boxtimes \mathcal{F}_{\mathbf{v}^m}) \qquad (3.1.7) \quad \text{phi-m}
$$

MacLane's coherence argument implies that this isomorphism is independent of the induction process.

Let $N(\mathfrak{S}_{\mathbf{v}_1,\ldots,\mathbf{v}_m})$ be the normalizer of the Young subgroup $\mathfrak{S}_{\mathbf{v}_1} \times \ldots \times \mathfrak{S}_{\mathbf{v}_m} \subset \mathfrak{S}_{\mathbf{v}_1+\ldots+\mathbf{v}_m}$. An element $\sigma \in N(\mathfrak{S}_{\mathbf{v}_1,\ldots,\mathbf{v}_m})$ induces a an automorphism τ_σ of $C^{\mathbf{v}_1} \times \ldots \times C^{\mathbf{v}_m}$. One has a commutative diagram

$$
\begin{array}{ccc}
C^{\mathrm{disj}}_{\mathbf{v}_1,\ldots,\mathbf{v}_m}/\prod_\alpha \mathfrak{S}_{\mathbf{v}_\alpha} & \xrightarrow{\quad\tau_\sigma\quad} & C^{\mathrm{disj}}_{\mathbf{v}_1,\ldots,\mathbf{v}_m}/\prod_\alpha \mathfrak{S}_{\mathbf{v}_\alpha} \\
& \searrow^{\imath_{\mathbf{v}_1,\ldots,\mathbf{v}_m}} \quad \swarrow^{\imath_{\mathbf{v}_1,\ldots,\mathbf{v}_m}} & \\
& C^{\mathbf{v}_1+\ldots+\mathbf{v}_m}/\mathfrak{S}_{\mathbf{v}_1+\ldots+\mathbf{v}_m} &
\end{array}
$$

24

Thus, there is a isomorphism of functors $\psi_\sigma : \iota^*_{\mathbf{v}_1,\ldots,\mathbf{v}_m} \xrightarrow{\sim} \tau^*_\sigma \circ \iota^*_{\mathbf{v}_1,\ldots,\mathbf{v}_m}$. Using that symmetric groups are generated by transpositions, from the commutativity constraint (3.1.6) one deduces that for any $m \geq 2$ and $\sigma \in N(\mathfrak{S}_{\mathbf{v}_1,\ldots,\mathbf{v}_m})$ the following diagram commutes

$$
\begin{array}{ccc}
\jmath^*_{\mathbf{v}_1,\ldots,\mathbf{v}_m}\iota^*_{\mathbf{v}_1,\ldots,\mathbf{v}_m}\mathcal{F}_{\mathbf{v}_1+\ldots+\mathbf{v}_m} & \xrightarrow{\varphi_{\mathbf{v}_1,\ldots,\mathbf{v}_m}} & \jmath^*_{\mathbf{v}_1,\ldots,\mathbf{v}_m}(\mathcal{F}_{\mathbf{v}_1}\boxtimes\ldots\boxtimes\mathcal{F}_{\mathbf{v}_m}) \\
\downarrow{\scriptstyle \jmath^*_{\mathbf{v}_1,\ldots,\mathbf{v}_m}(\psi_{\mathbf{v}_1,\ldots,\mathbf{v}_m})} & & \downarrow{\scriptstyle \cong} \\
\tau^*_\sigma\jmath^*_{\mathbf{v}_1,\ldots,\mathbf{v}_m}\iota^*_{\mathbf{v}_1,\ldots,\mathbf{v}_m}\mathcal{F}_{\mathbf{v}_1+\ldots+\mathbf{v}_m} & \xrightarrow{\tau^*_\sigma(\varphi_{\mathbf{v}_1,\ldots,\mathbf{v}_m})} & \tau^*_\sigma\jmath^*_{\mathbf{v}_1,\ldots,\mathbf{v}_m}(\mathcal{F}_{\mathbf{v}_1}\boxtimes\ldots\boxtimes\mathcal{F}_{\mathbf{v}_m})
\end{array}
\qquad (3.1.8) \quad \boxed{\text{nor}}
$$

$\boxed{\text{fact-rems}}$ *Remarks* 3.1.9. The following is clear

(1) Given a Weil (as opposed to a general absolutely convergent) sheaf $\mathcal{E}$ on a scheme (or a Deligne-Mumford stack), its Verdier dual is again a Weil sheaf, to be denoted $\mathcal{E}^\vee$. Let $\mathcal{F} = (\mathcal{F}_\mathbf{v})$ be a weak factorization sheaf where each $\mathcal{F}_\mathbf{v}$ is a Weil sheaf. Then, the collection $\mathcal{F}^\vee = (\mathcal{F}^\vee_\mathbf{v})$ is a weak factorization sheaf.

(2) For any pair $\mathcal{F} = (\mathcal{F}_\mathbf{v})$, $\mathcal{F}' = (\mathcal{F}'_\mathbf{v})$, of weak factorization sheaves on C, the collection $\mathcal{F} \otimes \mathcal{F}' = (\mathcal{F}_\mathbf{v} \otimes \mathcal{F}'_\mathbf{v})$ has the natural structure of a weak factorization sheaf on C.

(3) Let $\mathbb{C}^{\mathrm{sign}}_{C^\mathbf{v}}$ be a rank 1 constant sheaf on $C^\mathbf{v}$ equipped with a (unique) $\mathfrak{S}_\mathbf{v}$-equivariant structure such that the group $\mathfrak{S}_\mathbf{v}$ acts in the stalks over points of the principal diagonal of $C^\mathbf{v}$ via the sign character of the group $\mathfrak{S}_\mathbf{v}$. The collection $\{\mathbb{C}^{\mathrm{sign}}_{C^\mathbf{v}}[d_\mathbf{v}]\}$, where $d_\mathbf{v} = (\sum v_i)\cdot\dim C$ has the natural structure of a weak factorization sheaf. We remark that without homological shift by $d_\mathbf{v}$, the commutativity of (3.1.6) fails already in the case where the set I consists of 1 element, $C = \mathbb{A}$, and $v = w = 1$.

(4) For any factorization sheaf $\mathcal{F}$ on C and an open subset $C_0 \subset C$, the collection $\mathcal{F}|_{C_0} = (\mathcal{F}_\mathbf{v}|_{C_0^\mathbf{v}})$ is a weak factorization sheaf on C_0.

We often consider weak factorization sheaves $\mathcal{F} = (\mathcal{F}_\mathbf{v})$ such that each $\mathcal{F}_\mathbf{v}$ is an absolutely convergent sheaf and all the data involved in the above defininition is compatible with the Frobenius. Similarly, in the case where the ground field is $\mathsf{k} = \mathbb{C}$ we may (and will) use the category of mixed Hodge $\mathcal{D}$-modules as a replacement of the category of mixed ℓ-adic sheaves. We will not explicitly mention 'absolutely convergent', resp. 'mixed ℓ-adic' or 'mixed Hodge', when dealing with weak factorization sheaves in those settings.

There are several other, slightly different variants of the notion of weak factorization sheaf which are stronger than the one given in Definition 3.1.4. In particular, one has

$\boxed{\text{fac-str}}$ **Definition 3.1.10.** (i) A *factorization sheaf* on C is the data (1)-(2) that satisfies condition (B), as in the above definition, and (3) is replaced by isomorphisms $\bar{\varphi}_{\mathbf{v}^1,\ldots,\mathbf{v}^m} : \iota^*_{\mathbf{v}^1,\ldots,\mathbf{v}^m}\mathcal{F}_{\mathbf{v}^1\sqcup\ldots\sqcup\mathbf{v}^m} \to \mathcal{F}_{\mathbf{v}^1}\boxtimes\ldots\boxtimes\mathcal{F}_{\mathbf{v}^m}$ (of sheaves on $C^{\mathbf{v}^1} \times \ldots \times C^{\mathbf{v}^m}$ rather than on the open subset $C^{\mathrm{disj}}_{\mathbf{v}^1,\ldots,\mathbf{v}^m}$) such that the diagram in condition (B), with $\varphi_{\mathbf{v}^1,\ldots,\mathbf{v}^m}$ being replaced by $\bar{\varphi}_{\mathbf{v}^1,\ldots,\mathbf{v}^m}$, commutes.

(ii) A factorization sheaf $\mathcal{F}$ is called a *strong factorization sheaf* if each of the corresponding morphisms $\bar{\varphi}_{\mathbf{v}^1,\ldots,\mathbf{v}^m}$ is itself an isomorphism and condition (B) is replaced by the stronger equation $\left(\prod_\alpha \eta(\sigma_\alpha)\right) \circ \bar{\varphi}_{\mathbf{v}^1,\ldots,\mathbf{v}^m} = \bar{\varphi}_{K^1,\ldots,K^m} \circ \eta(\sqcup\sigma_\alpha)$.

Clearly, we have

strong factorization sheaves $\Rightarrow$ factorization sheaves $\Rightarrow$ weak factorization sheaves.

Remark 3.1.11. There are analogues of the notions of factorization sheaves, resp. weak and strong factorization sheaves, involving higher categories where condition (3.1.5) has an infinite sequence of higher analogues. In that context, analogues of weak factorization sheaves are are usually called 'factorization algebras', cf. e.g. [FG, §2.5]. These are sheaves on $\coprod_\mathbf{v} C^\mathbf{v}/\mathfrak{S}_\mathbf{v}$ where the latter is viewed as a factorization ∞-stack. Higher analogues of factorization sheaves considered in

25

Definition 3.1.10(i) are called *factorization E_1-algebras*, cf. [Lur]. These are associative algebras in the category of factorization algebras. Finally, objects analogous to strong factorization sheaves turn out to form a rather small subclass of objects which are determined by their components with dimension vectors $\mathbf{v} = 1_i$, $i \in I$. ◇

3.2. **A cohomology result on factorization sheaves.** Let $\mathbf{v}$ be a dimension vector and $\mathbb{A} \subset \mathbb{A}^{\mathbf{v}}$ the principal diagonal. We put $\mathfrak{t} = \mathbb{A}^{\mathbf{v}}/\mathbb{A}$. Using the identification $(\mathbb{A}^{\mathbf{v}})^* = \mathbb{A}^{\mathbf{v}}$, we have $\mathfrak{t}^* = \{(z_i^{\alpha})_{i \in I, \alpha \in [1, v_i]} \in \mathbb{A}^{\mathbf{v}} \mid \sum_{i, \alpha} z_i^{\alpha} = 0\}$, a codimension one hyperplane of $\mathbb{A}^{\mathbf{v}}$. We define a subset $\mathfrak{t}_{\mathbf{v}}^{\diamond} \subset \mathfrak{t}_{\mathbf{v}}^*$ as follows

$$\mathfrak{t}_{\mathbf{v}}^{\diamond} = \left\{ (z_i^{\alpha}) \in \mathbb{A}^{\mathbf{v}} \;\middle|\; \begin{array}{l} \text{(1) } \sum_{i \in I} \sum_{\alpha \in [v_i]} z_i^{\alpha} = 0; \quad \text{(2) } z_i^{\alpha} \neq z_i^{\beta} \text{ for any } \alpha \neq \beta; \text{ and} \\ \text{(3) for any } I\text{-tuple of subsets } J_i \subseteq [v_i], \; i \in I, \text{ such that } J_i \neq \emptyset, \\ \text{resp. } J_i \neq [v_i], \text{ for at least one } i, \text{ one has } \sum_{i \in I} \sum_{\alpha \in J_i} z_i^{\alpha} \neq 0. \end{array} \right\} \qquad (3.2.1) \quad \boxed{\text{de2}}$$

This is a $\mathbb{G} \times \mathfrak{S}_{\mathbf{v}}$-stable, Zariski open and dense subset of $\mathfrak{t}_{\mathbf{v}}^*$.

Remark 3.2.2. Condition (2) in (3.2.1) says that the $\mathfrak{S}_{\mathbf{v}}$-action on $\mathfrak{t}_{\mathbf{v}}^{\diamond}$ is free. The meaning of condition (3) will be explained in §3.4 below. We note that conditions (1) and (3) in the RHS of (3.2.1) are additive analogues of conditions (2) and (3), respectively, from section 1.5). ◇

Now, fix a a smooth scheme C with tangent, resp. cotangent, sheaf T_C, resp. T_C^*. The normal, resp. conormal, sheaf to $C \subset C^{\mathbf{v}}$, the principal diagonal, is canonically isomorphic to $\mathfrak{t} \otimes T_C$, resp. $\mathfrak{t}^* \otimes T_C^*$. Therefore, $\mathsf{N}_{\mathbf{v}}$, resp. $\mathsf{N}_{\mathbf{v}}^*$, the total space of the normal, resp. conormal, bundle on C is isomorphic to the total space of the vector bundle associated with the sheaf $\mathfrak{t} \otimes T_C$, resp. $\mathfrak{t}^* \otimes T_C^*$. We have $\mathrm{rk}\, \mathsf{N} = \dim \mathfrak{t} \cdot \dim C = (|\mathbf{v}| - 1) \dim C$. Let $T^{\times}C$ be the complement of the zero section in the cotangent bundle $T^*C \to C$. We define $\mathsf{N}_{\mathbf{v}}^{\diamond} := \mathfrak{t}_{\mathbf{v}}^{\diamond} \times_{\mathbb{G}} T^{\times}C$. This is a $\mathfrak{S}_{\mathbf{v}}$-stable, Zariski open and dense subset of $\mathsf{N}_{\mathbf{v}}^*$ that has been mentioned in §1.7. A geometric interpretation of the set $\mathsf{N}_{\mathbf{v}}^{\diamond}$ will become clear from Lemma 3.4.3.

Let $\Delta(\mathbf{v}) : C \to C^{\mathbf{v}}$ be the diagonal imbedding, $u : C \hookrightarrow \mathsf{N}_{\mathbf{v}}$ the zero section, and $v : \mathsf{N}_{\mathbf{v}}^* \to C$ the projection. Let $\mathrm{F}_{\mathsf{N}} : D_{\mathrm{abs}}(\mathsf{N}_{\mathbf{v}}) \to D_{\mathrm{abs}}(\mathsf{N}_{\mathbf{v}}^*)$ be the (relative) Fourier-Deligne transform functor along the fibers of the vector bundle $\mathsf{N} \to C$. We choose a self-dual normalization of F_{N} such that F_{N} takes perverse sheaves to perverse sheaves. Then, for any sheaf $\mathcal{E}$ on C, one has $\mathrm{F}_{\mathsf{N}}(u_! \mathcal{E}) = v^* \mathcal{E}[\mathrm{rk}\, \mathsf{N}_{\mathbf{v}}](\frac{\mathrm{rk}\, \mathsf{N}_{\mathbf{v}}}{2})$. There is also a similar functor $\mathrm{F}_{\mathsf{N}} : D_{\mathrm{abs}}(\mathsf{N}_{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}) \to D_{\mathrm{abs}}(\mathsf{N}_{\mathbf{v}}^*/\mathfrak{S}_{\mathbf{v}})$, between $\mathfrak{S}_{\mathbf{v}}$-equivariant categories. Let $\Phi_{\mathsf{N}} : D_{\mathrm{abs}}(C^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}) \to D_{\mathrm{abs}}(\mathsf{N}_{\mathbf{v}}^{\diamond}/\mathfrak{S}_{\mathbf{v}})$ denote the composition functor $\mathcal{F} \mapsto (\mathrm{F}_{\mathsf{N}} \circ \mathrm{sp}_{\mathsf{N}} \mathcal{F})|_{\mathsf{N}^{\diamond}/\mathfrak{S}_{\mathbf{v}}}$.

Note that $\mathfrak{S}_{\mathbf{v}}$-action on $\mathsf{N}_{\mathbf{v}}^{\diamond}$ is free and the setup of Section 2.5 applies to the natural maps $\mathsf{N}_{\mathbf{v}}^{\diamond} \xrightarrow{a} \mathsf{N}_{\mathbf{v}}^{\diamond}/\mathfrak{S}_{\mathbf{v}} \xrightarrow{b} C$ and $c = b \circ a$. The main result of section 3, which is the precise version of equation (1.7.3), reads

$\boxed{\text{expC}}$ **Theorem 3.2.3.** *Let $\mathcal{F} = (\mathcal{F}_{\mathbf{v}})_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^I}$ be a weak factorization sheaf on C such that for every $\mathbf{v}$ the sheaf $\Delta(\mathbf{v})^* \widetilde{\mathcal{F}}_{\mathbf{v}}$ is a local system on C. Then, we have*
   (i) For each $\mathbf{v}$, the sheaf $\Phi_{\mathsf{N}}(\mathcal{F}_{\mathbf{v}})$ is an object of $D_{\mathrm{abs}}^{\mathrm{unip}}(\mathsf{N}^{\diamond}/\mathfrak{S}_{\mathbf{v}}, c)$.
   (ii) In $\mathsf{K}_{\mathrm{abs}}^I$, one has the following equation

$$\bigoplus_{\mathbf{v}} \left[\mathrm{R}\Gamma_c(C^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}, \mathcal{F}_{\mathbf{v}})\right] = \mathrm{Sym}\left(\bigoplus_{\mathbf{v} \neq 0} \mathrm{R}\Gamma_c\left(C, [\Phi_{\mathsf{N}}(\mathcal{F}_{\mathbf{v}})]^{\langle\mathrm{triv}\rangle}\right)\{\dim C(|\mathbf{v}| - 1)\}\right).$$

Here, $\{n\} = [n](\frac{n}{2})$ and the symbol $[\ldots]^{\langle\mathrm{triv}\rangle}$ in the RHS of the formula of part (ii) refers to the map (2.5.4) in the case $\rho = \mathrm{triv}$. This makes sense thanks to the statement of part (i).
   The above theorem will be proved in §3.7.

3.3. **The case** $C = \mathbb{A}$. In this paper, we will only be interested in the case where the scheme C is either $C = \mathbb{A}$ or $C = \mathbb{G}$. Thus, $\mathbb{A}^{\mathbf{v}} := C^{\mathbf{v}}$ is an affine space, resp. $\mathbb{G}^{\mathbf{v}} := C^{\mathbf{v}}$ is an algebraic torus. Let $Aff = \mathbb{G} \ltimes \mathbb{G}_a$ be the group of affine-linear transformations $x \mapsto ax + b$, of $\mathbb{A}$. The natural *Aff*-action on $\mathbb{A}$ induces, for any dimension vector $\mathbf{v}$, the diagonal *Aff*-action on $\mathbb{A}^{\mathbf{v}}$ that descends to $\mathbb{A}^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}$. Let *Aff* act on $\mathfrak{t}$ through its quotient $\mathbb{G}$ by dilations and act on $\mathfrak{t} \times \mathbb{A}$, resp. $\mathfrak{t}^* \times \mathbb{A}$, diagonally. Then, there is a natural *Aff*-equivariant trivialization $\mathsf{N} = \mathfrak{t} \times \mathbb{A}$, resp. $\mathsf{N}^* = \mathfrak{t}^* \times \mathbb{A}$. Thus, we have $\mathsf{N}^\diamond := \mathbb{A} \times \mathfrak{t}_{\mathbf{v}}^\diamond$. The action of $\mathfrak{S}_{\mathbf{v}}$ on $\mathfrak{t}_{\mathbf{v}}^\diamond$ being free, the canonical map $\mathfrak{t}_{\mathbf{v}}^\diamond/\mathfrak{S}_{\mathbf{v}} \to \mathfrak{t}_{\mathbf{v}}^\diamond /\!\!/ \mathfrak{S}_{\mathbf{v}}$ is an isomorphism, so we will make no distinction between $\mathfrak{t}_{\mathbf{v}}^\diamond/\mathfrak{S}_{\mathbf{v}}$ and $\mathfrak{t}_{\mathbf{v}}^\diamond /\!\!/ \mathfrak{S}_{\mathbf{v}}$. Let $\delta : \mathsf{N}^*/\mathfrak{S}_{\mathbf{v}} = \mathfrak{t}^*/\mathfrak{S}_{\mathbf{v}} \times \mathbb{A} \to \mathfrak{t}_{\mathbf{v}}^*/\mathfrak{S}_{\mathbf{v}}$ be the first projection.

Let $\mathcal{F}_{\mathbf{v}}$ be an *Aff*-equivariant sheaf on $\mathbb{A}^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}$. Then, $\mathcal{F}_{\mathbf{v}}$ descends to a sheaf $\bar{\mathcal{F}}_{\mathbf{v}}$ on $(\mathbb{A}^{\mathbf{v}}/\mathbb{A})/\mathfrak{S}_{\mathbf{v}}$. Moreover, $\bar{\mathcal{F}}_{\mathbf{v}}$ is $\mathbb{G}$-equivariant with respect to the dilation action on $\mathbb{A}^{\mathbf{v}}/\mathbb{A}$. This implies a natural isomorphism $\mathrm{sp}_{\mathsf{N}}(\mathcal{F}_{\mathbf{v}}) \cong \mathcal{F}_{\mathbf{v}}$.

Let $\mathrm{F}_{\mathfrak{t}}$ denote the Fourier-Deligne transform $D_{\mathrm{abs}}(\mathfrak{t}_{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}) \to D_{\mathrm{abs}}(\mathfrak{t}_{\mathbf{v}}^*/\mathfrak{S}_{\mathbf{v}})$. The *Aff*-equivariant structure on $\mathcal{F}_{\mathbf{v}}$ provides the sheaf $\mathrm{F}_{\mathsf{N}}(\mathcal{F}_{\mathbf{v}})$, on $\mathsf{N}_{\mathbf{v}}^*/\mathfrak{S}_{\mathbf{v}} = \mathbb{A} \times \mathfrak{t}_{\mathbf{v}}^*/\mathfrak{S}_{\mathbf{v}}$, with a natural equivariant structure with respect to the action of the additive group on $\mathfrak{t}_{\mathbf{v}}^*/\mathfrak{S}_{\mathbf{v}} \times \mathbb{A}$ by translation along the second factor and one has a canonical isomorphism $\mathrm{F}_{\mathsf{N}}(\mathcal{F}_{\mathbf{v}}) \cong \delta^* \mathrm{F}_{\mathfrak{t}}(\bar{\mathcal{F}}_{\mathbf{v}})$. We put $\Phi_{\mathfrak{t}}(\mathcal{F}_{\mathbf{v}}) := \mathrm{F}(\bar{\mathcal{F}}_{\mathbf{v}})|_{\mathsf{N}_{\mathbf{v}}^\diamond/\mathfrak{S}_{\mathbf{v}}}$. By construction, we have $\Phi_{\mathsf{N}}(\mathcal{F}_{\mathbf{v}}) = \delta^* \Phi_{\mathfrak{t}}(\mathcal{F}_{\mathbf{v}})$.

Let $\mathcal{F} := (\mathcal{F}_{\mathbf{v}})_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^I}$ be be an *Aff*-equivariant weak factorization sheaf on $\mathbb{A}$. Then, each of the sheaves $\Delta(\mathbf{v})^* \widetilde{\mathcal{F}}_{\mathbf{v}}$ is an *Aff*-equivariant, hence a geometrically constant, sheaf on $\mathbb{A}$. Thus, applying part (i) of Theorem 3.2.3 we deduce that the local system $\Phi_{\mathfrak{t}}(\mathcal{F}_{\mathbf{v}})$ is unipotent relative to the constant map $\mathfrak{t}_{\mathbf{v}}^\diamond/\mathfrak{S}_{\mathbf{v}} \to \mathrm{pt}$. It follows that there is a canonical '*generalized* isotypic decomposition':

$$\Phi_{\mathfrak{t}}(\mathcal{F}_{\mathbf{v}}) = \oplus_{\rho \in \mathrm{Irr}(\mathfrak{S}_{\mathbf{v}})} \Phi_{\mathfrak{t}}^{\langle \rho \rangle}(\mathcal{F}_{\mathbf{v}}).$$

Choose an algebraic closure $\overline{\mathsf{k}(\mathfrak{t}_{\mathbf{v}}^\diamond/\mathfrak{S}_{\mathbf{v}})}$ of the fraction field of $\mathfrak{t}_{\mathbf{v}}^\diamond/\mathfrak{S}_{\mathbf{v}}$. Let $\eta = \mathrm{Spec}\, \overline{\mathsf{k}(\mathfrak{t}_{\mathbf{v}}^\diamond/\mathfrak{S}_{\mathbf{v}})}$ be the generic point of $\mathfrak{t}_{\mathbf{v}}^\diamond/\mathfrak{S}_{\mathbf{v}}$. Let $\Phi_{\mathfrak{t}}(\mathcal{F}_{\mathbf{v}})_\eta^{\langle \rho \rangle}$ be the restriction of $\Phi_{\mathfrak{t}}^{\langle \rho \rangle}(\mathcal{F}_{\mathbf{v}})$ to η. Let $\mathrm{Gal}(Y)$ denote the fundamental group of a k-scheme Y. The choice of a k-rational closed point $\eta_{\mathbf{v}} \in \mathfrak{t}_{\mathbf{v}}^\diamond/\mathfrak{S}_{\mathbf{v}}$ provides a a lift of the Frobenius element in $\mathrm{Gal}(\bar{\mathsf{k}}/\mathsf{k})$ to an element of $\mathrm{Gal}(\bar{\eta}/\eta)$. This defines, for each $\rho \in \mathrm{Irr}(\mathfrak{S}_{\mathbf{v}})$, a Frobenius action on $\Phi_{\mathfrak{t}}(\mathcal{F}_{\mathbf{v}})_\eta^{\langle \rho \rangle}$ which agrees with the one on $\Phi_{\mathfrak{t}}^{\langle \rho \rangle}(\mathcal{F}_{\mathbf{v}})|_{\eta_{\mathbf{v}}}$.

For every $\mathbf{v}$, one has a natural diagram $\mathbb{G}^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}} \xrightarrow{\varepsilon} \mathbb{A}^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}} \xleftarrow{i} 0/\mathfrak{S}_{\mathbf{v}}$, where the map ε is an open imbedding induced by the natural imbedding $\mathbb{G} \hookrightarrow \mathbb{A}$ and i is a closed imbedding. Let

$$D_{\mathrm{abs}}(\mathbb{G}^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}) \xleftarrow{\varepsilon^*} D_{\mathrm{abs}}(\mathbb{A}^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}) \xrightarrow{i^*} D_{\mathrm{abs}}(0/\mathfrak{S}_{\mathbf{v}})$$

be the corresponding restriction functors. For any *Aff*-equivariant weak factorization sheaf $\mathcal{F}$ on $\mathbb{A}$, the collection $\varepsilon^* \mathcal{F} = (\varepsilon^* \mathcal{F}_{\mathbf{v}})$ gives a $\mathbb{G}$-equivariant weak factorization sheaf on $\mathbb{G}$, cf. Remark 3.1.9(4).

The following result will be deduced from Theorem 3.2.3.

Theorem 3.3.1. *Let $\mathcal{F} = (\mathcal{F}_{\mathbf{v}})_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^I}$ be an Aff-equivariant weak factorization sheaf on $\mathbb{A}$.*

(i) *The monodromy action of the subgroup $\pi_1^{et}(\mathfrak{t}_{\mathbf{v}}^\diamond) \subset \pi_1^{et}(\mathfrak{t}_{\mathbf{v}}^\diamond/\mathfrak{S}_{\mathbf{v}})$ on $\Phi_{\mathfrak{t}}(\mathcal{F}_{\mathbf{v}})_\eta$ is unipotent.*

(ii) *In $\mathsf{K}_{\mathrm{abs}}^I$, one has the following equations*

$$\sum_{\mathbf{v}} z^{\mathbf{v}} \cdot [\mathrm{R}\Gamma_c(\mathbb{A}^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}, \mathcal{F}_{\mathbf{v}})] = \mathrm{Sym}\left(-\mathbb{L}^{\frac{1}{2}} \cdot \sum_{\mathbf{v} > 0} (-1)^{|\mathbf{v}|} \cdot z^{\mathbf{v}} \cdot \mathbb{L}^{\frac{|\mathbf{v}|}{2}} \cdot [\Phi_{\mathfrak{t}}(\mathcal{F}_{\mathbf{v}})_\eta]^{\langle \mathrm{triv} \rangle} \right) \qquad (3.3.2)$$

$$\sum_{\mathbf{v}} [\mathrm{R}\Gamma_c(\mathbb{G}^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}, \varepsilon^* \mathcal{F}_{\mathbf{v}})] = \mathrm{Sym}\left((\mathbb{L}^{-\frac{|\mathbf{v}|}{2}} - \mathbb{L}^{\frac{|\mathbf{v}|}{2}}) \cdot \sum_{\mathbf{v} > 0} (-1)^{|\mathbf{v}|} \cdot z^{\mathbf{v}} \cdot \mathbb{L}^{\frac{|\mathbf{v}|}{2}} \cdot [\Phi_{\mathfrak{t}}(\mathcal{F}_{\mathbf{v}})_\eta^{\langle \mathrm{triv} \rangle}] \right) \quad (3.3.3)$$

$$\sum_{\mathbf{v}} [\mathrm{R}\Gamma_c(0/\mathfrak{S}_{\mathbf{v}}, i^*\mathcal{F}_{\mathbf{v}})] = \mathrm{Sym}\left(-\mathbb{L}^{\frac{1}{2}} \cdot \sum_{\mathbf{v}>0} (-1)^{|\mathbf{v}|} \cdot z^{\mathbf{v}} \cdot \mathbb{L}^{-\frac{|\mathbf{v}|}{2}} \cdot [\Phi_t(\mathcal{F}_{\mathbf{v}})_\eta]^{\langle\mathrm{triv}\rangle}\right). \qquad (3.3.4) \quad \boxed{\texttt{4eqs}}$$

$\boxed{\texttt{diag}}$

3.4. Diagonal stratification. Given a dimension vector $\mathbf{v} \neq 0$, let $\widetilde{\mathfrak{P}}(\mathbf{v})$ be the set of decompositions $[\mathbf{v}] = \sqcup_\alpha J^\alpha$ into a disjoint union of *nonempty* subsets J^α. Let $\mathfrak{P}(\mathbf{v})$ be the set of unordered collections $(\mathbf{v}^\alpha)_\alpha$, of *nonzero* dimension vectors, such that $\mathbf{v} = \sum_\alpha \mathbf{v}^\alpha$. For $(\mathbf{v}^\alpha)_\alpha \in \mathfrak{P}(\mathbf{v})$, let $d_{\mathbf{w}}$ be the number of occurences of a dimension vector $\mathbf{w}$ as an element of the collection $(\mathbf{v}^\alpha)_\alpha$. We see that giving an element of $\mathfrak{P}(\mathbf{v})$ is equivalent to giving a collection $\{d_{\mathbf{w}} \in \mathbb{Z}_{\geq 0},\ \mathbf{w} \in Z_{\geq 0}^I \smallsetminus \{0\}\}$ such that $\sum_{\mathbf{w}} d_{\mathbf{w}} \cdot \mathbf{w} = \mathbf{v}$. Thus, the set $\mathfrak{P}(\mathbf{v})$ may be identified with the set of multi-partitions of $\mathbf{v}$.

For $\mathfrak{A}, \mathfrak{B} \in \widetilde{\mathfrak{P}}(\mathbf{v})$, resp. $\mathfrak{a}, \mathfrak{b} \in \mathfrak{P}(\mathbf{v})$, we write $\mathfrak{A} \leq \mathfrak{B}$, resp. $\mathfrak{a} \leq \mathfrak{b}$, whenever $\mathfrak{B}$ is a refinement of $\mathfrak{A}$, resp. $\mathfrak{b}$ is a refinement of $\mathfrak{a}$. This gives a partial order on $\widetilde{\mathfrak{P}}(\mathbf{v})$, resp. $\mathfrak{P}(\mathbf{v})$. By definition, $\mathfrak{B} \geq \mathfrak{A} = (J^\alpha)$ holds iff the parts of $\mathfrak{B}$ are obtained by partitioning further each of the sets J^α. For any tuple $\{\mathfrak{b}^\alpha = (\mathbf{w}^{\alpha,\beta})_\beta \in \mathfrak{P}(\mathbf{v}^\alpha),\ \alpha \in A\}$, where A is a finite set A, we have $\sum_{\alpha,\beta} \mathbf{w}^{\alpha,\beta} = \sum_\alpha \mathbf{v}^\alpha$, so the collection $(\mathbf{w}^{\alpha,\beta})_{\alpha,\beta}$ gives an element of $\mathfrak{P}(\sum_\alpha \mathbf{v}^\alpha)$. This way one obtains a bijection

$$\prod_\alpha \mathfrak{P}(\mathbf{v}^\alpha) \ \xrightarrow{\sim}\ \{\mathfrak{b} \in \mathfrak{P}(\textstyle\sum_\alpha \mathbf{v}^\alpha) \mid \mathfrak{b} \geq \mathfrak{a}\}, \quad \{(\mathbf{w}^{\alpha,\beta})_\beta \in \mathfrak{P}(\mathbf{v}^\alpha),\ \alpha \in A\} \mapsto (\mathbf{w}^{\alpha,\beta})_{\alpha,\beta}.$$

Any decomposition $\mathfrak{A} = (J^\alpha) \in \widetilde{\mathfrak{P}}(\mathbf{v})$ gives, for each $i \in I$, a decomposition $[v_i] = \sqcup_\alpha ([v_i] \cap J^\alpha)$. We put $\mathfrak{S}_{\mathfrak{A}} = \prod_{i,\alpha} \mathfrak{S}_{[v_i] \cap J^\alpha}$, a subgroup of $\mathfrak{S}_{\mathbf{v}}$. Further, let $\mathbf{v}^\alpha := (v_i^\alpha)_{i\in I}$ be a dimension vector with components $v_i^\alpha := \#([v_i] \cap J^\alpha)$. Since $\mathbf{v} = \sum_\alpha \mathbf{v}^\alpha$ we obtain a multi-partition $\mathfrak{a} = (\mathbf{v}^\alpha) \in \mathfrak{P}(\mathbf{v})$. The assignment $\mathfrak{A} \mapsto \mathfrak{a}$ provides a natural map $\widetilde{\mathfrak{P}}(\mathbf{v}) \to \mathfrak{P}(\mathbf{v})$.

The group $\mathfrak{S}_{\mathbf{v}}$ acts naturally on the set $\widetilde{\mathfrak{P}}(\mathbf{v})$. The stabilizer in $\mathfrak{S}_{\mathbf{v}}$ of a decomposition $\mathfrak{A} = (J^\alpha)_{\alpha \in A} \in \widetilde{\mathfrak{P}}(\mathbf{v})$ equals $N(\mathfrak{S}_{\mathfrak{A}})$, the normalizer of the subgroup $\mathfrak{S}_{\mathfrak{A}} \subset \mathfrak{S}_{\mathbf{v}}$. We have an isomorphism

$$N(\mathfrak{S}_{\mathfrak{A}}) \cong \prod_{\mathbf{w} \in \mathbb{Z}^I} \left((\mathfrak{S}_{\mathbf{v}})^{d_{\mathbf{w}}} \rtimes \mathfrak{S}_{d_{\mathbf{w}}}\right) \quad \text{where} \quad d_{\mathbf{w}} := \#\{\alpha \in A \mid \mathbf{w} = \mathbf{v}^\alpha\}. \qquad (3.4.1) \quad \boxed{\texttt{N(si)}}$$

Observe further that the fibers of the map $\widetilde{\mathfrak{P}}(\mathbf{v}) \to \mathfrak{P}(\mathbf{v})$ are precisely the $\mathfrak{S}_{\mathbf{v}}$-orbits. We will identify $\mathfrak{P}(\mathbf{v})$ with the set of $\mathfrak{S}_{\mathbf{v}}$-orbits in $\widetilde{\mathfrak{P}}(\mathbf{v})$.

Fix a smooth curve C. For $r > 0$ let $C^{r,\circ} \subset C^r$ be a subset formed by the r-tuples of pairwise distinct points of C. Given a dimension vector $\mathbf{v}$ and $\mathfrak{A} = \{J^\alpha,\ \alpha \in [r]\} \in \widetilde{\mathfrak{P}}(\mathbf{v})$, we consider a closed imbedding $C^r \hookrightarrow C^{\mathbf{v}}$ defined, using the decomposition $[\mathbf{v}] = \sqcup_\alpha J^\alpha$, by the assignment

$$z = ([r] \to C,\ \alpha \mapsto z(\alpha)) \ \longmapsto\ \bar{\Delta}(z) = (\sqcup_\alpha J^\alpha \to C,\ J^\alpha \mapsto z(\alpha)).$$

We will usually identify C^r, resp. $C^{r,\circ}$, with its image, to be denoted $C_{\mathfrak{A}}^{\mathbf{v}}$, resp. $C_{\mathfrak{A}}^{\mathbf{v},\circ}$. Thus, $C_{\mathfrak{A}}^{\mathbf{v}}$ is a smooth closed subscheme of $C^{\mathbf{v}}$ and $C_{\mathfrak{A}}^{\mathbf{v},\circ}$ is a Zariski open and dense subset of $C_{\mathfrak{A}}^{\mathbf{v}}$. Write $\Delta(\mathfrak{A}) : C^{r,\circ} = C_{\mathfrak{A}}^{\mathbf{v},\circ} \hookrightarrow C^{\mathbf{v}}$, resp. $\bar{\Delta}(\mathfrak{A}) : C^r = C_{\mathfrak{A}}^{\mathbf{v}} \hookrightarrow C^{\mathbf{v}}$, for the corresponding locally closed, resp. closed, imbedding. It is clear that, for $\mathfrak{A}, \mathfrak{B} \in \widetilde{\mathfrak{P}}(\mathbf{v})$, we have $\mathfrak{A} \leq \mathfrak{B}$ iff one has an inclusion $C_{\mathfrak{A}}^{\mathbf{v}} \subseteq C_{\mathfrak{B}}^{\mathbf{v}}$. In particular, $C_{\mathfrak{A}}^{\mathbf{v}} = C_{\mathfrak{B}}^{\mathbf{v}}$ holds iff $\mathfrak{A} = \mathfrak{B}$. Note also that the sets $C_{\mathfrak{A}}^{\mathbf{v},\circ}$ and $C_{\mathfrak{B}}^{\mathbf{v},\circ}$ are equal if $\mathfrak{A} = \mathfrak{B}$, and are disjoint otherwise. It follows that $C_{\mathfrak{B}}^{\mathbf{v}} = \bigsqcup_{\mathfrak{A} \leq \mathfrak{B}} C_{\mathfrak{A}}^{\mathbf{v},\circ}$. In particular, one has a stratification $C^{\mathbf{v}} = \bigsqcup_{\mathfrak{A} \in \widetilde{\mathfrak{P}}(\mathbf{v})} C_{\mathfrak{A}}^{\mathbf{v},\circ}$. The action of the group $\mathfrak{S}_{\mathbf{v}}$ on $C^{\mathbf{v}}$ respects the stratification and induces a stratification $C^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}$ with locally-closed smooth strata which are labeled by the set $\mathfrak{P}(\mathbf{v})$. These stratifications of either $C^{\mathbf{v}}$ or $C^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}$ will be called '*diagonal stratification*'.

In the special case where $r = 1$ and $\mathfrak{A} = \{J^1\}$ is the decomposition with a single part $J^1 = [\mathbf{v}]$, we have that $C_{\mathfrak{A}}^{\mathbf{v}} = C_\Delta^{\mathbf{v}}$ is the principal diagonal and $\Delta(\mathfrak{A}) = \bar{\Delta}(\mathfrak{A}) = \Delta(\mathbf{v})$ is the diagonal imbedding. In this case, abusing the notation, we write $\mathfrak{A} = \Delta_{\mathbf{v}}$ for the corresponding decomposition .

28

Observe that the principal diagonal is contained in the closure of any other stratum, and it is the unique closed stratum of the diagonal stratification.

For any $\mathfrak{A} \in \widetilde{\mathfrak{P}}(\mathbf{v})$, we have closed imbeddings $C \xrightarrow{\Delta(\mathbf{v})} C_{\mathfrak{A}}^{\mathbf{v}} \to C^{\mathbf{v}}$. Let $\mathrm{T}_C(C_{\mathfrak{A}}^{\mathbf{v}})$ denote the normal bundle to $\Delta(\mathbf{v})(C) \subset C_{\mathfrak{A}}^{\mathbf{v}}$, resp. $\mathrm{T}_C^{\perp}(C_{\mathfrak{A}}^{\mathbf{v}})$ a vector sub-bundle of $\mathsf{N}_{\mathbf{v}}^*$ whose fibers are the annihilators of the corresponding fibers of $\mathrm{T}_C(C_{\mathfrak{A}}^{\mathbf{v}})$. Thus, one has natural diagrams

$$C \xleftarrow{u_{\mathfrak{A}}} \mathrm{T}_C(C_{\mathfrak{A}}^{\mathbf{v}}) \xhookrightarrow{v_{\mathfrak{A}}} \mathrm{T}_C(C^{\mathbf{v}}) = \mathsf{N}_{\mathbf{v}}, \qquad C \xleftarrow{u_{\mathfrak{A}}^{\perp}} \mathrm{T}_C^{\perp}(C_{\mathfrak{A}}^{\mathbf{v}}) \xhookrightarrow{v_{\mathfrak{A}}^{\perp}} \mathrm{T}_C^*(C^{\mathbf{v}}) = \mathsf{N}_{\mathbf{v}}^*. \quad (3.4.2) \quad \boxed{\text{TT}}$$

The following result is straighforward; it explains the geometric meaning of the set $\mathsf{N}_{\mathbf{v}}^{\diamond}$.

$\boxed{\text{veryreg}}$ **Lemma 3.4.3.** *The set $\mathsf{N}_{\mathbf{v}}^{\diamond}$ is equal to the set of elements $z \in \mathsf{N}_{\mathbf{v}}^*$ such that $w(z) \neq z$ for any $w \in \mathfrak{S}_{\mathbf{v}}$, $w \neq 1$, and, moreover, $z \notin \mathrm{T}_C^{\perp}(C_{\mathfrak{A}}^{\mathbf{v}})$ for any $\mathfrak{A} \in \mathfrak{P}(\mathbf{v})$, $\mathfrak{A} \neq \Delta$.* $\square$

$\boxed{\text{AB}}$

3.5. Fix $\mathfrak{A} \in \widetilde{\mathfrak{P}}(\mathbf{v})$ and let $\mathfrak{a} = (\mathbf{v} = \mathbf{v}^1 + \ldots + \mathbf{v}^r) \in \mathfrak{P}(\mathbf{v})$ be an associated multi-partition. We have a product of diagonal imbeddings $\prod_{\alpha \in [r]} \Delta(\mathbf{v}^{\alpha}) : C^r \hookrightarrow \prod_{\alpha \in [r]} C^{\mathbf{v}^{\alpha}}$. Note that the image of the open subset $C^{r,\circ} \subset C^r$ is contained in $C_{\mathbf{v}^1,\ldots,\mathbf{v}^r}^{\text{disj}}$, cf. $(\overset{\text{hec}}{3.1.2})$. We obtain a commutative diagram

$$\begin{array}{ccccc} C_{\mathbf{v}^1,\ldots,\mathbf{v}^r}^{\text{disj}} & \xleftarrow{\prod_{\alpha \in [r]} \Delta(\mathbf{v}^{\alpha})} & C^{r,\circ} = C_{\mathfrak{A}}^{\mathbf{v},\circ} & \xrightarrow{\Delta(\mathfrak{A})} & C^{\mathbf{v}} \\ \downarrow{\scriptstyle \varpi_{\mathbf{v}^1,\ldots,\mathbf{v}^r}} & & & & \downarrow{\scriptstyle \varpi_{\mathbf{v}}} \\ C_{\mathbf{v}^1,\ldots,\mathbf{v}^r}^{\text{disj}}/(\prod_{\alpha=1}^r \mathfrak{S}_{\mathbf{v}^{\alpha}}) & \xhookrightarrow{\jmath_{\mathbf{v}^1,\ldots,\mathbf{v}^r}} & \prod_{\alpha=1}^r (C^{\mathbf{v}^{\alpha}}/\mathfrak{S}_{\mathbf{v}^{\alpha}}) & \xrightarrow{\imath_{\mathbf{v}^1,\ldots,\mathbf{v}^r}} & C^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}. \end{array} \quad (3.5.1) \quad \boxed{\text{circ}}$$

It is often convenient to choose and fix an identification $[\mathbf{v}^1] \sqcup \ldots \sqcup [\mathbf{v}^r] \cong [\mathbf{v}]$. Such an identification determines a lift of the bijection $\mathfrak{P}(\mathbf{v}^1) \times \ldots \times \mathfrak{P}(\mathbf{v}^r) \xrightarrow{\sim} \{\mathfrak{b} \in \mathfrak{P}(\mathbf{v}) \mid \mathfrak{b} \geq \mathfrak{a}\}$ to a bijection $\gamma : \widetilde{\mathfrak{P}}(\mathbf{v}^1) \times \ldots \times \widetilde{\mathfrak{P}}(\mathbf{v}^r) \xrightarrow{\sim} \{\mathfrak{B} \in \widetilde{\mathfrak{P}}(\mathbf{v}) \mid \mathfrak{B} \geq \mathfrak{A}\}$. This provides an identification $\prod_{\alpha} C^{\mathbf{v}^{\alpha}} \xrightarrow{\sim} C^{\mathbf{v}}$ and, therefore, an identification of $C_{\mathbf{v}^1,\ldots,\mathbf{v}^r}^{\text{disj}}$ with an open subset of $C^{\mathbf{v}}$ that lifts the map $\imath_{\mathbf{v}^1,\ldots,\mathbf{v}^r} \circ \jmath_{\mathbf{v}^1,\ldots,\mathbf{v}^r}$. It is immediate to see that this way, for any $\mathfrak{B}^1 \in \widetilde{\mathfrak{P}}(\mathbf{v}^1),\ldots,\mathfrak{B}^r \in \widetilde{\mathfrak{P}}(\mathbf{v}^r)$, we get

$$C_{\mathbf{v}^1,\ldots,\mathbf{v}^r}^{\text{disj}} \cap (C_{\mathfrak{B}^1}^{\mathbf{v}^1,\circ} \times \ldots \times C_{\mathfrak{B}^r}^{\mathbf{v}^r,\circ}) = C_{\gamma(\mathfrak{B}^1,\ldots,\mathfrak{B}^r)}^{\mathbf{v},\circ}. \quad (3.5.2) \quad \boxed{\text{strat1}}$$

It will sometimes be convenient to use a separate notation $C_{\mathfrak{B}^1,\ldots,\mathfrak{B}^r}^{\circ}$ for the set on the left of this equation whenever we want to think of it as a Zariski open subset of $C_{\mathfrak{B}^1}^{\mathbf{v}^1,\circ} \times \ldots \times C_{\mathfrak{B}^r}^{\mathbf{v}^r,\circ}$.

Thus, the diagonal stratification of $C^{\mathbf{v}}$ induces the following stratification

$$C_{\mathbf{v}^1,\ldots,\mathbf{v}^r}^{\text{disj}} = \bigsqcup_{\mathfrak{B}^1 \in \widetilde{\mathfrak{P}}(\mathbf{v}^1),\ldots,\mathfrak{B}^r \in \widetilde{\mathfrak{P}}(\mathbf{v}^r)} C_{\mathfrak{B}^1,\ldots,\mathfrak{B}^r}^{\circ} = \bigsqcup_{\mathfrak{B} \geq \mathfrak{A}} C_{\mathfrak{B}}^{\mathbf{v},\circ}. \quad (3.5.3) \quad \boxed{\text{strat2}}$$

Below, we will freely use the above identifications without further mention.

$\boxed{\text{hec-rem}}$ *Remark* 3.5.4. Note that we have $C_{\Delta(\mathbf{v}^1),\ldots,\Delta(\mathbf{v}^r)}^{\circ} = C_{\mathfrak{A}}^{\mathbf{v}} \cong C^{r,\circ}$ is the unique closed stratum of the stratification.

Let $\mathcal{F} = (\mathcal{F}_{\mathbf{v}})$ be a weak factorization sheaf on C and recall the notation $\widetilde{\mathcal{F}} = (\widetilde{\mathcal{F}}_{\mathbf{v}})$, $\widetilde{\mathcal{F}}_{\mathbf{v}} = \varpi_{\mathbf{v}}^* \mathcal{F}_{\mathbf{v}}$.

We say that $\mathcal{F}$ is diagonally constructible, resp. diagonally constant, if for all $\mathbf{v}$ and $\mathfrak{A} \in \widetilde{\mathfrak{P}}(\mathbf{v})$ the sheaf $\Delta(\mathfrak{A})^* \widetilde{\mathcal{F}}_{\mathbf{v}}$ is a locally constant, resp. *geometrically* constant, sheaf on $C_{\mathfrak{A}}^{\mathbf{v},\circ}$.

$\boxed{\text{constr}}$ **Lemma 3.5.5.** *For $\mathcal{F}$ to be diagonally constructible, resp. diagonally constant, it is sufficient that $\Delta(\mathbf{v})^* \widetilde{\mathcal{F}}_{\mathbf{v}}$, the restriction of $\widetilde{\mathcal{F}}_{\mathbf{v}}$ to the principal diagonal, be a locally constant, resp. geometrically constant, sheaf for all $\mathbf{v}$.*

Proof. Fix $\mathbf{v}$ and $\mathfrak{A} = \{J^\alpha, \alpha \in [r]\} \in \widetilde{\mathfrak{P}}(\mathbf{v})$. Let $(\mathbf{v} = \mathbf{v}^1 + \ldots + \mathbf{v}^r) \in \mathfrak{P}(\mathbf{v})$ be the decomposition associated with $\mathfrak{A}$ and for each $\alpha = 1, \ldots, r$ let $\mathfrak{B}^\alpha \in \mathfrak{P}(\mathbf{v}^\alpha)$.

Observe that the stratum $C_{\mathfrak{A}}^{\mathbf{v}}$ is stable under the action of the group $N(\mathfrak{S}_{\mathfrak{A}}) \subset \mathfrak{S}_{\mathbf{v}}$. Using (3.1.7) and isomorphisms (3.5.2)-(3.5.3) we deduce an isomorphism

$$\left(\Delta(\gamma(\mathfrak{B}^1, \ldots, \mathfrak{B}^r))^* \widetilde{\mathcal{F}}_{\mathbf{v}}\right)\big|_{C_{\mathbf{v}^1, \ldots, \mathbf{v}^r}^{\mathrm{disj}}} \ \xrightarrow{\sim}\ \left(\Delta(\mathfrak{B}^1)^* \widetilde{\mathcal{F}}_{\mathbf{v}^1} \boxtimes \ldots \boxtimes \Delta(\mathfrak{B}^r)^* \widetilde{\mathcal{F}}_{\mathbf{v}^r}\right)\big|_{C_{\mathbf{v}^1, \ldots, \mathbf{v}^r}^{\mathrm{disj}}}. \qquad (3.5.6) \quad \boxed{\text{BBB}}$$

Moreover, diagram (3.1.8) says that this isomorphism is compatible with the natural action of the group $N(\mathfrak{S}_{\mathfrak{A}}) \cong N(\mathfrak{S}_{\mathbf{v}_1, \ldots, \mathbf{v}^r})$ on each side of (3.5.6).

If $\Delta(\mathbf{v})^* \widetilde{\mathcal{F}}_{\mathbf{v}}$ is locally constant for all $\mathbf{v}$ then so is the sheaf $\left(\Delta(\mathbf{v}^1)^* \widetilde{\mathcal{F}}_{\mathbf{v}^1} \boxtimes \ldots \boxtimes \Delta(\mathbf{v}^r)^* \widetilde{\mathcal{F}}_{\mathbf{v}^r}\right)\big|_{C_{\ldots, \Delta}^\circ}$, for all decompositions $\mathbf{v}^1 + \ldots + \mathbf{v}^r = \mathbf{v}$, $r \geq 1$. Combining Remark 3.5.4 and isomorphism (3.5.6) in the special case $(\mathfrak{B}^1, \ldots, \mathfrak{B}^r) = (\Delta(\mathbf{v}^1), \ldots, \Delta(\mathbf{v}^r))$, we deduce that $\Delta(\mathfrak{A})^* \widetilde{\mathcal{F}}_{\mathbf{v}}$ is a locally constant sheaf on $C_{\mathfrak{A}}^{\mathbf{v}, \circ}$ for all $\mathfrak{A} \in \widetilde{\mathfrak{P}}(\mathbf{v})$. An identical argument applies in the geometrically constant case as well. $\qquad \square$

3.6. Semisimple factorization sheaves.
Let Z be a geometrically connected variety. An object of $D_{\mathrm{abs}}(Z)$ is called *a semisimple local system* if it is isomorphic to a finite direct sum of objects of the form $V \otimes \mathcal{L}$, where $V \in D_{\mathrm{abs}}(\mathrm{pt})$ is an absolutely convergent complex with zero differential and $\mathcal{L}$ is a finite dimensional *geometrically* irreducible local system on Z.

Definition 3.6.1. A sheaf on $C^{\mathbf{v}}$ is said to be *nice* if it is isomorphic to a direct sum of the form $\bigoplus_{\mathfrak{A} \in \widetilde{\mathfrak{P}}(\mathbf{v})} \bar{\Delta}(\mathfrak{A})_* \mathcal{E}^{\mathfrak{A}}$, where $\mathcal{E}^{\mathfrak{A}}$ is a semisimple local system on $C_{\mathfrak{A}}^{\mathbf{v}}$. A weak factorization sheaf $\mathcal{F}$ is called nice, resp. semisimple, if so is the sheaf $\widetilde{\mathcal{F}}_{\mathbf{v}}$ for every $\mathbf{v}$.

Lemma 3.6.2. (i) *Let $\mathcal{F} = (\mathcal{F}_{\mathbf{v}})$ be a nice weak factorization sheaf so, for each $\mathbf{v}$, we have $\mathcal{F}_{\mathbf{v}} = \bigoplus_{\mathfrak{A} \in \widetilde{\mathfrak{P}}(\mathbf{v})} \mathcal{F}^{\mathfrak{A}}$, where $\mathcal{F}^{\mathfrak{A}} = \bar{\Delta}(\mathfrak{A})_* \mathcal{E}^{\mathfrak{A}}$ for some semisimple local system $\mathcal{E}^{\mathfrak{A}}$ on $C_{\mathfrak{A}}^{\mathbf{v}}$. Then, for every $\mathbf{v}$ and $\mathfrak{A} \in \widetilde{\mathfrak{P}}(\mathbf{v})$, there is an isomorphism*

$$\mathcal{E}^{\mathfrak{A}} \cong \boxtimes_{\alpha \in [r]}\, \mathcal{E}^{\Delta(\mathbf{v}^\alpha)}, \qquad (3.6.3) \quad \boxed{\text{ssfact}}$$

of $N(\mathfrak{S}_{\mathfrak{A}})$-equivariant local systems on $C_{\mathfrak{A}}^{\mathbf{v}} \cong \prod_{\alpha \in [r]} C^{\Delta(\mathbf{v}^\alpha)}$, where $\mathbf{v} = \mathbf{v}^1 + \ldots + \mathbf{v}^r$ is the decomposition associated with $\mathfrak{A}$ and r is the number of parts.

(ii) *Let $\mathcal{F} = (\mathcal{F}_{\mathbf{v}})$ be a semisimple weak factorization sheaf such that $\Delta(\mathbf{v})^* \widetilde{\mathcal{F}}_{\mathbf{v}}$ is a semisimple local system for any $\mathbf{v}$. Then $\mathcal{F}$ is nice.*

Proof. (ii) Fix $\mathfrak{A} \in \widetilde{\mathfrak{P}}(\mathbf{v})$ and use the notation as above. Given an r-tuple $\mathfrak{B}^\alpha \in \widetilde{\mathfrak{P}}(\mathbf{v}^\alpha)$, $\alpha \in [r]$, let

$$C_{\mathfrak{B}^1, \ldots, \mathfrak{B}^r} := C_{\mathbf{v}^1, \ldots, \mathbf{v}^r}^{\mathrm{disj}} \cap (C_{\mathfrak{B}^1}^{\mathbf{v}^1} \times \ldots \times C_{\mathfrak{B}^r}^{\mathbf{v}^r}) = C_{\mathbf{v}^1, \ldots, \mathbf{v}^r}^{\mathrm{disj}} \cap C_{\gamma(\mathfrak{B}^1, \ldots, \mathfrak{B}^r)}.$$

Thus, the set $C_{\mathfrak{B}^1, \ldots, \mathfrak{B}^r}$ equals the closure of the set $C_{\mathfrak{B}^1, \ldots, \mathfrak{B}^r}^\circ = C_{\mathbf{v}^1, \ldots, \mathbf{v}^r}^{\mathrm{disj}} \cap (C_{\mathfrak{B}^1}^{\mathbf{v}^1, \circ} \times \ldots \times C_{\mathfrak{B}^r}^{\mathbf{v}^r, \circ})$ in $C_{\mathbf{v}^1, \ldots, \mathbf{v}^r}^{\mathrm{disj}}$, cf. (3.5.3). Let $\Delta^{\mathrm{disj}}(\mathfrak{B}^1, \ldots, \mathfrak{B}^r) : C_{\mathfrak{B}^1, \ldots, \mathfrak{B}^r} \hookrightarrow C_{\mathbf{v}^1, \ldots, \mathbf{v}^r}^{\mathrm{disj}}$ denote the closed imbedding.

We have the following diagram

Hence, we find

$$\widetilde{\mathcal{F}}_{\mathbf{v}}\big|_{C^{\mathrm{disj}}_{\mathbf{v}^1,\dots,\mathbf{v}^r}} = \bigoplus_{\mathfrak{B}\in\mathfrak{P}(\mathbf{v})} \mathcal{F}^{\mathfrak{B}}\big|_{C^{\mathrm{disj}}_{\mathbf{v}^1,\dots,\mathbf{v}^r}} = \bigoplus_{\mathfrak{B}\geq\mathfrak{A}} (\bar\Delta(\mathfrak{B})_*\mathcal{E}^{\mathfrak{B}})\big|_{C^{\mathrm{disj}}_{\mathbf{v}^1,\dots,\mathbf{v}^r}}$$

$$= \bigoplus_{(\mathfrak{B}^1,\dots,\mathfrak{B}^r)\in\widetilde{\mathfrak{P}}(\mathbf{v}^1)\times\dots\times\widetilde{\mathfrak{P}}(\mathbf{v}^r)} (\Delta^{\mathrm{disj}}(\mathfrak{B}^1,\dots,\mathfrak{B}^r)_* \big(\mathcal{E}^{\gamma(\mathfrak{B}^1,\dots,\mathfrak{B}^r)}\big|_{C_{\mathfrak{B}^1,\dots,\mathfrak{B}^r}}\big).$$

Similarly, we have

$$(\boxtimes_\alpha \widetilde{\mathcal{F}}_{\mathbf{v}^\alpha})\big|_{C^{\mathrm{disj}}_{\mathbf{v}^1,\dots,\mathbf{v}^r}} = \Big(\boxtimes_\alpha \big(\bigoplus_{\mathfrak{B}^\alpha\in\mathfrak{P}(\mathbf{v}^\alpha)} \mathcal{F}^{\mathfrak{B}^\alpha}\big)\Big)\big|_{C^{\mathrm{disj}}_{\mathbf{v}^1,\dots,\mathbf{v}^r}}$$

$$= \bigoplus_{(\mathfrak{B}^1,\dots,\mathfrak{B}^r)\in\widetilde{\mathfrak{P}}(\mathbf{v}^1)\times\dots\times\widetilde{\mathfrak{P}}(\mathbf{v}^r)} \big(\boxtimes_\alpha \bar\Delta(\mathfrak{B}^\alpha)_*\mathcal{E}^{\mathfrak{B}^\alpha}\big)\big|_{C^{\mathrm{disj}}_{\mathbf{v}^1,\dots,\mathbf{v}^r}}$$

$$= \bigoplus_{(\mathfrak{B}^1,\dots,\mathfrak{B}^r)\in\widetilde{\mathfrak{P}}(\mathbf{v}^1)\times\dots\times\widetilde{\mathfrak{P}}(\mathbf{v}^r)} (\Delta^{\mathrm{disj}}(\mathfrak{B}^1,\dots,\mathfrak{B}^r)_* \big((\boxtimes_\alpha \mathcal{E}^{\mathfrak{B}^\alpha})\big|_{C_{\mathfrak{B}^1,\dots,\mathfrak{B}^r}}\big).$$

Comparing the above isomorphisms, we see that the factorization isomorphism $\varphi_{\mathbf{v}^1,\dots,\mathbf{v}^r}$ yields, for each tuple $(\mathfrak{B}^1,\dots,\mathfrak{B}^r)\in\mathfrak{P}(\mathbf{v}^1,\dots,\mathbf{v}^r)$, an isomorphism

$$\mathcal{E}^{\gamma(\mathfrak{B}^1,\dots,\mathfrak{B}^r)}\big|_{C_{\mathfrak{B}^1,\dots,\mathfrak{B}^r}} \cong (\boxtimes_\alpha \mathcal{E}^{\mathfrak{B}^\alpha})\big|_{C_{\mathfrak{B}^1,\dots,\mathfrak{B}^r}},$$

of local systems on $C_{\mathfrak{B}^1,\dots,\mathfrak{B}^r}$. The local system on each side of the isomorphism is a restriction of a local system on $C^{\mathbf{v}}_{\gamma(\mathfrak{B}^1,\dots,\mathfrak{B}^r)}$. The homomorphism $\pi_1^{et}\big(C_{\mathfrak{B}^1,\dots,\mathfrak{B}^r}\big) \to \pi_1^{et}(C^{\mathbf{v}}_{\gamma(\mathfrak{B}^1,\dots,\mathfrak{B}^r)})$ being surjective, we conclude that the isomorphism of local systems on $C_{\mathfrak{B}^1,\dots,\mathfrak{B}^r}$ extends to an isomorphism $\mathcal{E}^{\gamma(\mathfrak{B}^1,\dots,\mathfrak{B}^r)} \cong \boxtimes_\alpha \mathcal{E}^{\mathfrak{B}^\alpha}$ of the corresponding local systems on $C^{\mathbf{v}}_{\gamma(\mathfrak{B}^1,\dots,\mathfrak{B}^r)}$.

In the special case where $(\mathfrak{B}^1,\dots,\mathfrak{B}^r) = (\Delta(\mathbf{v}^1),\dots,\Delta(\mathbf{v}^r))$ one has $\gamma(\Delta(\mathbf{v}^1),\dots,\Delta(\mathbf{v}^r)) = \mathfrak{A}$. Thus, $\mathcal{E}^{\gamma(\mathfrak{B}^1,\dots,\mathfrak{B}^r)} \cong \boxtimes_\alpha \mathcal{E}^{\mathfrak{B}^\alpha}$ reduces to an isomorphism as in (3.6.3). Furthermore, one checks that this isomorphism respects the $N(\mathfrak{S}_\mathfrak{A})$-equivariant structures.

(ii) From the factorization isomorphism $(\Delta(\mathbf{v}^1)^*\widetilde{\mathcal{F}}_{\mathbf{v}^1}\boxtimes\dots\boxtimes\Delta(\mathbf{v}^r)^*\widetilde{\mathcal{F}}_{\mathbf{v}^r})\big|_{C^\circ_{\Delta,\dots,\Delta}} \cong \Delta(\mathfrak{A})^*\widetilde{\mathcal{F}}_{\mathbf{v}}$ we deduce that $\mathcal{F}$ is diagonally constructible, cf. proof of Lemma 3.5.5. Therefore, the semisimple sheaf $\widetilde{\mathcal{F}}_{\mathbf{v}}$ is isomorphic to $\oplus_\mathfrak{A} \Delta(\mathfrak{A})_{!*}\mathcal{E}^{\mathfrak{A},\circ}$, a direct sum of intersection cohomology complexes associated to some local systems $\mathcal{E}^{\mathfrak{A},\circ}$ on $C^{\mathfrak{A},\circ}_{\mathbf{v}}$. Furthermore, since the local system $(\Delta(\mathbf{v}^1)^*\widetilde{\mathcal{F}}_{\mathbf{v}^1}\boxtimes \dots \boxtimes \Delta(\mathbf{v}^r)^*\widetilde{\mathcal{F}}_{\mathbf{v}^r})$ is semisimple and $C^{\mathbf{v}}_\mathfrak{A} \cong C_{\Delta(\mathbf{v}^1)}\times\dots\times C_{\Delta(\mathbf{v}^r)}$, we deduce that the local system $\Delta(\mathfrak{A})^*\widetilde{\mathcal{F}}_{\mathbf{v}}$ extends to a semisimple local system on $C^{\mathbf{v}}_\mathfrak{A}$. The local system $\mathcal{E}^{\mathfrak{A},\circ}$ is a direct summand of $\Delta(\mathfrak{A})^*\widetilde{\mathcal{F}}_{\mathbf{v}}$. It follows that $\mathcal{E}^{\mathfrak{A},\circ}$ extends to a semisimple local system $\mathcal{E}^\mathfrak{A}$ on $C^{\mathbf{v}}_\mathfrak{A}$. Hence, we have $\Delta(\mathfrak{A})_{!*}\mathcal{E}^{\mathfrak{A},\circ} = \bar\Delta(\mathfrak{A})_*\mathcal{E}^\mathfrak{A}$, proving that $\mathcal{F}_{\mathbf{v}}$ is nice. $\qquad\square$

In the nice case we have the following stronger version of Theorem 3.2.3. Recall the projection $c : \mathsf{N}^\diamond_{\mathbf{v}} = \mathrm{T}^*_C(C^{\mathbf{v}})^\diamond \to C$.

Proposition 3.6.4. *Let $\mathcal{F} = (\mathcal{F}_{\mathbf{v}})_{\mathbf{v}\in\mathbb{Z}^I_{\geq 0}}$ be a nice weak factorization sheaf on C. Then, writing $\widetilde{\mathcal{F}}_{\mathbf{v}} = \bigoplus_{\mathfrak{A}\in\widetilde{\mathfrak{P}}(\mathbf{v})} (\bar\Delta(\mathfrak{A}))_*\mathcal{E}^\mathfrak{A}$, one has:*

(i) *For each $\mathbf{v}$ we have $(\mathrm{F}_{\mathsf{N}}\circ \mathrm{sp}_{C^{\mathbf{v}}/C}(\widetilde{\mathcal{F}}_{\mathbf{v}}))\big|_{\mathsf{N}^\diamond_{\mathbf{v}}} \cong c^*\mathcal{E}^{\Delta(\mathbf{v})}[\mathrm{rk}\,\mathsf{N}_{\mathbf{v}}](\frac{\mathrm{rk}\,\mathsf{N}_{\mathbf{v}}}{2})$.*

(ii) *In $D^I_{\mathrm{abs}}(\mathrm{pt})$ (rather than in K^I_{abs}), one has a natural isomorphism*

$$\bigoplus_{\mathbf{v}} \mathrm{R}\Gamma_c(C^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}},\, \mathcal{F}_{\mathbf{v}}) = \mathrm{Sym}\Big(\bigoplus_{\mathbf{w}\neq 0} \mathrm{R}\Gamma_c(C_{\Delta(\mathbf{w})},\, \mathcal{E}^{\Delta(\mathbf{w})})^{\mathfrak{S}_{\mathbf{w}}}\Big).$$

31

Proof. The construction of Verdier specialization involves a deformation to the normal bundle, see [?]. From that deformation and the factorization isomorphism (3.6.3), it is easy to see that one has $\mathrm{sp}_{C_{\mathfrak{A}}/C_{\Delta}}(\mathcal{E}^{\mathfrak{A}}) = u_{\mathfrak{A}}^*(\mathcal{E}^{\mathfrak{A}}|_{C_{\Delta}})$, where we use the notation of diagram (3.4.2). It follows that $\mathrm{sp}_{C^{\mathbf{v}}/C_{\Delta}}(\widetilde{\mathcal{F}}^{\mathfrak{A}}) = (v_{\mathfrak{A}})_* u_{\mathfrak{A}}^*(\mathcal{E}^{\mathfrak{A}}|_{C_{\Delta}})$. By the properties of Fourier-Deligne transform, this implies that

$$\mathrm{F}_{\mathsf{N}} \circ \mathrm{sp}_{C^{\mathbf{v}}/C_{\Delta}}(\widetilde{\mathcal{F}}^{\mathfrak{A}}) = (v_{\mathfrak{A}}^{\perp})_* (u_{\mathfrak{A}}^{\perp})^*(\mathcal{E}^{\mathfrak{A}}|_{C_{\Delta}})[\mathrm{rk}\,\mathsf{N}_{\mathbf{v}}](\tfrac{1}{2}\,\mathrm{rk}\,\mathsf{N}_{\mathbf{v}}). \tag{3.6.5}$$

fdsp

The sheaf above is supported on $\mathrm{T}_C^{\perp}(C_{\mathfrak{A}}^{\mathbf{v}})$. We see that the restriction of this sheaf to the open set $\mathsf{N}_{\mathbf{v}}^{\diamond} = \mathrm{T}_C^*(C^{\mathbf{v}})^{\diamond}$ vanishes unless $\mathfrak{A} = \Delta(\mathbf{v})$ and $C_{\mathfrak{A}}^{\mathbf{v}} = C$ is the principal diagonal. In the latter case, we have $\mathrm{sp}_{C_{\Delta}/C_{\Delta}}(\mathcal{E}^{\Delta(\mathbf{v})}) = (u_{\Delta}^{\perp})^* \mathcal{E}^{\Delta(\mathbf{v})}$ and $\mathrm{T}_C^{\perp}(C_{\Delta(\mathbf{v})}^{\mathbf{v}})^{\diamond} = \mathsf{N}_{\mathbf{v}}^{\diamond}$. Thus, we obtain

$$(\mathrm{F}_{\mathsf{N}} \circ \mathrm{sp}_{C^{\mathbf{v}}/C} \mathcal{F}_{\mathbf{v}})|_{\mathsf{N}_{\mathbf{v}}^{\diamond}} = \bigoplus\nolimits_{\mathfrak{A} \in \widetilde{\mathfrak{P}}(\mathbf{v})} (\mathrm{F}_{\mathsf{N}} \circ \mathrm{sp}_{C^{\mathbf{v}}/C}(\widetilde{\mathcal{F}}^{\mathfrak{A}}))|_{\mathsf{N}_{\mathbf{v}}^{\diamond}}$$

$$= (u_{\Delta}^{\perp})^* \mathcal{E}^{\Delta(\mathbf{v})}[\mathrm{rk}\,\mathsf{N}_{\mathbf{v}}](\tfrac{\mathrm{rk}\,\mathsf{N}_{\mathbf{v}}}{2})|_{\mathsf{N}_{\mathbf{v}}^{\diamond}} = c^* \mathcal{E}^{\Delta(\mathbf{v})}[\mathrm{rk}\,\mathsf{N}_{\mathbf{v}}](\tfrac{\mathrm{rk}\,\mathsf{N}_{\mathbf{v}}}{2}).$$

This proves (i).

To prove (ii), let $\widetilde{\mathfrak{P}}(\mathbf{v}, \mathfrak{a})$ denote the fiber of the canonical projection $\widetilde{\mathfrak{P}}(\mathbf{v}) \to \mathfrak{P}(\mathbf{v})$ over a decomposition $\mathfrak{a} \in \mathfrak{P}(\mathbf{v})$. Further, we put $\widetilde{\mathcal{F}}^{\mathfrak{a}} := \bigoplus_{\mathfrak{B} \in \widetilde{\mathfrak{P}}(\mathbf{v}, \mathfrak{a})} \widetilde{\mathcal{F}}_{\mathbf{v}}^{\mathfrak{B}}$. This is a $\mathfrak{S}_{\mathbf{v}}$-invariant direct summand of $\widetilde{\mathcal{F}}_{\mathbf{v}}$. Therefore it descends to a direct summand $\mathcal{F}_{\mathbf{v}}^{\mathfrak{a}}$ of $\mathcal{F}_{\mathbf{v}}$.

Choose a decomposition $\mathfrak{A} \in \widetilde{\mathfrak{P}}(\mathbf{v}, \mathfrak{a})$. The group $\mathfrak{S}_{\mathbf{v}}$ acts transitively on $\widetilde{\mathfrak{P}}(\mathbf{v}, \mathfrak{a})$ and the stabilizer of the element $\mathfrak{A}$ equals $N(\mathfrak{S}_{\mathfrak{A}})$. Therefore, $\mathrm{R}\Gamma_c(C^{\mathbf{v}}, \widetilde{\mathcal{F}}^{\mathfrak{a}}) = \bigoplus_{\mathfrak{B} \in \widetilde{\mathfrak{P}}(\mathbf{v}, \mathfrak{a})} \mathrm{R}\Gamma_c(C^{\mathbf{v}}, \widetilde{\mathcal{F}}_{\mathbf{v}}^{\mathfrak{B}})$, viewed as a representation of $\mathfrak{S}_{\mathbf{v}}$, is induced from the $N(\mathfrak{S}_{\mathfrak{A}})$-representation $\mathrm{R}\Gamma(C^{\mathbf{v}}, \widetilde{\mathcal{F}}^{\mathfrak{A}})$. Thus, using Lemma 3.6.3 and the notation therein we get

$$\mathrm{R}\Gamma_c(C^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}, \mathcal{F}^{\mathfrak{a}}) = \mathrm{R}\Gamma_c(C^{\mathbf{v}}, \widetilde{\mathcal{F}}^{\mathfrak{a}})^{\mathfrak{S}_{\mathbf{w}}} = \mathrm{R}\Gamma_c(C^{\mathbf{v}}, \widetilde{\mathcal{F}}^{\mathfrak{A}})^{N(\mathfrak{S}_{\mathfrak{A}})}$$

$$= \mathrm{R}\Gamma_c(\textstyle\prod_\alpha C^{\mathbf{v}^\alpha}, \boxtimes_\alpha \mathcal{F}^{\Delta(\mathbf{v}^\alpha)})^{N(\mathfrak{S}_{\mathfrak{A}})} = \big(\textstyle\bigotimes_\alpha \mathrm{R}\Gamma_c(C_{\Delta(\mathbf{v}^\alpha)}, \mathcal{E}^{\Delta(\mathbf{v}^\alpha)})\big)^{N(\mathfrak{S}_{\mathfrak{A}})}.$$

Observe next that the set $\mathfrak{P}(\mathbf{v})$, of multipartitions, has a natural identification with the set of collections $\{d_{\mathbf{w}}\}_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^I}$ such that one has $\sum_{\mathbf{w}} d_{\mathbf{w}} \cdot \mathbf{w} = \mathbf{v}$. Let $\{d_{\mathbf{w}}\}$ be the collection that corresponds to a decomposition $\mathfrak{a} \in \mathfrak{P}(\mathbf{v})$. Then, by (3.4.1) we have $N(\mathfrak{S}_{\mathfrak{A}}) \cong \prod_{\mathbf{w}} \big((\mathfrak{S}_{\mathbf{w}})^{d_{\mathbf{w}}} \rtimes \mathfrak{S}_{d_{\mathbf{w}}}\big)$. Hence, we get

$$\bigg(\bigotimes_{\alpha \in [r]} \mathrm{R}\Gamma_c(C_{\Delta(\mathbf{v}^\alpha)}, \mathcal{E}^{\Delta(\mathbf{v}^\alpha)})\bigg)^{N(\mathfrak{S}_{\mathbf{v}^1, \dots, \mathbf{v}^r})} = \bigg(\bigotimes_{\mathbf{w}} \mathrm{R}\Gamma_c(C_{\Delta(\mathbf{w})}, \mathcal{E}^{\Delta(\mathbf{w})})^{\otimes d_{\mathbf{w}}}\bigg)^{\prod_{\mathbf{w}} \big((\mathfrak{S}_{\mathbf{w}})^{d_{\mathbf{w}}} \rtimes \mathfrak{S}_{d_{\mathbf{w}}}\big)}$$

$$= \bigotimes_{\mathbf{w}} \mathrm{Sym}^{d_{\mathbf{w}}}\big(\mathrm{R}\Gamma_c(C_{\Delta(\mathbf{w})}, \mathcal{E}^{\Delta(\mathbf{w})})^{\mathfrak{S}_{\mathbf{w}}}\big).$$

Thus, using the direct sum decomposition $\mathcal{F}_{\mathbf{v}} = \bigoplus_{\mathfrak{a} \in \mathfrak{P}(\mathbf{v})} \mathcal{F}_{\mathbf{v}}^{\mathfrak{a}}$, we compute

$$\bigoplus_{\mathbf{v}} \mathrm{R}\Gamma_c(C^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}, \mathcal{F}_{\mathbf{v}}) = \bigoplus_{\mathbf{v}} \bigoplus_{\mathfrak{a} \in \mathfrak{P}(\mathbf{v})} \mathrm{R}\Gamma_c(C^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}, \mathcal{F}_{\mathbf{v}}^{\mathfrak{a}})$$

$$= \bigoplus_{\mathbf{v}} \bigoplus_{\{(d_{\mathbf{w}})_{\mathbf{w} \in \mathbb{Z}_{\geq 0}^I \smallsetminus \{0\}} \,|\, \sum_{\mathbf{w}} d_{\mathbf{w}} \cdot \mathbf{w} = \mathbf{v}\}} \bigg(\bigotimes_{\mathbf{w}} \mathrm{Sym}^{d_{\mathbf{w}}}\big(\mathrm{R}\Gamma_c(C_{\Delta(\mathbf{w})}, \mathcal{E}^{\Delta(\mathbf{w})})^{\mathfrak{S}_{\mathbf{w}}}\big)\bigg)$$

$$= \bigoplus_{(d_{\mathbf{w}})_{\mathbf{w} \in \mathbb{Z}_{\geq 0}^I \smallsetminus \{0\}}} \bigg(\bigotimes_{\mathbf{w}} \mathrm{Sym}^{d_{\mathbf{w}}}\big(\mathrm{R}\Gamma_c(C_{\Delta(\mathbf{w})}, \mathcal{E}^{\Delta(\mathbf{w})})^{\mathfrak{S}_{\mathbf{w}}}\big)\bigg)$$

$$= \mathrm{Sym}\big(\bigoplus_{\mathbf{w} \neq 0} \mathrm{R}\Gamma_c(C_{\Delta(\mathbf{w})}, \mathcal{E}^{\Delta(\mathbf{w})})^{\mathfrak{S}_{\mathbf{w}}}\big).$$

32

Part (ii) follows. □

3.7. **Proofs of Theorem 3.2.3 and Theorem 3.3.1.** Let $D_{\mathfrak{P}}(C^{\mathbf{v}})$ be a full subcategory of $D_{\mathrm{abs}}(C^{\mathbf{v}})$ whose objects are constructible with respect to the diagonal stratification and let $K_{\mathrm{abs}}(C^{\mathbf{v}})$ be the corresponding Grothendieck group. The semisimplification, $\mathcal{F}^{\mathrm{ss}}$, of any object $\mathcal{F} \in D_{\mathfrak{P}}(C^{\mathbf{v}})$ is again an object of $D_{\mathfrak{P}}(C^{\mathbf{v}})$. The semisimplification operation extends to the $\mathfrak{S}_{\mathbf{v}}$-equivariant setting.

Lemma 3.7.1. *For any weak factorization sheaf* $\mathcal{F} = (\mathcal{F}_{\mathbf{v}})$ *the collection* $\mathcal{F}^{\mathrm{ss}} := (\mathcal{F}_{\mathbf{v}}^{\mathrm{ss}})_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^I}$ *has the natural structure of a weak factorization sheaf.*

Sketch of Proof. Since the operation of open restriction preserves the weight filtration, we can consider semisimplification more generally for sheaves on any open substack $U \subset C^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}$. Furthermore, it is clear that for any such U, and $\mathcal{E} \in D_{\mathfrak{P}}(C^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}})$, one has $(\mathcal{E}^{\mathrm{ss}})|_U = (\mathcal{E}|_U)^{\mathrm{ss}}$. In particular, for any decomposition $\mathbf{v} = \mathbf{v}^1 + \ldots + \mathbf{v}^r$ we have an isomorphism $(\jmath_{\mathbf{v}^1,\ldots,\mathbf{v}^r}^* \imath_{\mathbf{v}^1,\ldots,\mathbf{v}^r}^* \mathcal{F}_{\mathbf{v}})^{\mathrm{ss}} = \jmath_{\mathbf{v}^1,\ldots,\mathbf{v}^r}^* \imath_{\mathbf{v}^1,\ldots,\mathbf{v}^r}^* (\mathcal{F}^{\mathrm{ss}})$, of sheaves on $C_{\mathbf{v}^1,\ldots,\mathbf{v}^r}^{\mathrm{disj}}$. Also, it is clear that $(\mathcal{F}_{\mathbf{v}^1} \boxtimes \ldots \boxtimes \mathcal{F}_{\mathbf{v}^r})^{\mathrm{ss}} = \mathcal{F}_{\mathbf{v}^1}^{\mathrm{ss}} \boxtimes \ldots \boxtimes \mathcal{F}_{\mathbf{v}^r}^{\mathrm{ss}}$. Since the only operations with sheaves involved in the definition of factorization sheaf are the operations $\jmath_{\mathbf{v}^1,\ldots,\mathbf{v}^r}^* \imath_{\mathbf{v}^1,\ldots,\mathbf{v}^r}^*$ and $\boxtimes$, and the operation of taking semisimplification is functorial, it follows that for any collection $(\mathcal{F}_{\mathbf{v}})_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^I}$ that satisfies that definition the collection $(\mathcal{F}_{\mathbf{v}}^{\mathrm{ss}})_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^I}$ satisfies it as well. □

Proof of Theorem 3.2.3. Each of the functors sp_N and F_N is t-exact with respect to the perverse t-structure. Hence, for any weak factorization sheaf $\mathcal{F}$, we have $(\Phi_N(\mathcal{F}_{\mathbf{v}}))^{\mathrm{ss}} = \Phi_N(\mathcal{F}_{\mathbf{v}}^{\mathrm{ss}})$. Further, part (i) of Proposition 3.6.4 implies that $(\Phi_N(\widetilde{\mathcal{F}}_{\mathbf{v}}))^{\mathrm{ss}}$ is a pull-back of a semisimple local system on C via the projection $N_{\mathbf{v}}^{\diamond} \to C$. We deduce that the local system $\Phi_N(\widetilde{\mathcal{F}}_{\mathbf{v}})$ is unipotent along the map $c : N_{\mathbf{v}}^{\diamond} \to C$. Part (i) of Theorem 3.2.3 follows.

Observe next that in $K_{\mathrm{abs}}(C^{\mathbf{v}})$ we have $[\mathcal{E}] = [\mathcal{E}^{\mathrm{ss}}]$. The equation of part (ii) of Theorem 3.2.3 is an equation in K_{abs}^I. Such an equation is unaffected by replacing each of the sheaves $\mathcal{F}_{\mathbf{v}}$ by $\mathcal{F}_{\mathbf{v}}^{\mathrm{ss}}$. The sheaf $\mathcal{F}_{\mathbf{v}}^{\mathrm{ss}}$ is nice. For such a sheaf, the equation of part (ii) of Theorem 3.2.3 is an immediate consequence of the formula of Proposition 3.6.4(ii) and the isomorphism of part (i) of that proposition, using that $\mathrm{rk}\, N_{\mathbf{v}} = \dim C(|\mathbf{v}| - 1)$. □

Proof of Theorem 3.3.1. Part (i) of the theorem follows directly from Theorem 3.2.3(i). Also, it is clear from the equivariance of the sheaf $\mathcal{F}_{\mathbf{v}}$ under the diagonal translations that one has $[\Phi_N(\mathcal{F}_{\mathbf{v}})]^{\langle \rho \rangle} = [\mathrm{R}\Gamma_c(\mathbb{A}, \underline{\mathbf{C}}_{\mathbb{A}})] \otimes [\Phi_t(\mathcal{F}_{\mathbf{v}})_\eta^{\langle \rho \rangle}] = \mathbb{L} \cdot [\Phi_t(\mathcal{F}_{\mathbf{v}})_\eta^{\langle \rho \rangle}]$, resp. $[\Phi_N(\varepsilon^* \mathcal{F}_{\mathbf{v}})]^{\langle \rho \rangle} = \mathrm{R}\Gamma_c(\mathbb{G}, \underline{\mathbf{C}}_{\mathbb{A}}) \otimes [\Phi_t(\mathcal{F}_{\mathbf{v}})_\eta^{\langle \rho \rangle}] = (\mathbb{L} - 1) \cdot [\Phi_t(\mathcal{F}_{\mathbf{v}})_\eta^{\langle \rho \rangle}]$, for any $\rho \in \mathrm{Irr}(\mathfrak{S}_{\mathbf{v}})$. Thus, equations (3.3.2) and (3.3.3) follow from Theorem 3.2.3(ii).

We now prove (3.3.4). Since this is an equation in K_{abs}^I it suffices to prove it for semisimple weak factorization sheaves $\mathcal{F}$. Furthermore, without loss of generality one may assume in addition that each of the corresponding local systems $\mathcal{E}^{\mathfrak{A}}$ that appear in Definition 3.6.1 has *finite* rank, that is, the semisimple sheaves $\mathcal{F}_{\mathbf{v}}$ are Weil sheaves. In that case, the collection $\mathcal{F}^{\vee} = (\mathcal{F}_{\mathbf{v}}^{\vee})$ is a semisimple weak factorization sheaf again, cf. Remark 3.1.9(1). Thus, we are in a position to apply equation (3.3.2) in the case of the factorization sheaf $\mathcal{F}^{\vee}$. Note that $\overline{\mathcal{F}_{\mathbf{v}}^{\vee}} = (\bar{\mathcal{F}}_{\mathbf{v}})^{\vee}[2](1)$. Using that the Fourier-Deligne functor F_t commutes with the Verdier duality we deduce $[\Phi_t(\mathcal{F}_{\mathbf{v}}^{\vee})_\eta] = [\Phi_t(\mathcal{F}_{\mathbf{v}})_\eta^*]\{2\} = \mathbb{L}^{-1} \cdot [\Phi_t(\mathcal{F}_{\mathbf{v}})_\eta^*]$. Also, we one has $\mathrm{R}\Gamma_c(\mathbb{A}^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}, \mathcal{F}_{\mathbf{v}}^{\vee}) = \mathrm{R}\Gamma(\mathbb{A}^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}, \mathcal{F}_{\mathbf{v}})^*$. Thus, using (3.3.2) for $\mathcal{F}^{\vee}$, we compute

$$\sum_{\mathbf{v}} z^{\mathbf{v}} \cdot [\mathrm{R}\Gamma(\mathbb{A}^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}, \mathcal{F}_{\mathbf{v}})^*] = \sum_{\mathbf{v}} z^{\mathbf{v}} \cdot [\mathrm{R}\Gamma(\mathbb{A}^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}, \mathcal{F}_{\mathbf{v}}^{\vee})]$$

33

$$= \mathrm{Sym}\left(- \mathbb{L}^{\frac{1}{2}} \cdot \sum_{\mathbf{v}>0} (-1)^{\mathbf{v}} \cdot z^{\mathbf{v}} \cdot \mathbb{L}^{\frac{|\mathbf{v}|}{2}} \cdot [\Phi_{\mathfrak{t}}(\mathcal{F}_{\mathbf{v}}^{\vee})_{\eta}]^{\langle \mathrm{triv}\rangle} \right)$$

$$= \mathrm{Sym}\left(- \mathbb{L}^{\frac{1}{2}} \cdot \mathbb{L}^{-1} \cdot \sum_{\mathbf{v}>0} (-1)^{\mathbf{v}} \cdot z^{\mathbf{v}} \cdot \mathbb{L}^{\frac{|\mathbf{v}|}{2}} \cdot \left[(\Phi_{\mathfrak{t}}(\mathcal{F}_{\mathbf{v}})_{\eta}^{*} \right]^{\langle \mathrm{triv}\rangle} \right).$$

Now, the sheaf $\mathcal{F}_{\mathbf{v}}$ being $\mathbb{G}$-equivariant, the well known result of Springer yields a canonical isomorphism $\mathrm{R}\Gamma(\mathbb{A}^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}, \mathcal{F}_{\mathbf{v}}) \cong \mathrm{R}\Gamma(0/\mathfrak{S}_{\mathbf{v}}, i_{\mathbf{v}}^{*}\mathcal{F}_{\mathbf{v}})$. Applying $(-)^{*}$ to the displayed formula above yields the required equation (3.3.4). $\qquad\square$

4. Factorization sheaves on GL

GtoT

4.1. **From G to T.** We will use the notation pt_X for a constant map $X \to \mathrm{pt}$.

Let G be a split connected reductive group. For all Borel subgroups B of G, the tori $T = B/[B,B]$ are canonically isomorphic to each other and we write W for the corresponding Weyl group. Let $\mathfrak{g}$, $\mathfrak{b}$, and $\mathfrak{t}$, be the Lie algebras of G, B, and T, respectively. The imbedding $\mathfrak{b} \hookrightarrow \mathfrak{g}$, resp. the projection $\mathfrak{b} \twoheadrightarrow \mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$, gives a morphism of quotient stacks $\kappa_{\mathfrak{b}} : \mathfrak{b}/B \to \mathfrak{g}/G$, resp. $\nu_{\mathfrak{b}} : \mathfrak{b}/B \to \mathfrak{t}/T$. Observe that the action of T on $\mathfrak{t}$ being trivial, there is a canonical isomorphism $\mathfrak{t}/T \cong \mathfrak{t} \times \mathrm{pt}/T$. This isomorphism respects the natural actions of the Weyl group W on each side, where W acts on $\mathfrak{t} \times \mathrm{pt}/T$ diagonally. By the Chevalley isomorphism $\mathfrak{g}/\!/G \cong \mathfrak{t}/\!/W$, we have a commutative diagram:

$$\begin{array}{ccccccccc}
0/G & \overset{i_{\mathfrak{g}}}{\hookrightarrow} & \mathfrak{g}/G & \overset{\kappa_{\mathfrak{b}}}{\hookleftarrow} & \mathfrak{b}/B & \overset{\nu_{\mathfrak{b}}}{\twoheadrightarrow} & \mathfrak{t}/T = \mathfrak{t}\times\mathbb{B}T & \overset{i_{\mathfrak{t}}}{\hookleftarrow} & \{0\}\times\mathbb{B}T = 0/T \\
\downarrow{\scriptstyle \mathrm{pt}_{0/G}} & {\scriptstyle \mathfrak{w}_{\mathfrak{g}}} & \downarrow & & & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle \mathrm{pt}_{0/W}} \\
\mathrm{pt} & & \mathfrak{g}/\!/G & = & \mathfrak{t}/\!/W & \overset{\mathfrak{w}_{\mathfrak{t}}}{\longleftarrow} & \mathfrak{t}/W & \overset{q}{\longleftarrow} & \mathfrak{t} & & \mathrm{pt}
\end{array} \qquad (4.1.1) \quad \boxed{\text{ww}}$$

We claim that for any sheaf $\mathcal{F}$ on $\mathfrak{g}/G$, the sheaf $r_{!}(\nu_{\mathfrak{b}})_{!}\kappa_{\mathfrak{b}}^{*}\mathcal{F}$ on $\mathfrak{t}$ has a canonical W-equivariant structure. This seems to be well known (its version for parabolic restriction of character sheaves is essentially due to Lusztig [L2]), but we have been unable to find the statement in the generality that we need in the literature.

To state our result, put $n_G = \dim G/B = \dim \mathfrak{b} - \dim \mathfrak{t}$, resp. $\mathrm{rk} = \dim T$. Write $\mathrm{sign} \otimes (-)$ for a twist by the sign character of the W-action on a W-representation or, more generally, a twist of the equivariant structure on a W-equivariant sheaf. In particular, one has a functor $\mathrm{sign} \otimes (-)$ on $D_{\mathrm{abs}}(\mathfrak{t}/W)$ induced by tensoring with a rank 1 local system *Sign*.

In the proofs below, we will freely use the following known results about *compactly supported cohomology* of G/T, $\mathbb{B}T$, and $\mathbb{B}G$. The top degree nonzero compactly supported cohomology groups are $H_c^{4n_G}(G/T)$, resp. $H_c^{-2\,\mathrm{rk}}(\mathbb{B}T)$ and $H_c^{-2\dim G}(\mathbb{B}G) = H_c^{-4n_G - 2\,\mathrm{rk}}(\mathbb{B}G)$. These groups are 1-dimensional, furthermore, the Weyl group acts on $H_c^{4n_G}(G/T)$ and $H_c^{-2\,\mathrm{rk}}(\mathbb{B}T)$ via the sign character, where we have used the isomorphisms $H^{\bullet}(G/B) = H_c^{\bullet}(G/B) = H_c^{\bullet+2n_G}(G/T)$. Also, each of the groups $H_c^{\mathrm{odd}}(G/B)$, $H_c^{\mathrm{odd}}(\mathbb{B}T)$, and $HH_c^{\mathrm{odd}}(\mathbb{B}G)$, vanishes. Therefore, the spectral sequence for the fibration $\mathbb{B}T \overset{G/T}{\longrightarrow} \mathbb{B}G$ collapses, yielding a W-equivariant isomorphism $H_c^{\mathrm{ev}}(\mathbb{B}T) = H_c^{\mathrm{ev}}(G/T) \otimes H_c^{\mathrm{ev}}(\mathbb{B}G)$. In particular, one has W-equivariant isomorphisms

$$H_c^{\bullet}(\mathbb{B}T)^{W} = H_c^{\bullet + 2n_G}(\mathbb{B}G) \quad \text{resp.} \quad H_c^{\bullet}(\mathbb{B}T)^{\mathrm{sign}} = \mathrm{sign} \otimes H_c^{\bullet + 4n_G}(\mathbb{B}G).$$

The functors constructed in the following lemma are called parabolic restriction functors.

xi **Lemma 4.1.2.** (i) *There exists a functor* $\mathfrak{R}_{\mathfrak{t}} \colon D_{\mathrm{abs}}(\mathfrak{g}/G) \to D_{\mathrm{abs}}(\mathfrak{t}/W)$ *such that one has an isomorphism of functors* $r_{!}(\nu_{\mathfrak{b}})_{!}\,\kappa_{\mathfrak{b}}^{*} \cong q^{*}\,\mathfrak{R}_{\mathfrak{t}}$, *resp.*

$$r_{!}(\nu_{\mathfrak{b}})_{!}\,\kappa_{\mathfrak{b}}^{*} \cong q^{*}\,\mathfrak{R}_{\mathfrak{t}}, \quad (\mathfrak{w}_{\mathfrak{g}})_{!} \cong ((\mathfrak{w}_{\mathfrak{t}})_{!}\,\mathfrak{R}_{\mathfrak{t}})^{W}, \quad (\mathrm{pt}_{0/G})_{!}\,i_{\mathfrak{g}}^{*} \cong ((\mathrm{pt}_{0/W})_{!}\,(i_{\mathfrak{t}}^{*}\,\mathfrak{R}_{\mathfrak{t}} \otimes \mathrm{sign}))\{-2n_G\}.$$

$$\qquad\qquad\qquad (4.1.3) \quad \boxed{\text{xieq}}$$

(ii) *There is also a group analogue, a functor $\mathfrak{R}_T : D_{\mathrm{abs}}(G/_{\mathrm{Ad}}G) \to D_{\mathrm{abs}}(T/W)$ with similar properties.*

(iii) *For any smooth affine curve C there is a functor $\mathfrak{R}_C : D_{\mathrm{abs}}(\mathrm{Coh}_{\mathbf{v}}\, C) \to D_{\mathrm{abs}}(C^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}})$ such that analogues of isomorphisms (4.1.3) hold. Given an open imbedding $\varepsilon : C_1 \hookrightarrow C$, one has an isomorphism of functors*

$$\varepsilon_{\mathrm{Sym}}^* \circ \mathfrak{R}_C \cong \mathfrak{R}_{C_1} \circ \varepsilon_{\mathrm{Coh}}^*, \tag{4.1.4} \quad \boxed{\texttt{epep}}$$

where $\varepsilon_{\mathrm{Coh}}^* : D_{\mathrm{abs}}(\mathrm{Coh}_{\mathbf{v}}\, C) \to D_{\mathrm{abs}}(\mathrm{Coh}_{\mathbf{v}}\, C_1)$, resp. $\varepsilon_{\mathrm{Sym}}^* : D_{\mathrm{abs}}(C^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}) \to D_{\mathrm{abs}}(C_1^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}})$, is the natural restriction.

(iv) *The functor $\mathfrak{R}_C$ reduces to the functor $\mathfrak{R}_{\mathfrak{t}}$, resp. $\mathfrak{R}_T$, in the case $G = \mathrm{GL}_{\mathbf{v}}$ and $C = \mathbb{A}$, resp. $C = \mathbb{G}$.*

Let $\widetilde{\mathfrak{g}}$ be the variety of pairs $(\mathfrak{b}, g)$ such that $\mathfrak{b}$ is a Borel subalgebra and g is an element of $\mathfrak{b}$. It is convenient to reformulate the lemma in terms of the Grothendieck-Springer resolution $\widetilde{\mathfrak{g}} \to \mathfrak{g},\ (\mathfrak{b}, g) \mapsto g$. This map is G-equivariant, hence descends to a morphism $\widetilde{\kappa} : \widetilde{\mathfrak{g}}/G \to \mathfrak{g}/G$. There is a natural isomorphism $\widetilde{\mathfrak{g}}/G \cong \mathfrak{b}/B$, of quotient stacks. Therefore, one also has a morphism $\widetilde{\nu} : \widetilde{\mathfrak{g}}/G = \mathfrak{b}/B \to \mathfrak{t}/T = \mathfrak{t} \times \mathrm{pt}/T,\ (\mathfrak{b}, g) \mapsto \nu_{\mathfrak{b}}(g)$. Thus we have the following commutative diagram:

$$\boxed{(4.1.5) \quad \texttt{GGTT}}$$

Proof of Lemma 4.1.2. We have a chain of canonical isomorphisms of functors:

$$r_!\,(\nu_{\mathfrak{b}})_!\,\kappa_{\mathfrak{b}}^* = r_!\,\widetilde{\nu}_!\,\widetilde{\kappa}^* = (\mathrm{pr}_2)_!\,\pi_*\,\widetilde{\kappa}^* = (\mathrm{pr}_2)_!\,\pi_*\,\pi^*\,\kappa^* = (\mathrm{pr}_2)_!\,\big(\nu_!\,\kappa^* \otimes \pi_!\underline{\mathbf{C}}_{\widetilde{\mathfrak{g}}/G}\big) \tag{4.1.6} \quad \boxed{\texttt{rnu}}$$

where the last isomorphism follows from the projection formula and we have used that $\pi_! = \pi_*$.

The sheaf $\pi_!\underline{\mathbf{C}}_{\widetilde{\mathfrak{g}}/G}$ can be equipped with a W-equivariant structure as follows. Let $\mathfrak{g}^{rs}$ be the set of regular semisimple elements of $\mathfrak{g}$ and $U := (\mathfrak{g}^{rs}/G) \times_{\mathfrak{t}/\!/W} \mathfrak{t}$. The morphism $\bar{\pi} : \widetilde{\mathfrak{g}} \to \mathfrak{g} \times_{\mathfrak{t}/\!/W} \mathfrak{t}$, a non-stacky counterpart of π, is a proper morphism which is an isomorphism over U, moreover, this morphism is known to be small [L1, Section 3]. It follows that $\bar{\pi}_!\underline{\mathbf{C}}_{\widetilde{\mathfrak{g}}} = \mathrm{IC}(U)[-\dim U]$, where $\mathrm{IC}(U)$ is the IC-sheaf corresponding to the constant sheaf $\underline{\mathbf{C}}_U$. The natural $G \times W$-equivariant structure on $\underline{\mathbf{C}}_U$ induces one on $\mathrm{IC}(U)$ and, hence, on $\bar{\pi}_!\underline{\mathbf{C}}_{\widetilde{\mathfrak{g}}} = \mathrm{IC}(U)[-\dim U]$. It is clear that the sheaf $\pi_!\underline{\mathbf{C}}_{\widetilde{\mathfrak{g}}/G}$ on $(\mathfrak{g}/G) \times_{\mathfrak{t}/\!/W} \mathfrak{t}$ is, up to a shift, obtained from $\bar{\pi}_!\underline{\mathbf{C}}_{\widetilde{\mathfrak{g}}}$ by G-equivariant descent. Since the actions of G and W on $\mathfrak{g} \times_{\mathfrak{t}/\!/W} \mathfrak{t}$ commute, the W-equivariant structure on $\bar{\pi}_!\underline{\mathbf{C}}_{\widetilde{\mathfrak{g}}}$ induces a W-equivariant structure on $\pi_!\underline{\mathbf{C}}_{\widetilde{\mathfrak{g}}/G}$. Thus, there is a canonically defined sheaf IC on $(\mathfrak{g}/G) \times_{\mathfrak{t}/\!/W} (\mathfrak{t}/W)$ such that $\pi_!\underline{\mathbf{C}}_{\widetilde{\mathfrak{g}}/G} = (\pi')^*\,\mathrm{IC}$.

We now define the required funtor $\mathfrak{R}_{\mathfrak{t}}$ as follows

$$\mathfrak{R}_{\mathfrak{t}}(\mathcal{F}) := (\mathrm{pr}_2')_!((\kappa')^*\mathcal{F} \otimes \mathrm{IC}).$$

Using base change with respect to the cartesian square at the bottom right corner of diagram (4.1.5) we deduce

$$\begin{aligned}
q^*\,\mathfrak{R}_{\mathfrak{t}} = q^*(\mathrm{pr}_2')_!\big((\kappa')^* \otimes \mathrm{IC}\big) &= (\mathrm{pr}_2)_!(\pi')^*\big((\kappa')^* \otimes \mathrm{IC}(U/W)\big) \\
&= (\mathrm{pr}_2)_!\big((\pi')^*(\kappa')^* \otimes (\pi')^*\,\mathrm{IC}\big) \\
&= (\mathrm{pr}_2)_!\big(\kappa^* \otimes \mathrm{IC}(U)\big) = (\mathrm{pr}_2)_!\big(\kappa^* \otimes \pi_!\underline{\mathbf{C}}_{\widetilde{\mathfrak{g}}/G}\big) \overset{(4.1.6)}{=\!=} r_!\,(\nu_{\mathfrak{b}})_!\,\kappa_{\mathfrak{b}}^*,
\end{aligned}$$

proving the first isomorphism in (4.1.3).

To prove the second isomorphism in ($\overset{\text{xieg}}{4.1.3}$), we observe that the natural W-action on $\kappa_!\pi_!\underline{\mathbf{C}}_{\widetilde{\mathfrak{g}}/G}$ gives, for any sheaf $\mathcal{F} \in D_{\mathrm{abs}}(\mathfrak{g}/G)$, a W-action on

$$\widetilde{\kappa}_!\widetilde{\kappa}^*\mathcal{F} = \mathcal{F} \otimes \widetilde{\kappa}_!\widetilde{\kappa}^*\underline{\mathbf{C}}_{\widetilde{\mathfrak{g}}/G} = \mathcal{F} \otimes \kappa_!\pi_!\pi^*\underline{\mathbf{C}}_{\mathfrak{g}/G}.$$

Furthermore, the well known isomorphism $(\kappa_!\pi_!\underline{\mathbf{C}}_{\widetilde{\mathfrak{g}}/G})^W = \underline{\mathbf{C}}_{\widetilde{\mathfrak{g}}/G}$ yields

$$(\widetilde{\kappa}_!\widetilde{\kappa}^*\mathcal{F})^W = \mathcal{F} \otimes (\kappa_!\pi_!\underline{\mathbf{C}}_{\widetilde{\mathfrak{g}}/G})^W = \mathcal{F} \otimes \underline{\mathbf{C}}_{\mathfrak{g}/G} = \mathcal{F}.$$

We deduce that $(\mathfrak{w}_{\mathfrak{g}})_!\mathcal{F} = (\mathfrak{w}_{\mathfrak{g}})_!(\widetilde{\kappa}_!\,\widetilde{\kappa}^*\mathcal{F})^W = ((\mathfrak{w}_{\mathfrak{g}}\circ\widetilde{\kappa})_!\,\widetilde{\kappa}^*\mathcal{F})^W$. On the other hand, diagram ($\overset{\text{GGTT}}{4.1.5}$) shows that $\mathfrak{w}_{\mathfrak{g}}\circ\widetilde{\kappa} = \mathfrak{w}_{\mathfrak{t}}\circ q\circ r\circ\widetilde{\nu}$. Therefore, we have an isomorphism of functors

$$(\mathfrak{w}_{\mathfrak{g}}\circ\widetilde{\kappa})_!\,\widetilde{\kappa}^* = (\mathfrak{w}_{\mathfrak{t}})_!\,q_!\,r_!\,\widetilde{\nu}_!\widetilde{\kappa}^* = (\mathfrak{w}_{\mathfrak{t}})_!\,q_!\,q^*\,\mathfrak{R}_{\mathfrak{t}},$$

by the first isomorphism in ($\overset{\text{xieg}}{4.1.3}$). The proof of the second isomorphism in ($\overset{\text{xieg}}{4.1.3}$) is now completed by the following chain of isomorphisms of functors

$$(\mathfrak{w}_{\mathfrak{g}})_! = ((\mathfrak{w}_{\mathfrak{g}}\circ\widetilde{\kappa})_!\,\widetilde{\kappa}^*(-))^W = ((\mathfrak{w}_{\mathfrak{t}})_!\,q_!\,q^*\,\mathfrak{R}_{\mathfrak{t}}(-))^W$$
$$= (\mathfrak{w}_{\mathfrak{t}})_!\,(\mathfrak{R}_{\mathfrak{t}}(-) \otimes (q_!q^*\underline{\mathbf{C}}_{\mathfrak{t}/W})^W) = ((\mathfrak{w}_{\mathfrak{t}})_!\,(\mathfrak{R}_{\mathfrak{t}}(-) \otimes \underline{\mathbf{C}}_{\mathfrak{t}/W}) = (\mathfrak{w}_{\mathfrak{t}})_!\,\mathfrak{R}_{\mathfrak{t}}\,.$$

Next, let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent variety. Let $i_0 : 0/G \hookrightarrow \mathcal{N}/G$ and $i_{\mathcal{N}} : \mathcal{N}/G \hookrightarrow \mathfrak{g}/G$ be the natural imbeddings, and $\mathrm{pt}_{\mathcal{N}} : \mathcal{N}/G \to \mathrm{pt}$. According to the theory of Springer representations, one has

$$i_{\mathcal{N}}^*\kappa_!'\big(\mathrm{IC}\otimes(pr_2')^*Sign\big) \;=\; (i_0)_*\underline{\mathbf{C}}_{\{0/G\}}[-2n](-n),$$

where $n = n_G$ and the shift is due to the fact that $\dim\mathcal{N} = 2n$. We deduce

$$i_{\mathcal{N}}^*\kappa_!'\big((\kappa')^*\mathcal{F} \otimes \mathrm{IC}\otimes(pr_2')^*Sign\big) \;=\; (i_0)_*i_{\mathfrak{g}}^*\mathcal{F}[-2n](-n).$$

Note that $(\kappa')^{-1}(\mathcal{N}/G) = \mathcal{N}/G \times 0/W = (pr_2')^{-1}(0/W)$. Let $\iota : \mathcal{N}/G \times 0/W \hookrightarrow (\mathfrak{g}/G) \times_{\mathfrak{t}/\!/W} (\mathfrak{t}/W)$ be the imbedding. Since $(\mathrm{pt}_{\mathfrak{g}/G})_!i_{\mathfrak{g}}^* = (\mathrm{pt}_{\mathcal{N}})_!\,(i_0)_!\,i_{\mathfrak{g}}^*$, using the isomorphism above and base change, we find

$$(\mathrm{pt}_{\mathfrak{g}/G})_!i_{\mathfrak{g}}^*\mathcal{F} = (\mathrm{pt}_{\mathcal{N}})_!\,(i_0)_*\,i_{\mathfrak{g}}^*\mathcal{F} = (\mathrm{pt}_{\mathcal{N}})_!\,i_{\mathcal{N}}^*\kappa_!'\big((\kappa')^*\mathcal{F} \otimes \mathrm{IC}\otimes(pr_2')^*Sign\big)[2n](n)$$
$$= (\mathrm{pt}_{\mathcal{N}} \boxtimes \mathrm{pt}_{\mathfrak{t}/W})_!\iota^*\big[(\kappa')^*\mathcal{F} \otimes \mathrm{IC}\otimes(pr_2')^*Sign\big][2n](n)$$
$$= (\mathrm{pt}_{\mathfrak{t}/W})_!\,i_{\mathfrak{t}}^*\,(pr_2')_!\big((\kappa')^*\mathcal{F} \otimes \mathrm{IC}\otimes(pr_2')^*Sign\big)[2n](n)$$
$$= (\mathrm{pt}_{\mathfrak{t}/W})_!\,i_{\mathfrak{t}}^*\,(\mathfrak{R}_{\mathfrak{t}}\,\mathcal{F} \otimes Sign)[2n](n).$$

The third isomorphism in ($\overset{\text{xieg}}{4.1.3}$) follows. The proof of part (ii) is left for the reader.

To prove (iii) one replaces $\widetilde{\mathfrak{g}}/G$ by the stack $\widetilde{\mathrm{Coh}}_{\mathbf{v}}C$ be the stack whose objects are complete flags $0 = M_0 \subset M_1 \subset \ldots \subset M_d = M$, where M_j is a length j coherent sheaf on C. One has a natural diagram $\mathrm{Coh}_{\mathbf{v}}\,C \overset{\widetilde{\kappa}}{\leftarrow} \widetilde{\mathrm{Coh}}_{\mathbf{v}}C \overset{\widetilde{\nu}}{\to} C^{\mathbf{v}}$ that reduces to the diagram $\mathfrak{g}/G \overset{\widetilde{\kappa}}{\leftarrow} \widetilde{\mathfrak{g}}/G \overset{\widetilde{\nu}}{\to} \mathfrak{t}$ in the case where $C = \mathbb{A}$ and $\mathfrak{g} = \mathfrak{gl}_{\mathbf{v}}$. The diagram for a general curve C is isomorphic, étale locally, to the diagram for $C = \mathbb{A}$. In particular, the map $\widetilde{\mathrm{Coh}}_{\mathbf{v}}C \to \mathrm{Coh}_{\mathbf{v}}\,C \times_{C^{\mathbf{v}}/\!/\mathfrak{S}_{\mathbf{v}}} C^{\mathbf{v}}$ is small. The isomorphism of part (iii) is an immediate consequence of base change in the following diagram with cartesian squares:

$$
\begin{array}{ccccccc}
\mathrm{Coh}_{\mathbf{v}}\,C_1 & \longleftarrow & \widetilde{\mathrm{Coh}}_{\mathbf{v}}C_1 & \longrightarrow & C_1^{\mathbf{v}} & \longrightarrow & C_1^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{Coh}_{\mathbf{v}}\,C & \longleftarrow & \widetilde{\mathrm{Coh}}_{\mathbf{v}}C & \longrightarrow & C^{\mathbf{v}} & \longrightarrow & C^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}
\end{array}
$$

Generalization to the the I-graded case is straightforward, as well as the proof of (iv), is straightforward. $\qquad\square$

Remark 4.1.7. One can also define an induction functor $\mathrm{ind} : D_{\mathrm{abs}}(\mathfrak{t}/W) \to D_{\mathrm{abs}}(\mathfrak{g}/G)$ by the formula $\mathrm{ind}(\mathcal{F}) = (\kappa'_!((pr'_2)^*\mathcal{F} \otimes \mathrm{IC}))^W$. ◇

Next, we dualize the maps in the top row of diagram (4.1.1). Thus, writing $(-)^\top$ for the transposed map, we get a diagram

Let $\mathrm{F}_{\mathfrak{g}/G} : D_{\mathrm{abs}}(\mathfrak{g}/G) \to D_{\mathrm{abs}}(\mathfrak{g}^*/G)$ be the Fourier-Deligne transform, that is, a relative Fourier trasform on the vector bundle $\mathfrak{g}/G \to \mathrm{pt}/G$. With obvious modifications of the notation of Lemma 4.1.2 we have the following result.

Lemma 4.1.8. *There exists a functor* $\mathfrak{R}_{\mathfrak{t}^*} : D(\mathfrak{g}^*/G) \to D(\mathfrak{t}^*/W)$, *such that analogues of the three isomorphisms in* (4.1.3) *hold and, in addition, there is an isomorphism of functors*

$$\mathrm{F}_{\mathfrak{t}/W} \circ \mathfrak{R}_{\mathfrak{t}} \cong \mathrm{sign} \otimes (\mathfrak{R}_{\mathfrak{t}^*} \circ \mathrm{F}_{\mathfrak{g}/G}). \tag{4.1.9}$$

Proof. The construction of the functor $\mathfrak{R}_{\mathfrak{t}^*}$ mimics the proof of Lemma 4.1.2. Further, let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ be the nilradical of $\mathfrak{b}$. The proof of (4.1.9) is well known, it is based on the fact that the Killing form provides an identification of the following natural diagram

$$\begin{array}{ccccc}
\mathfrak{g}^*/G & \longleftarrow \mathfrak{n}^\perp/B \longrightarrow & \mathfrak{t}^*/T \mathrel{=\!\!=} \mathfrak{t}^* \times \mathbb{B}T \\
\mathfrak{w}_{\mathfrak{g}^*} \downarrow & & \downarrow r_{\mathfrak{t}^*} \\
\mathfrak{g}^* /\!/ G \mathrel{=\!\!=} \mathfrak{t}^* /\!/ W & \overset{\mathfrak{w}_{\mathfrak{t}^*}}{\longleftarrow} \mathfrak{t}^*/W & \overset{q_{\mathfrak{t}^*}}{\longleftarrow} \mathfrak{t}^*
\end{array}$$

with the rectangle in the middle of diagram (4.1.1). We refer to Brylinski's paper [Br] for details. □

Remark 4.1.10. It is instructive to illustrate isomorphism (4.1.9) by the following computation. Let $i_{0/G} : 0/G \hookrightarrow \mathfrak{g}^*/G$, resp. $i_{0/W} : 0/W \hookrightarrow \mathfrak{t}^*/W$, denote the imbedding and write U for the unipotent radical of a Borel subgroup of G.

The Fourier-Deligne transform functor $\mathrm{F}_{\mathfrak{g}} : D_{\mathrm{abs}}(\mathfrak{g}/G) \to D_{\mathrm{abs}}(\mathfrak{g}^*/G)$ respects perverse t-structures. Therefore, one has $\mathrm{F}_{\mathfrak{g}}(\underline{\mathbf{C}}_{\mathfrak{g}/G}[\dim \mathfrak{g}/G]) = (i_{0/G})_!\underline{\mathbf{C}}_{0/G}[\dim 0/G]$. Note that $\dim(\mathrm{pt}/G) = -\dim G$, resp. $\dim \mathfrak{g}/G = 0$ and $2\dim U = \dim G - \mathrm{rk}\,G$. Thus, disregarding W-equivariant structures and Tate twists, we compute

$$\begin{aligned}
\mathfrak{R}_{\mathfrak{t}^*} \mathrm{F}_{\mathfrak{g}}\underline{\mathbf{C}}_{\mathfrak{g}/G} &= \mathfrak{R}_{\mathfrak{t}^*}(i_{0/G})_!\underline{\mathbf{C}}_{0/G}[\dim 0/G] \\
&= H_c^\bullet(\mathrm{pt}/T) \otimes H_c^\bullet(0/U) \otimes (i_{0/W})_!\underline{\mathbf{C}}_{0/W}[-\dim G] \\
&= H_c^\bullet(\mathrm{pt}/T) \otimes (i_{0/W})_!\underline{\mathbf{C}}_{0/W}[2\dim U][-\dim G] = H_c^\bullet(\mathrm{pt}/T) \otimes (i_{0/W})_!\underline{\mathbf{C}}_{0/W}[-\mathrm{rk}\,G].
\end{aligned}$$

Similarly, using that $\mathrm{F}_{\mathfrak{t}}(\underline{\mathbf{C}}_{\mathfrak{t}/W}[\mathrm{rk}\,G]) = (i_{0/W})_!\underline{\mathbf{C}}_{0/W}$, we find

$$\mathrm{F}_{\mathfrak{t}}\,\mathfrak{R}_{\mathfrak{t}}\,\underline{\mathbf{C}}_{\mathfrak{g}/G} = \mathrm{F}_{\mathfrak{t}}\big(H_c^\bullet(\mathrm{pt}/T) \otimes \underline{\mathbf{C}}_{\mathfrak{t}/W}\big) = H_c^\bullet(\mathrm{pt}/T) \otimes \mathrm{F}_{\mathfrak{t}}\underline{\mathbf{C}}_{\mathfrak{t}/W} = H_c^\bullet(\mathrm{pt}/T) \otimes (i_{0/W})_!\underline{\mathbf{C}}_{0/W}[-\mathrm{rk}\,G].$$

Later on, we will also need the following result. Let $\chi : G \to \mathbb{G}$ be a morphism of algebraic groups and $\wp$ a local system on $\mathbb{G}$. The local system $\chi^*\wp$, on G, is clearly $\mathrm{Ad}\,G$-equivariant, hence it descends to a local system $\wp^G$ on $G/_{\mathrm{Ad}}G$. The morphism χ gives, by restriction, a W-invariant morphism $\chi_T : T \to \mathbb{G}$. The corresponding local system $\chi_T^*\wp$ is therefore W-equivariant, hence it descends to a local system $\wp^T$ on T/W.

Corollary 4.1.11. *With the above notation, for any* $\mathcal{F} \in D_{\mathrm{abs}}(G/_{\mathrm{Ad}}G)$, *there is a canonical isomorphism*

$$\mathrm{R}\Gamma_c(G_{\mathbf{v}}/_{\mathrm{Ad}}G_{\mathbf{v}}, \wp^G \otimes \varepsilon_G^*\mathcal{F}) \cong \mathrm{R}\Gamma_c(T_{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}, \wp^T \otimes \varepsilon_T^*\mathfrak{R}_{\mathfrak{t}}\mathcal{F}). \tag{4.1.12}$$

Proof. Let $\chi_B : B \to \mathbb{G}$ be the restriction of χ to B. As above, the local system $\chi_B^* \wp$ descends to a local system $\wp^B$ on $B/_{\mathrm{Ad}} B$. Clearly, we have $\mu_B^* \wp^G \cong \wp^B \cong \nu_B^* \wp^T$. Using the composite isomorphism $\mu_B^* \wp^G \cong \nu_B^* \wp^T$, for any $\mathcal{E} \in D_{\mathrm{abs}}(G/_{\mathrm{Ad}} G)$, we compute

$$(\nu_B)_! \mu_B^*(\wp^G \otimes \mathcal{E}) = (\nu_B)_!(\mu_B^* \wp^G \otimes \mu_B^* \mathcal{E}) = (\nu_B)_!(\nu_B^* \wp^T \otimes \mu_B^* \mathcal{E}) = \wp^T \otimes (\nu_B)_! \mu_B^* \mathcal{E},$$

were the last isomorphism is the projection formula. We deduce that $\mathfrak{R}_T(\wp^G \otimes \mathcal{E}) \cong \wp^T \otimes \mathfrak{R}_T(\mathcal{E})$ canonically.

Now, given $\mathcal{F} \in D_{\mathrm{abs}}(\mathfrak{g}/G)$, we apply the above isomorphism for $\mathcal{E} := \varepsilon_G^* \mathcal{F}$. We compute

$$\mathrm{R}\Gamma_c(G/_{\mathrm{Ad}} G, \ \wp^G \otimes \varepsilon_G^* \mathcal{F}) = \mathrm{R}\Gamma_c(T/W, \ \mathfrak{R}_T(\wp^G \otimes \varepsilon_G^* \mathcal{F})) \qquad (\text{Lemma } 4.1.2)$$
$$= \mathrm{R}\Gamma_c(T/W, \ \wp^T \otimes (\mathfrak{R}_T \varepsilon_G^* \mathcal{F}))$$
$$= \mathrm{R}\Gamma_c(T/W, \ \wp^T \otimes (\varepsilon_T^* \mathfrak{R}_{\mathfrak{t}} \mathcal{F})). \qquad (\text{Lemma } 4.1.8) \qquad \square$$

Remark 4.1.13. Below, we will use analogues of the above results in the case of $\mathrm{GL}_{\mathbf{v}}$-equivariant (rather than $\mathrm{PGL}_{\mathbf{v}}$-equivariant) sheaves on $\mathfrak{pgl}_{\mathbf{v}}$, resp. $\mathrm{PGL}_{\mathbf{v}}$. In such a case, the stack $\widetilde{\mathfrak{g}}/G$ must be replaced by $(\mathrm{GL}_{\mathbf{v}} \times_{B_{\mathbf{v}}} \mathfrak{pb}_{\mathbf{v}})/\mathrm{GL}_{\mathbf{v}} \cong \mathfrak{pb}_{\mathbf{v}}/B_{\mathbf{v}}$, where $B_{\mathbf{v}} \subset \mathrm{GL}_{\mathbf{v}}$ is the Borel subgroup of $\mathrm{GL}_{\mathbf{v}}$ and $\mathfrak{pb}_{\mathbf{v}} \subset \mathfrak{pgl}_{\mathbf{v}}$ is the image of $\mathrm{Lie}\, B_{\mathbf{v}}$ in $\mathfrak{pgl}_{\mathbf{v}}$, a Borel subalgebra of $\mathfrak{pgl}_{\mathbf{v}}$. Similarly, the stack $\mathfrak{t}_{\mathbf{v}}/\mathbb{G}^{\mathbf{v}}$, where $\mathfrak{t}_{\mathbf{v}} = \mathbb{A}^{\mathbf{v}}/\mathbb{A}$ is the Cartan subalgebra of $\mathfrak{pgl}_{\mathbf{v}}$, plays the role of $\mathfrak{t}/T$.

4.2. **Factorization sheaves on** GL **and** $\mathfrak{gl}$. In this subsection we write $G_{\mathbf{v}}$ for $\mathrm{GL}_{\mathbf{v}}$, resp. $\mathfrak{g}_{\mathbf{v}}$ for $\mathfrak{gl}_{\mathbf{v}}$.

We will consider factorization sheaves on the collection of Lie algebras $\mathfrak{g}_{\mathbf{v}}$, resp. groups $G_{\mathbf{v}}$. To this end, for any $\mathbf{v}_1, \mathbf{v}_2$, we define

$$\mathfrak{g}_{1,2}^{\mathrm{disj}} = \{(a, b) \in \mathfrak{g}_{\mathbf{v}_1} \times \mathfrak{g}_{\mathbf{v}_2} \mid a = (a_i)_{i \in I}, \ b = (b_i)_{i \in I}, \ \mathrm{spec}(a_i) \cap \mathrm{spec}(b_j) = \emptyset \ \forall\, i, j \in I\}, \quad (4.2.1) \quad \boxed{\texttt{g12}}$$

where $\mathrm{spec}(g)$ stands for the set of eigenvalues of a matrix g. It is clear that $\mathfrak{g}_{1,2}^{\mathrm{disj}}$ is an affine, $G_{\mathbf{v}_1} \times G_{\mathbf{v}_2}$-stable Zariski open and dense subset of $\mathfrak{g}_{\mathbf{v}_1} \times \mathfrak{g}_{\mathbf{v}_2}$. Thus, we have natural maps

$$\mathfrak{g}_{\mathbf{v}_1, \mathbf{v}_2}^{\mathrm{disj}} \xrightarrow{\jmath_{\mathbf{v}_1, \mathbf{v}_2}} \mathfrak{g}_{\mathbf{v}_1}/G_{\mathbf{v}_1} \times \mathfrak{g}_{\mathbf{v}_2}/G_{\mathbf{v}_2} \xrightarrow{\imath_{\mathbf{v}_1, \mathbf{v}_2}} \mathfrak{g}_{\mathbf{v}_1 + \mathbf{v}_2}/G_{\mathbf{v}_1 + \mathbf{v}_2},$$

where the first, resp. second, map is an open, resp. closed, imbedding. For any triple $\mathbf{v}', \mathbf{v}'', \mathbf{v}'''$, of dimension vectors, we let $\mathfrak{g}_{\mathbf{v}', \mathbf{v}'', \mathbf{v}'''}^{\mathrm{disj}}$ be a subset of $\mathfrak{g}_{\mathbf{v}'} \times \mathfrak{g}_{\mathbf{v}''} \times \mathfrak{g}_{\mathbf{v}'''}$ formed by the triples $\left((a_i)_{i \in I}, (b_i)_{i \in I}, (c_i)_{i \in I}\right)$, of I-tuples, where $a_i \in \mathfrak{g}_{v_i'}, b_i \in \mathfrak{g}_{v_i''}, c_i \in \mathfrak{g}_{v_i'''}$ such that any two of the three sets $\cup_i \mathrm{spec}(a_i)$, $\cup_i \mathrm{spec}(b_j)$, and $\cup_i \mathrm{spec}(c_i)$, are disjoint.

We have an open imbedding $\mathrm{GL}_{\mathbf{v}} \hookrightarrow \mathfrak{gl}_{\mathbf{v}}$ of invertible matrices into the space of all matrices, and induced imbeddings $G_{\mathbf{v}_1} \times G_{\mathbf{v}_2} \hookrightarrow \mathfrak{g}_{\mathbf{v}_1} \times \mathfrak{g}_{\mathbf{v}_2}$, etc. Let $G_{\mathbf{v}_1, \mathbf{v}_2}^{\mathrm{disj}} := \mathfrak{g}_{\mathbf{v}_1, \mathbf{v}_2}^{\mathrm{disj}} \cap (G_{\mathbf{v}_1} \times G_{\mathbf{v}_2})$, resp. $G_{\mathbf{v}', \mathbf{v}'', \mathbf{v}'''}^{\mathrm{disj}} := \mathfrak{g}_{\mathbf{v}', \mathbf{v}'', \mathbf{v}'''}^{\mathrm{disj}} \cap (G_{\mathbf{v}'} \times G_{\mathbf{v}''} \times G_{\mathbf{v}'''})$. One can equivalently define $\mathfrak{g}_{\mathbf{v}', \mathbf{v}'', \mathbf{v}'''}^{\mathrm{disj}}$, resp. $G_{\mathbf{v}_1, \mathbf{v}_2}^{\mathrm{disj}}$, as the preimage of $\mathbb{A}_{\mathbf{v}_1, \mathbf{v}_2}^{\mathrm{disj}}/\!\!/(\mathfrak{S} \times \mathfrak{S}_{\mathbf{v}_2}$, resp $\mathbb{G}_{\mathbf{v}_1, \mathbf{v}_2}^{\mathrm{disj}}$, under the Chevalley isomorphism $\mathfrak{g}_{\mathbf{v}_1}/\!\!/ G_{\mathbf{v}_1} \times \mathfrak{g}_{\mathbf{v}_2}/\!\!/ G_{\mathbf{v}_2} \cong \mathbb{A}/\!\!/ \mathfrak{S}_{\mathbf{v}_1} \times \mathbb{A}^{\mathbf{v}_2}/\!\!/ \mathfrak{S}_{\mathbf{v}_2}$, resp. $G_{\mathbf{v}_1}/\!\!/_{\mathrm{Ad}} G_{\mathbf{v}_1} \times G_{\mathbf{v}_2}/\!\!/_{\mathrm{Ad}} G_{\mathbf{v}_2} \cong \mathbb{G}^{\mathbf{v}_1}/\!\!/ \mathfrak{S}_{\mathbf{v}_1} \times \mathbb{G}^{\mathbf{v}_2}/\!\!/ \mathfrak{S}_{\mathbf{v}_2}$.

The general notion of factorization sheaves on $\mathrm{Coh}\, C$ in the case $C = \mathbb{A}$, resp. $C = \mathbb{G}$, translates into the following

Definition 4.2.2. (i) A *factorization sheaf* $\mathcal{F}$ on $\mathfrak{gl}$ is the data of a collection $(\mathcal{F}_{\mathbf{v}})_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^I}$, where $\mathcal{F}_{\mathbf{v}} \in D(\mathfrak{g}_{\mathbf{v}}/G_{\mathbf{v}})$, equipped, for each pair $\mathbf{v}', \mathbf{v}''$, with a morphism $\bar{\varphi}_{\mathbf{v}', \mathbf{v}''} : \imath_{\mathbf{v}', \mathbf{v}''}^* \mathcal{F}_{\mathbf{v}' + \mathbf{v}''} \to \mathcal{F}_{\mathbf{v}'} \boxtimes \mathcal{F}_{\mathbf{v}''}$ such that analogues of (3.1.5) and (3.1.6), where φ is replaced by $\bar{\varphi}$ and $C_{\mathbf{v}_1, \mathbf{v}_2}^{\mathrm{disj}}/(\mathfrak{S}_{\mathbf{v}_1} \times \mathfrak{S}_{\mathbf{v}_2})$ is replaced by $\mathfrak{g}_{\mathbf{v}_1, \mathbf{v}_2}^{\mathrm{disj}}/(G_{\mathbf{v}_1} \times G_{\mathbf{v}_2})$, hold.

(ii) Equivariant factorization sheaves and factorization sheaves on GL are defined similarly.

Remark 4.2.3. We will not use the notion of 'weak factorization sheaf' on either G or $\mathfrak{g}$. $\diamondsuit$

We let the group $\mathbb{G}$ act on $G_{\mathbf{v}}$, resp. $\mathfrak{g}_{\mathbf{v}}$, by scalar multiplication, resp. *Aff* act on $\mathfrak{g}_{\mathbf{v}}$ by "$az + b$" : $g \mapsto ag + b \cdot \mathrm{Id}$. These actions commute with the $G_{\mathbf{v}}$-action, hence descend to well-defined actions on the corresponding quotient stacks $\mathfrak{g}_{\mathbf{v}}/G_{\mathbf{v}}$ and $G_{\mathbf{v}}/_{\mathrm{Ad}}G_{\mathbf{v}}$.

Lemma 4.2.4. *The functor $\mathfrak{R}_{\mathfrak{t}}$, resp. $\mathfrak{R}_T$, sends factorization sheaves on $\mathfrak{g}$, resp. G, to factorization sheaves on $\mathbb{A}$, resp. $\mathbb{G}$. A similar statement holds for Aff-equivariant, resp. $\mathbb{G}$-equivariant, factorization sheaves in the Lie algebra resp. group, case.*

Proof. Let $\mathfrak{g}_1 = \mathfrak{g}_{\mathbf{v}_1}$, $\mathfrak{g}_2 = \mathfrak{g}_{\mathbf{v}_2}$, let $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \hookrightarrow \mathfrak{g} := \mathfrak{g}_{v_1+v_2}$ be a block diagonal imbedding., and put $\mathfrak{g}_{1,2}^{\mathrm{disj}} = \mathfrak{g}_{\mathbf{v}_1,\mathbf{v}_2}^{\mathrm{disj}}$. Let $\mathfrak{b}_i = \mathfrak{t}_i + \mathfrak{u}_i$, $i = 1,2$, be a Borel subalgebra in $\mathfrak{g}_i$, let $\mathfrak{u}$ be the upper right $\mathbf{v}_1 \times \mathbf{v}_2$-block. Thus $\mathfrak{t} = \mathfrak{t}_1 \times \mathfrak{t}_2$ is a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{p} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{u}$ is a parabolic subalgebra of $\mathfrak{g}$ with nilradical $\mathfrak{u}$, and $\mathfrak{b} = \mathfrak{b}_1 \times \mathfrak{b}_2 \times \mathfrak{u}$ is a Borel subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$. Let G_i, B_i, U, etc. be the subgroups of G corresponding to the Lie algebras $\mathfrak{g}_i, \mathfrak{b}_i, \mathfrak{u}$, etc. We put

$$\mathfrak{g}^{\mathrm{disj}} = \mathfrak{g}_{\mathbf{v}_1,\mathbf{v}_2}^{\mathrm{disj}}, \quad \mathfrak{t}^{\mathrm{disj}} = \mathfrak{t}_{\mathbf{v}_1,\mathbf{v}_2}^{\mathrm{disj}}, \quad (\mathfrak{b}_1 \oplus \mathfrak{b}_2)^{\mathrm{disj}} = \mathfrak{t}_{\mathbf{v}_1,\mathbf{v}_2}^{\mathrm{disj}} \times (\mathfrak{u}_1 \oplus \mathfrak{u}_2), \quad \mathfrak{b}^{\mathrm{disj}} = \mathfrak{t}^{\mathrm{disj}} \times \mathfrak{u}.$$

Then, the variety $\mathfrak{b}^{\mathrm{disj}}$ is stable under the adjoint action of B, furthermore, for any

Claim 4.2.5. The variety $\mathfrak{b}^{\mathrm{disj}}$ is B-stable and the action the map yields an isomorphism

$$B \times_{B_1 \times B_2} (\mathfrak{b}_1 \oplus \mathfrak{b}_2)^{\mathrm{disj}} \to \mathfrak{b}^{\mathrm{disj}}.$$

Proof. For any $b \in \mathfrak{b}^{\mathrm{disj}}$, the variety $b + \mathfrak{u}$ is U-stable. The differential of the adjoint action $U \to b+\mathfrak{u}, u \mapsto u(b)$ is injective. This is clear if $b \in \mathfrak{t}^{\mathrm{disj}}$ since in that case all the weights of the $\mathrm{ad}\, b$-action in $\mathfrak{u}$ are nonzero by the definition of the set $\mathfrak{t}^{\mathrm{disj}}$. The case of a general $b \in \mathfrak{b}^{\mathrm{disj}}$ reduces to the one where $b \in \mathfrak{t}^{\mathrm{disj}}$ by considering Jordan decomposition. Thus, the map $u \mapsto u(b)$ gives an imbedding $U \hookrightarrow b + \mathfrak{u}$. The image of this map is open since $\dim U = \dim(b + \mathfrak{u})$, and it is also closed in $b + \mathfrak{u}$ since orbits of a unipotent acting on an affine variety are known to be closed. We deduce that the action map $U \times_{U_1 \times U_2} (\mathfrak{b}_1 \oplus \mathfrak{b}_2)^{\mathrm{disj}} \to \mathfrak{b}^{\mathrm{disj}}$ is an isomorphism. This implies the claim. $\qquad\square$

Let $Z_i := \mathfrak{g}_i \times_{\mathfrak{g}_i /\!/ G_i} \mathfrak{t}_i$, resp. $Z := \mathfrak{g} \times_{\mathfrak{g}/\!/G} \mathfrak{t}$, and let $\widetilde{\mathfrak{g}}_i$, resp. $\widetilde{\mathfrak{g}}$, denote the Grothendieck-Springer resolution. For any scheme (or stack) $\mathcal{G}$ over $\mathfrak{t}$ we write $\mathcal{G}^{\mathrm{disj}} := \mathcal{G} \times_{\mathfrak{t}} \mathfrak{t}^{\mathrm{disj}}$. We have a commutative diagram:

$$
\begin{array}{ccc}
G \times_{G_1 \times G_2} (\widetilde{\mathfrak{g}}_1 \times \widetilde{\mathfrak{g}}_2)^{\mathrm{disj}} & \xrightarrow{\ \mathrm{Id}_G \times \boldsymbol{\mu}_1 \times \boldsymbol{\mu}_2\ } & G \times_{G_1 \times G_2} (Z_1 \times Z_2)^{\mathrm{disj}} \\
{\scriptstyle \widetilde{f}}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle f} \\
\widetilde{\mathfrak{g}}^{\mathrm{disj}} & \xrightarrow{\qquad\qquad \boldsymbol{\mu} \qquad\qquad} & Z^{\mathrm{disj}}
\end{array}
$$

Note that

$$G \times_{G_1 \times G_2} (\widetilde{\mathfrak{g}}_1 \times \widetilde{\mathfrak{g}}_2) = G \times_{G_1 \times G_2} ((G_1 \times G_2) \times_{B_1 \times B_2} (\mathfrak{b}_1 \oplus \mathfrak{b}_2)) = G \times_B (B \times_{B_1 \times B_2} (\mathfrak{b}_1 \oplus \mathfrak{b}_2)).$$

Therefore, Claim $\overset{\text{u-act}}{4.2.5}$ implies that the map $\widetilde{f}$ in the commutative diagram is an isomorphism. Further, it is known that one has $(\boldsymbol{\mu}_1 \times \boldsymbol{\mu}_2)_* \mathcal{O}_{\widetilde{\mathfrak{g}}_1 \times \widetilde{\mathfrak{g}}_2} = \mathcal{O}_{Z_1 \times Z_2}$, resp. $\boldsymbol{\mu}_* \mathcal{O}_{\widetilde{\mathfrak{g}}} = \mathcal{O}_Z$. Hence, we deduce an isomorphism

$$f_* \mathcal{O}_{G \times_{G_1 \times G_2} (Z_1 \times Z_2)^{\mathrm{disj}}} \overset{\sim}{\to} \mathcal{O}_{Z^{\mathrm{disj}}}.$$

The morphism f being affine, it follows that this morphism is an isomorphism. Thus, taking quotients by the G-action in the diagram above yields the following commutative diagram

$$
\begin{array}{ccccc}
\mathfrak{t}_1 \times \mathfrak{t}_2 & \xleftarrow{pr_{\mathfrak{t}_1} \times pr_{\mathfrak{t}_2}} & (Z_1 \times Z_2)^{\mathrm{disj}}/(G_1 \times G_2) & \xrightarrow{pr_{\mathfrak{g}_1} \times pr_{\mathfrak{g}_2}} & (\mathfrak{g}_1 \times \mathfrak{g}_2)^{\mathrm{disj}}/(G_1 \times G_2) \\
\jmath_{\mathfrak{t}} \downarrow \cong & & f \downarrow \cong & & \downarrow \jmath_{\mathfrak{g}} \\
\mathfrak{t} & \xleftarrow{pr_{\mathfrak{t}}} & Z^{\mathrm{disj}}/G & \xrightarrow{pr_{\mathfrak{g}}} & \mathfrak{g}/G
\end{array}
$$

Recall the open sets $U_i := \mathfrak{g}^{rs} \times_{\mathfrak{g}_i /\!/ G_i} \mathfrak{t}_i$, resp. $U := \mathfrak{g}^{rs} \times_{\mathfrak{g} /\!/ G} \mathfrak{t}$. The morphism f being an isomorphism, we deduce an isomorphism $\mathrm{IC}(U_1/G_1 \times U_2/G_2) \xrightarrow{\sim} f^* \mathrm{IC}(U/G)$. Now, for any $\mathcal{F} \in D_{\mathrm{abs}}(\mathfrak{g}/G)$, we compute

$$
\begin{aligned}
(\mathfrak{R}_{\mathfrak{t}_1} \times \mathfrak{R}_{\mathfrak{t}_2})(\jmath_{\mathfrak{g}}^* \mathcal{F}) &= (pr_{\mathfrak{t}_1} \times pr_{\mathfrak{t}_2})_! ((pr_{\mathfrak{g}_1} \times pr_{\mathfrak{g}_2})^* \jmath_{\mathfrak{g}}^* \mathcal{F} \otimes \mathrm{IC}(U_1/G_1 \times U_2/G_2)) \\
&= (pr_{\mathfrak{t}_1} \times pr_{\mathfrak{t}_2})_! (f^* \mathrm{IC}(U/G) \otimes f^* pr_{\mathfrak{g}}^* \mathcal{F}) \\
&= \jmath_{\mathfrak{t}}^* (pr_{\mathfrak{t}})_! (\mathrm{IC}(U/G) \otimes pr_{\mathfrak{g}}^* \mathcal{F}) = \jmath_{\mathfrak{t}}^* \mathfrak{R}_{\mathfrak{t}}(\mathcal{F}).
\end{aligned}
$$

The sheaf $\mathrm{IC}(U/G)$ being W-equivariant, the functors $\jmath_{\mathfrak{t}}^*$ and $f^* \mathrm{IC}(U/G) \otimes f^*(-)$ clearly respect $W_1 \times W_2$-equivariant structures, proving the lemma. $\qquad \square$

Remark 4.2.6. An argument similar to the proof of Lemma $\overset{\mathrm{res}}{4.2.4}$ shows that each of the two functors $D_{\mathrm{abs}}(\mathrm{Coh}_{\mathbf{v}}) \underset{\mathrm{ind}}{\overset{\mathrm{res}}{\rightleftarrows}} D_{\mathrm{abs}}(C^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}})$ sends factorization sheaves to factorization sheaves. We will neither use nor prove this result.

4.3. *Aff*-equivariant factorization sheaves on $\mathfrak{gl}$.

We are going to formulate a $\mathfrak{gl}$-counterpart of the result from §$\overset{\mathrm{sec3}}{3}$ about the cohomology of *Aff*-equivariant factorization sheaves on $\mathbb{A}$.

We identify $\mathfrak{t}_{\mathbf{v}} = \mathbb{A}^{\mathbf{v}}/\mathbb{A}$ with the Cartan subalgebra of $\mathfrak{pgl}_{\mathbf{v}}$ and also identify $\mathfrak{pgl}_{\mathbf{v}}^* /\!/ \mathrm{PGL}_{\mathbf{v}}$ with $\mathfrak{t}_{\mathbf{v}}^* /\!/ \mathfrak{S}_{\mathbf{v}}$ via the Chevalley isomorphism.

Definition 4.3.1. Let $\mathfrak{pgl}_{\mathbf{v}}^{\diamond}$ be the preimage of the set $\mathfrak{t}_{\mathbf{v}}^{\diamond} /\!/ \mathfrak{S}_{\mathbf{v}}$, see $(\overset{\mathrm{de2}}{3.2.1})$, under the quotient map $\mathfrak{pgl}_{\mathbf{v}}^* \to \mathfrak{pgl}_{\mathbf{v}}^* /\!/ \mathrm{PGL}_{\mathbf{v}} = \mathfrak{t}_{\mathbf{v}}^* /\!/ \mathfrak{S}_{\mathbf{v}}$.

The group $\mathrm{GL}_{\mathbf{v}}$ acts on $\mathfrak{pgl}_{\mathbf{v}}$, resp. $\mathfrak{pgl}_{\mathbf{v}}^*$, through its quotient $\mathrm{PGL}_{\mathbf{v}}$ and the imbedding $\mathfrak{t}_{\mathbf{v}}^{\diamond} \hookrightarrow \mathfrak{pgl}_{\mathbf{v}}^{\diamond}$ induces an isomorphism $\mathfrak{t}_{\mathbf{v}}^{\diamond}/\mathbb{G}^{\mathbf{v}} \xrightarrow{\sim} \mathfrak{pgl}_{\mathbf{v}}^{\diamond}/\mathrm{GL}_{\mathbf{v}}$, where $\mathbb{G}^{\mathbf{v}}$, the maximal torus of $\mathrm{GL}_{\mathbf{v}}$, acts trivially on $\mathfrak{t}_{\mathbf{v}}^{\diamond}$. Thus, the canonical map $\mathfrak{w} : \mathfrak{pgl}_{\mathbf{v}}^{\diamond}/\mathrm{GL}_{\mathbf{v}} \to \mathfrak{pgl}_{\mathbf{v}}^{\diamond} /\!/ \mathrm{GL}_{\mathbf{v}} = \mathfrak{pgl}_{\mathbf{v}}^{\diamond} /\!/ \mathrm{PGL}_{\mathbf{v}} = \mathfrak{t}_{\mathbf{v}}^{\diamond} /\!/ \mathfrak{S}_{\mathbf{v}}$ may be identified with the first projection $\mathfrak{t}_{\mathbf{v}}^{\diamond} /\!/ \mathfrak{S}_{\mathbf{v}} \times \mathrm{pt}/\mathbb{G}^{\mathbf{v}} \to \mathfrak{t}_{\mathbf{v}}^{\diamond} /\!/ \mathfrak{S}_{\mathbf{v}}$.

The group *Aff* acts on $\mathfrak{gl}_{\mathbf{v}}$ by $ax + b : g \mapsto a \cdot g + b \cdot \mathrm{Id}$. This action commutes with the $\mathrm{GL}_{\mathbf{v}}$-action, hence, it descends to $\mathfrak{gl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}$. An *Aff*-equivariant sheaf $\mathcal{F}_{\mathbf{v}}$ on $\mathfrak{gl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}$ descends to a $\mathbb{G}$-equivariant sheaf $\bar{\mathcal{F}}_{\mathbf{v}}$ on $\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}$, where $\mathbb{G}$ acts on $\mathfrak{pgl}_{\mathbf{v}}$ by dilations. We put $\Phi_{\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}}(\mathcal{F}_{\mathbf{v}}) = \mathfrak{w}_! (\mathrm{F}_{\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}}(\bar{\mathcal{F}}_{\mathbf{v}})|_{\mathfrak{pgl}_{\mathbf{v}}^{\diamond}/\mathrm{GL}_{\mathbf{v}}})$, where $\mathrm{F}_{\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}}$ denotes the Fourier-Deligne transform on $\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}$. Thus, $\Phi_{\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}}(\mathcal{F}_{\mathbf{v}})$ is a sheaf on $\mathfrak{t}_{\mathbf{v}}^{\diamond} /\!/ \mathfrak{S}_{\mathbf{v}}$ and we let $\Phi_{\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}}(\mathcal{F}_{\mathbf{v}})_{\eta}$ be the restriction of $\Phi_{\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}}(\mathcal{F}_{\mathbf{v}})$ to $\eta = \mathrm{Spec}\,\overline{\mathsf{k}(\mathfrak{t}_{\mathbf{v}}^{\diamond} /\!/ \mathfrak{S}_{\mathbf{v}})}$.

We have natural imbeddings $\mathrm{GL}_{\mathbf{v}}/_{\mathrm{Ad}}\mathrm{GL}_{\mathbf{v}} \xrightarrow{\varepsilon} \mathfrak{gl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}} \xleftarrow{i} 0/\mathrm{GL}_{\mathbf{v}}$. The image of each imbedding is stable under the action of the subgroup $\mathbb{G} \subset$ *Aff*.

Theorem 4.3.2. *For any Aff-equivariant factorization sheaf $\mathcal{F} = (\mathcal{F}_{\mathbf{v}})_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^I}$ on $\mathfrak{gl}$ we have:*

(i) *The monodromy action of the subgroup $\pi_1^{et}(\mathfrak{t}_{\mathbf{v}}^{\diamond}) \subset \pi_1^{et}(\mathfrak{t}_{\mathbf{v}}^{\diamond}/\mathfrak{S}_{\mathbf{v}})$ on $\Phi_{\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}}(\mathcal{F}_{\mathbf{v}})$ is unipotent and there is a Frobenius stable generalized isotypic decomposition*

$$
\Phi_{\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}}(\mathcal{F}_{\mathbf{v}})_{\eta} = \oplus_{\rho \in \mathrm{Irr}(\mathfrak{S}_{\mathbf{v}})} \Phi_{\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}}(\mathcal{F}_{\mathbf{v}})_{\eta}^{\langle \rho \rangle}.
$$

(ii) *In* $\mathsf{K}^{I}_{\mathrm{abs}}$, *one has the following equations*

$$\sum_{\mathbf{v}} z^{\mathbf{v}} \cdot [\mathrm{R}\Gamma_c(\mathfrak{gl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}, \mathcal{F}_{\mathbf{v}})] = \mathrm{Sym}\left(-\mathbb{L}^{\frac{1}{2}} \cdot \sum_{\mathbf{v}>0} (-1)^{|\mathbf{v}|} \cdot z^{\mathbf{v}} \cdot \mathbb{L}^{\frac{|\mathbf{v}|}{2}} \cdot [\Phi_{\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}}(\mathcal{F}_{\mathbf{v}})_\eta]^{\langle \mathrm{sign}\rangle}\right); \quad (4.3.3) \quad \boxed{\texttt{2eqsG}}$$

$$\sum_{\mathbf{v}} z^{\mathbf{v}} \cdot [\mathrm{R}\Gamma_c(\mathrm{GL}_{\mathbf{v}}/_{\mathrm{Ad}}\mathrm{GL}_{\mathbf{v}}, \varepsilon^* \mathcal{F}_{\mathbf{v}})] = \mathrm{Sym}\left((\mathbb{L}^{-\frac{1}{2}} - \mathbb{L}^{\frac{1}{2}}) \cdot \sum_{\mathbf{v}>0} (-1)^{|\mathbf{v}|} \cdot z^{\mathbf{v}} \cdot \mathbb{L}^{\frac{|\mathbf{v}|}{2}} \cdot [\Phi_{\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}}(\mathcal{F}_{\mathbf{v}})_\eta]^{\langle \mathrm{sign}\rangle}\right);$$

$$(4.3.4) \quad \boxed{\texttt{3eqsG}}$$

$$\sum_{\mathbf{v}} z^{\mathbf{v}} \cdot \mathbb{L}^{-\frac{\mathbf{v}\cdot\mathbf{v}}{2}} \cdot \left[\mathrm{R}\Gamma_c(0/\mathrm{GL}_{\mathbf{v}}, i^*\mathcal{F}_{\mathbf{v}})\right] = \mathrm{Sym}\left(\sum_{\mathbf{v}>0} z^{\mathbf{v}} \cdot \mathbb{L}^{-\frac{|\mathbf{v}|}{2}} \cdot [\Phi_{\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}}(\mathcal{F}_{\mathbf{v}})_\eta]^{\langle \mathrm{triv}\rangle}\right). \quad (4.3.5) \quad \boxed{\texttt{4eqsG}}$$

Proof. First, applying the functor $\mathrm{R}\Gamma_c(\mathbb{A}^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}, -)$ to the second isomorphism in ($\overset{\texttt{xieq}}{4.1.3}$) in the case of the group $G = \mathrm{GL}_{\mathbf{v}}$, we obtain

$$\mathrm{R}\Gamma_c(\mathrm{GL}_{\mathbf{v}}/_{\mathrm{Ad}}\mathrm{GL}_{\mathbf{v}}, \varepsilon^*\mathcal{F}_{\mathbf{v}}) = \mathrm{R}\Gamma_c(\mathrm{GL}_{\mathbf{v}}/\!/_{\mathrm{Ad}}\mathrm{GL}_{\mathbf{v}}, (\mathfrak{w}_{\mathrm{GL}_{\mathbf{v}}})_! \varepsilon^*\mathcal{F}_{\mathbf{v}})$$
$$= \mathrm{R}\Gamma_c(\mathbb{G}^{\mathbf{v}}/\!/\mathfrak{S}_{\mathbf{v}}, ((\mathfrak{w}_{\mathbb{G}^{\mathbf{v}}})_! \, \mathfrak{R}_{\mathbb{G}}\, \varepsilon^*\mathcal{F}_{\mathbf{v}})^{\mathfrak{S}_{\mathbf{v}}}) = \mathrm{R}\Gamma_c(\mathbb{G}^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}, \mathfrak{R}_{\mathbb{G}}\, \varepsilon^*\mathcal{F}_{\mathbf{v}}) = \mathrm{R}\Gamma_c(\mathbb{G}^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}, \varepsilon^* \mathfrak{R}_{\mathbb{A}}\, \mathcal{F}_{\mathbf{v}}).$$

Next, let $T_{\mathbf{v}} = \mathbb{G}^{\mathbf{v}}/\mathbb{G}$ be the Cartan torus, resp. $\mathfrak{t}_{\mathbf{v}} = \mathbb{A}^{\mathbf{v}}/\mathbb{A}$ the Cartan subalgebra, for $\mathrm{PGL}_{\mathbf{v}}$. Abusing the notation, we write $\mathfrak{R}_{\mathfrak{t}}$ for the functor $D_{\mathrm{abs}}(\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}) \to D_{\mathrm{abs}}(\mathfrak{t}_{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}})$ described in Remark $\overset{\texttt{GL-pgl}}{4.1.13}$. For each $\mathbf{v}$, the sheaf $\mathcal{F}_{\mathbf{v}}$, resp. $\mathfrak{R}_{\mathbb{A}}\,\mathcal{F}_{\mathbf{v}}$, descends, thanks to the *Aff*-equivariance, to a sheaf $\bar{\mathcal{F}}_{\mathbf{v}}$ on $\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}$, resp. $\overline{\mathfrak{R}_{\mathbb{A}}\,\mathcal{F}_{\mathbf{v}}}$ on $\mathfrak{t}_{\mathbf{v}}$, furthermore, one has $\mathfrak{R}_{\mathfrak{t}}\,\bar{\mathcal{F}}_{\mathbf{v}} = \overline{\mathfrak{R}_{\mathbb{A}}\,\mathcal{F}_{\mathbf{v}}}$. Applying the analogue of the second isomorphism in ($\overset{\texttt{xieq}}{4.1.3}$) in the case of $\mathfrak{pgl}^*_{\mathbf{v}}$ and using Lemma $\overset{\texttt{qFourier2}}{4.1.8}$, we find

$$(\mathfrak{w}_{\mathfrak{pgl}^*_{\mathbf{v}}})_! \mathrm{F}_{\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}}(\bar{\mathcal{F}}_{\mathbf{v}}) = (\mathfrak{w}_{\mathfrak{t}^*_{\mathbf{v}}})_! \, \mathfrak{R}_{\mathfrak{t}^*} \, \mathrm{F}_{\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}}(\bar{\mathcal{F}}_{\mathbf{v}})$$
$$= \mathrm{sign} \otimes (\mathfrak{w}_{\mathfrak{t}^*_{\mathbf{v}}})_! \mathrm{F}_{\mathfrak{t}_{\mathbf{v}}}(\mathfrak{R}_{\mathfrak{t}}\,\bar{\mathcal{F}}_{\mathbf{v}}) = \mathrm{sign} \otimes (\mathfrak{w}_{\mathfrak{t}^*_{\mathbf{v}}})_! \mathrm{F}_{\mathfrak{t}_{\mathbf{v}}}\overline{\mathfrak{R}_{\mathbb{A}}\,\mathcal{F}_{\mathbf{v}}}.$$

We deduce that for any $\rho \in \mathrm{Irr}(\mathfrak{S}_{\mathbf{v}})$ one has

$$\Phi_{\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}}(\mathcal{F}_{\mathbf{v}})^{\langle\rho\rangle}_\eta = ((\mathfrak{w}_{\mathfrak{pgl}^*_{\mathbf{v}}})_! \mathrm{F}_{\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}}\bar{\mathcal{F}}_{\mathbf{v}})^{\langle\rho\rangle}_\eta \qquad (4.3.6) \quad \boxed{\texttt{fw-rho}}$$
$$= \left(\mathrm{sign} \otimes (\mathfrak{w}_{\mathfrak{t}^*_{\mathbf{v}}})_! \mathrm{F}_{\mathfrak{t}_{\mathbf{v}}}\overline{\mathfrak{R}_{\mathbb{A}}\,\mathcal{F}_{\mathbf{v}}}\right)^{\langle\rho\rangle}_\eta = \Phi_{\mathfrak{t}_{\mathbf{v}}}(\mathfrak{R}_{\mathbb{A}}\,\mathcal{F}_{\mathbf{v}})^{\langle \mathrm{sign}\otimes\rho\rangle}_\eta.$$

By Lemma $\overset{\texttt{res}}{4.2.4}$, the collection $\mathfrak{R}_{\mathbb{A}}\,\mathcal{F} = (\mathfrak{R}_{\mathbb{A}}\,\mathcal{F}_{\mathbf{v}})$ is an *Aff*-equivariant factorization sheaf on $\mathbb{A}$. Thus, using ($\overset{\texttt{3eqs}}{3.3.3}$), we compute

$$\sum_{\mathbf{v}} z^{\mathbf{v}} \cdot [\mathrm{R}\Gamma_c(\mathrm{GL}_{\mathbf{v}}/_{\mathrm{Ad}}\mathrm{GL}_{\mathbf{v}}, \varepsilon^*\mathcal{F}_{\mathbf{v}})] = \sum_{\mathbf{v}} z^{\mathbf{v}} \cdot [\mathrm{R}\Gamma_c(\mathbb{G}^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}}, \varepsilon^* \mathfrak{R}_{\mathbb{A}}\,\mathcal{F}_{\mathbf{v}})]$$
$$= \mathrm{Sym}\left((\mathbb{L}^{-\frac{1}{2}} - \mathbb{L}^{\frac{1}{2}}) \cdot \sum_{\mathbf{v}>0} (-1)^{|\mathbf{v}|} \cdot z^{\mathbf{v}} \cdot \mathbb{L}^{\frac{|\mathbf{v}|}{2}} \cdot [\Phi_{\mathfrak{t}}(\mathfrak{R}_{\mathbb{A}}\,\mathcal{F}_{\mathbf{v}})_\eta]^{\langle \mathrm{triv}\rangle}\right)$$
$$= \mathrm{Sym}\left((\mathbb{L}^{-\frac{1}{2}} - \mathbb{L}^{\frac{1}{2}}) \cdot \sum_{\mathbf{v}>0} (-1)^{|\mathbf{v}|} \cdot z^{\mathbf{v}} \cdot \mathbb{L}^{\frac{|\mathbf{v}|}{2}} \cdot [\Phi_{\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}}(\mathcal{F}_{\mathbf{v}})^{\langle \mathrm{sign}\rangle}_\eta]\right).$$

This proves ($\overset{\texttt{3eqsG}}{4.3.4}$). The proofs of ($\overset{\texttt{2eqsG}}{4.3.3}$) and ($\overset{\texttt{4eqsG}}{4.3.5}$) are similar using the corresponding equation of Theorem $\overset{\texttt{expC}}{3.2.3}$. For example, to prove formula ($\overset{\texttt{4eqsG}}{4.3.5}$) recall first that $n_{\mathrm{GL}_{\mathbf{v}}} = n_{\mathrm{GL}_{\mathbf{v}}} = \frac{\mathbf{v}\cdot\mathbf{v}-|\mathbf{v}|}{2}$. We compute

$$[\mathrm{R}\Gamma_c(0/\mathrm{GL}_{\mathbf{v}}, i^*\mathcal{F}_{\mathbf{v}})] \cdot \mathbb{L}^{-n_{\mathrm{GL}_{\mathbf{v}}}} = [(\mathrm{pt}_{0/\mathrm{GL}_{\mathbf{v}}})_! i^*\mathcal{F}_{\mathbf{v}}]\{n_{\mathrm{GL}_{\mathbf{v}}}\}$$
$$= [((\mathrm{pt}_{0/\mathfrak{S}_{\mathbf{v}}})_! i^* \mathfrak{R}_{\mathbb{A}}\,\mathcal{F}_{\mathbf{v}})]^{\mathrm{sign}}\{n_{\mathrm{GL}_{\mathbf{v}}} - n_{\mathrm{GL}_{\mathbf{v}}}\} = [\mathrm{R}\Gamma_c(0/\mathfrak{S}_{\mathbf{v}}, i^*(\mathrm{sign} \otimes \mathfrak{R}_{\mathbb{A}}\,\mathcal{F}_{\mathbf{v}}))].$$

By Remarks $\overset{\texttt{fact-rems}}{3.1.9}$(2)-(3), the collection $(\mathrm{sign} \otimes \mathfrak{R}_{\mathbb{A}}\,\mathcal{F}_{\mathbf{v}}[|\mathbf{v}|])$ is a factorization sheaf on $\mathbb{A}$. Thus, using ($\overset{\texttt{fw-rho}}{4.3.6}$), we compute

41

$$\sum_{\mathbf{v}} (-1)^{|\mathbf{v}|} \cdot \mathbb{L}^{-\frac{\mathbf{v} \cdot \mathbf{v} - |\mathbf{v}|}{2}} \cdot \left[R\Gamma_c(0/\mathrm{GL}_\mathbf{v}, \, i^* \mathcal{F}_\mathbf{v}) \right]$$

$$= \sum_{\mathbf{v}} \left[(\mathrm{pt}_{0/\mathrm{GL}_\mathbf{v}})_! i^* \mathcal{F}_\mathbf{v} \right] \{ n_{\mathrm{GL}_\mathbf{v}} \} \left[|\mathbf{v}| \right] = \sum_{\mathbf{v}} \left[R\Gamma_c(0/\mathfrak{S}_\mathbf{v}, \, i^*(\mathrm{sign} \otimes \mathfrak{R}_\mathbb{A} \, \mathcal{F}_\mathbf{v} \left[|\mathbf{v}| \right])) \right]$$

$$= \mathrm{Sym} \left(- \mathbb{L}^{\frac{1}{2}} \cdot \sum_{\mathbf{v}>0} (-1)^{|\mathbf{v}|} \cdot z^\mathbf{v} \cdot \mathbb{L}^{-\frac{|\mathbf{v}|}{2}} \cdot \left[\Phi_\mathfrak{t}(\mathrm{sign} \otimes \mathfrak{R}_\mathbb{A} \, \mathcal{F}_\mathbf{v} \left[|\mathbf{v}| \right])_\eta \right]^{\langle \mathrm{triv} \rangle} \right)$$

$$= \mathrm{Sym} \left(- \mathbb{L}^{\frac{1}{2}} \cdot \sum_{\mathbf{v}>0} (-1)^{|\mathbf{v}|} \cdot z^\mathbf{v} \cdot \mathbb{L}^{-\frac{|\mathbf{v}|}{2}} \cdot \left[\Phi_\mathfrak{t}(\mathfrak{R}_\mathbb{A} \, \mathcal{F}_\mathbf{v})_\eta \right]^{\langle \mathrm{sign} \rangle} \left[|\mathbf{v}| \right] \right)$$

$$= \mathrm{Sym} \left(- \mathbb{L}^{\frac{1}{2}} \cdot \sum_{\mathbf{v}>0} z^\mathbf{v} \cdot \mathbb{L}^{-\frac{|\mathbf{v}|}{2}} \cdot \left[\Phi_{\mathfrak{pgl}_\mathbf{v}/\mathrm{GL}_\mathbf{v}}(\mathcal{F}_\mathbf{v})_\eta \right]^{\langle \mathrm{triv} \rangle} \right).$$

Finally, it is known that in $\mathsf{K}^I_{\mathrm{abs}}$ the following two equations are equivalent

$$\mathrm{Sym} \left(\sum_{\mathbf{v}>0} a_\mathbf{v} \right) = \sum_{\mathbf{v}} f_\mathbf{v} \quad \Leftrightarrow \quad \mathrm{Sym} \left(- \mathbb{L}^{-1/2} \cdot \sum_{\mathbf{v}>0} a_\mathbf{v} \right) = \sum_{\mathbf{v}} (-1)^{|\mathbf{v}|} \cdot \mathbb{L}^{-\frac{|\mathbf{v}|}{2}} \cdot f_\mathbf{v}. \qquad (4.3.7) \quad \boxed{\texttt{a-exp}}$$

We apply this in the case where $a_\mathbf{v} = (-1)^{|\mathbf{v}|} \cdot \mathbb{L}^{-\frac{|\mathbf{v}|}{2}} \cdot \left[\Phi_{\mathfrak{pgl}_\mathbf{v}/\mathrm{GL}_\mathbf{v}}(\mathcal{F}_\mathbf{v})_\eta \right]^{\langle \mathrm{triv} \rangle}$ and formula (4.3.5) follows. $\qquad \boxed{\texttt{4eqsG}}$ $\square$

5. Inertia stacks

$\boxed{\texttt{gstaek}}$

5.1. G-stacks. Given an algebraic group G and a G-stack Y, over k, we write $q_Y : Y \to Y/G$ for the quotient morphism. The categories $Y(\bar{\mathsf{k}})$ and $(Y/G)(\bar{\mathsf{k}})$ have the same objects and the quotient morphism $q_Y : Y \to Y/G$ is the identity on objects. The groups of morphisms between the corresponding objects of Y and Y/G may differ, in general. In particular, for $y \in Y(\bar{\mathsf{k}})$, the automorphism group of the object $q_Y(y) \in (Y/G)(\bar{\mathsf{k}})$ equals

$$\mathrm{Aut}(q_Y(y)) = \{ (g, f) \mid g \in G, f \in \mathrm{Mor}(y, gy) \},$$

where the group operation is given by $(g, f) \cdot (g', f') = (gg', g'(f) \circ f')$.

There is a natural exact sequence

$$1 \longrightarrow \mathrm{Aut}(y) \xrightarrow{f \mapsto (1, f)} \mathrm{Aut}(q_Y(y)) \xrightarrow{p_y : (g,f) \mapsto g} G, \qquad (5.1.1) \quad \boxed{\texttt{algaut}}$$

of algebraic groups. The group $G_y := \mathrm{Im}(p_y)$ may be viewed as the stabilizer of y in G.

Recall that a morphism $Y \to X$, where Y is a G-stack and X is a stack, is called a G-torsor (on X) if, for any test scheme S and an S-point $S \to X$, the first projection $S \times_X Y \to S$ is a G-torsor on S. Any morphism of G-torsors on X is an isomorphism. For any G-stack Y, the quotient morphism $q_Y : Y \to Y/G$ is a G-torsor. Conversely, for any G-torsor $Y \to X$ the canonical morphism $Y/G \to X$ is an isomorphism of stacks.

One has the universal G-torsor $\mathrm{pt} \to \mathrm{pt}/G$, where $\mathrm{pt}/G = \mathbb{B}G$ is the classifying stack of the group G. For any G-torsor $Y \to X$ the constant map $Y \to \mathrm{pt}$ induces a map $Y/G \to \mathrm{pt}/G$ It follows, since $Y/G = X$ and any morphism of G-torsors on X is an isomorphism, that the G-torsor $Y \to X$ may be obtained from the universal G-torsor by base change, equivalently, we have $Y \cong (Y/G) \times_{\mathrm{pt}/G} \mathrm{pt}$.

Let $X \times \mathsf{k}^v \to X$ be a trivial rank v vector bundle on a stack X. For any rank v vector bundle $\mathcal{V}$ on X, let $Y(\mathcal{V})$ be the sheaf (on X) of vector bundle isomorphisms $X \times \mathsf{k}^v \xrightarrow{\sim} \mathcal{V}$. There is a GL_v-action on $Y(\mathcal{V})$ induced by the natural GL_v-action on k^v. The projection $Y(\mathcal{V}) \to X$ is a GL_v-torsor on X, to be called the 'frame bundle' of the vector bundle $\mathcal{V}$.

42

5.2. G-inertia stacks. The *inertia stack*, $I(X)$, of a stack X is defined as a fiber product $I(X) = X \times_{X \times X} X$, where each of the two maps $X \to X \times X$ is the diagonal morphism. An object of the inertia stack is a pair (x, f) where $x \in X$ and $f \in \mathrm{Aut}(x)$. Morphisms $(x, f) \to (x', f')$ are defined as morphisms $\varphi : x \to x'$ such that $f' \circ \varphi = \varphi \circ f$.

Let G be an algebraic group with Lie algebra $\mathfrak{g}$, and fix a G-stack Y. One defines equivariant analogues of the inertia stack as follows. The morphism

$$a_G : G \times Y \to Y \times Y, \quad \text{resp.} \quad a_\mathfrak{g} : \mathfrak{g} \times Y \to \mathrm{T}Y,$$

given by $(g, y) \mapsto (gy, y)$, makes $G \times Y$ a stack over $Y \times Y$, resp. $\mathfrak{g} \times Y$ a stack over $\mathrm{T}Y$. We define the *G-inertia stack*, resp. *infinitesimal G-inertia stack*, of Y, by

$$I(Y, G) = Y \times_{Y \times Y} (G \times Y), \quad \text{resp.} \quad I(Y, \mathfrak{g}) = Y \times_{\mathrm{T}Y} (\mathfrak{g} \times Y),$$

where $Y \to Y \times Y$ is the diagonal morphism, resp. $Y \to \mathrm{T}Y$ is the zero section. In the special case where Y is a G-scheme, viewed as a stack, the G-inertia stack becomes a scheme and the corresponding set of closed points can also be defined in the following equivalent, more familiar way

$$I(Y, G) = \{(y, g) \in Y \times G \mid gy = y\} = \{(y, g) \in Y \times G \mid y \in Y^g\}. \tag{5.2.1}$$

We return to the general case. It is easy to see that objects of the stack $I(Y, G)$ are triples of the form (y, g, f), where y is an object of Y, $g \in G$, and $f : y \to g(y)$ is a morphism. Similarly, an object of $I(Y, \mathfrak{g})$ is a triple (y, g, f) where $y \in Y$, $g \in \mathfrak{g}$, and $f : gy \to 0$ is an isomorphism in the groupoid $\mathrm{T}_y Y$.

We let the group G act on itself, resp. its Lie algebra, via the adjoint action. Then, it is clear that $I(Y, G)$, resp. $I(Y, \mathfrak{g})$, is a G-stack. The first projection $\mathrm{pr}_Y : I(Y, G) \to Y$, resp. the composition, pr_G, of the second projection $I(Y, G) \to G \times Y$ and the first projection $G \times Y \to G$, are morphisms of G-stacks. Further, it follows from definitions that the composition $I(Y, G) \to Y \times Y \to Y/G \times Y/G$ factors through a morphism $q_I : I(Y, G) \to I(Y/G)$, furthermore, the latter morphism factors through $I(Y, G)/G \to I(Y/G)$.

Lemma 5.2.2. (i) *The morphism $I(Y, G)/G \to I(Y/G)$ is an isomorphism.*

(ii) *The morphism q_I is a G-torsor that fits into a diagram of cartesian squares*

$$\begin{array}{ccccc}
Y & \xleftarrow{\;\mathrm{pr}_Y\;} & I(Y, G) & \xrightarrow{\;\mathrm{pr}_G\;} & G \\
\downarrow{\scriptstyle q_Y} & \square & \downarrow{\scriptstyle q_I} & \square & \downarrow{\scriptstyle q_G} \\
Y/G & \xleftarrow{\;p_{Y/G}\;} & I(Y/G) & \xrightarrow{\;p_G\;} & G/_{\mathrm{Ad}}G
\end{array} \tag{5.2.3}$$

Proof. One has natural isomorphisms $Y = (Y/G) \times_{\mathrm{pt}/G} \mathrm{pt}$, resp. $G = \mathrm{pt} \times_{\mathrm{pt}/G} \mathrm{pt}$, and also $G \times Y = Y \times_{Y/G} Y$. Using these isomorphisms we find

$$I(Y, G) = Y \times_{Y \times Y} (G \times Y) = Y \times_{Y \times Y} (Y \times_{Y/G} Y) = Y \times_{Y \times Y} (Y \times Y) \times_{Y/G \times Y/G} (Y/G)$$

$$= Y \times_{Y/G \times Y/G} (Y/G) = Y \times_{Y/G} (Y/G) \times_{Y/G \times Y/G} (Y/G) = Y \times_{Y/G} I(Y/G).$$

The resulting isomorphism $I(Y, G) \cong Y \times_{Y/G} I(Y/G)$ yields the left cartesian square in (5.2.3). Further, since $Y \to Y/G$ is a G-torsor, it follows that $I(Y, G) \to I(Y/G)$ is a G-torsor. This implies (i).

By the universal property of quotient stacks, the map $\mathrm{pr}_G \times \mathrm{pr}_Y : I(Y, G) \to G \times Y$ followed by the projection to $(G \times Y)/G$ factors through $I(Y, G)/G$, which is isomorphic to $I(Y/G)$ by (1). Thus, we obtain a diagram

$$I(Y/G) \xeq{\;\text{part (1)}\;} I(Y, G)/G \xrightarrow{\;\mathrm{pr}_G \times \mathrm{pr}_Y\;} (G \times Y)/G \xrightarrow{\;\mathrm{pr}_1\;} G/_{\mathrm{Ad}}G. \tag{5.2.4}$$

Let $p_G : I(Y/G) \to G/_{\mathrm{Ad}} G$ be the composite map. Note that the map $q_I \times p_G : I(Y, G) \to I(Y/G) \times_{G/_{\mathrm{Ad}} G} G$ is a morphism of G-torsors on $I(Y/G)$, hence it is an isomorphism. It follows that the right square in (5.2.3) is cartesian, completing the proof. $\qquad\square$

In the Lie algebra case, we put $I^+(Y/G) := Y/G \times_{(\mathrm{T}Y)/G} (\mathfrak{g} \times Y)/G$, where $Y/G \to (\mathrm{T}Y)/G$ is given by the zero section. The Lie algebra counterpart of diagram (5.2.3) reads

$$
\begin{array}{ccccc}
Y & \xleftarrow{\quad \mathrm{pr}_Y^+ \quad} & I(Y, \mathfrak{g}) & \xrightarrow{\quad \mathrm{pr}_{\mathfrak{g}} \quad} & \mathfrak{g} \\
\downarrow{\scriptstyle q_Y} & \square & \downarrow{\scriptstyle q_I^+} & \square & \downarrow{\scriptstyle q_G^+} \\
Y/G & \xleftarrow{\quad p_{Y/G}^+ \quad} & I^+(Y/G) & \xrightarrow{\quad p_{\mathfrak{g}} \quad} & \mathfrak{g}/G.
\end{array}
\qquad (5.2.5) \quad \boxed{\texttt{llie}}
$$

In particular, the map q_I^+ is a G-torsor, so one has a canonical isomorphism $I^+(Y/G) = I(Y, \mathfrak{g})/G$.

5.3. A trace formula. Below, we will freely use the notation of diagram (5.2.3) and put $X = Y/G$. Thus, we have the projections $p_X : I(X) \to X$ and $p_G : I(X) \to G/_{\mathrm{Ad}} G$, respectively. Recall that an object of $I(X)$ is a pair $z = (x, f)$ where $x \in X$ and $f \in \mathrm{Aut}(x)$.

$\boxed{\texttt{tr\_prop}}$ **Proposition 5.3.1.** *Let k be a finite field. Then, for any $\mathcal{E} \in D_{\mathrm{abs}}(X)$ and a locally constant sheaf $\mathcal{F} \in D_{\mathrm{abs}}(G/_{\mathrm{Ad}} G)$, we have*

$$
\mathsf{Tr}_{\mathsf{Fr}}|\, \mathrm{R\Gamma}_c(G/_{\mathrm{Ad}} G,\ (p_G)_!p_X^*\mathcal{E}\otimes\mathcal{F}) \;=\; \sum_{[x]\in[X(\mathsf{k})]}\ \sum_{f\in\mathrm{Aut}(x)(\mathsf{k})} \frac{1}{\#\,\mathrm{Aut}(x)(\mathsf{k})}\cdot(\mathsf{Tr}_{\mathsf{Fr}}|\,\mathcal{E}_x)\cdot(\mathsf{Tr}_{\mathsf{Fr}}|\,\mathcal{F}_{p_G(x,f)}). \quad (5.3.2) \quad \boxed{\texttt{tr2}}
$$

Proof. Let $\mathscr{J} := p_X^*\mathcal{E} \otimes p_G^*\mathcal{F}$. This is a sheaf on $I(X)$ and we compute the trace of Frobenius on $\mathrm{R\Gamma}_c(I(X), \mathscr{J})$ in two ways as follows.

On the one hand, using the projection formula, we find

$$
\begin{aligned}
\mathsf{Tr}_{\mathsf{Fr}}|\, \mathrm{R\Gamma}_c(I(X), \mathscr{J}) &= \mathsf{Tr}_{\mathsf{Fr}}|\, \mathrm{R\Gamma}_c(G/_{\mathrm{Ad}} G,\ (p_G)_!(p_X^*\mathcal{E} \otimes p_G^*\mathcal{F})) \\
&= \mathsf{Tr}_{\mathsf{Fr}}|\, \mathrm{R\Gamma}_c(G/_{\mathrm{Ad}} G,\ (p_G)_!p_X^*\mathcal{E} \otimes \mathcal{F}) \;=\; \text{LHS}\,(5.3.2).
\end{aligned}
$$

On the other hand, by the Grothendieck-Lefschetz formula for stacks due to Behrend [B1, Theorem 6.4.9], one has

$$
\mathsf{Tr}_{\mathsf{Fr}}|\, \mathrm{R\Gamma}_c(I(X), \mathscr{J}) \;=\; \sum_{[z]\in[I(X)(\mathsf{k})]} \frac{1}{\#\,\mathrm{Aut}(z)} \mathsf{Tr}_{\mathsf{Fr}}|\, \mathscr{J}_z.
$$

We write $z = (x, f)$ and put $A(x, f) := \mathrm{Aut}(z)(\mathsf{k})$. An object $z' = (x', f') \in I(X)$ is isomorphic to (x, f) iff there is an isomorphism $\varphi : x \to x'$ such that $f' = \varphi \circ f \circ \varphi^{-1}$. Hence, the set $[I(X)(\mathsf{k})]$ is formed by the pairs $([x], C)$ where $[x] \in [X(\mathsf{k})]$ and C is a conjugacy class in the group $A(x) := \mathrm{Aut}(x)(\mathsf{k})$. Furthermore, for an object $z \in I(X)(\mathsf{k})$ in the isomorphism class $([x], C)$ the group $\mathrm{Aut}(z)(\mathsf{k})$ is isomorphic to $A(x, C)$, the centralizer in $A(x)$ of an element of the conjugacy class C. Thus, writing $\mathrm{Cl}\, A(x)$ for the set of conjugacy classes of the group $A(x)$, we compute

$$
\begin{aligned}
\mathsf{Tr}_{\mathsf{Fr}}|\, \mathrm{R\Gamma}_c(I(X), \mathscr{J}) &= \sum_{([x],f)\in[I(X)(\mathsf{k})]} \frac{1}{\#\,\mathrm{Aut}(x,f)(\mathsf{k})} \cdot \mathsf{Tr}_{\mathsf{Fr}}|\, (\mathcal{E}_x \otimes \mathcal{F}_{p_G(x,f)}) && (5.3.3) \quad \boxed{\texttt{I-tr}} \\
&= \sum_{[x]\in[X(\mathsf{k})]}\ \sum_{C\in\mathrm{Cl}\, A(x)}\ \sum_{f\in C} \frac{1}{\#A(x,C)} \cdot (\mathsf{Tr}_{\mathsf{Fr}}|\,\mathcal{E}_x)\cdot(\mathsf{Tr}_{\mathsf{Fr}}|\,\mathcal{F}_C) \\
&= \sum_{[x]\in[X(\mathsf{k})]} \frac{1}{\#A(x)} \cdot (\mathsf{Tr}_{\mathsf{Fr}}|\,\mathcal{E}_x)\cdot\left(\sum_{C\in\mathrm{Cl}\, A(x)} \frac{\#A(x)}{\#A(x,C)} \cdot \mathsf{Tr}_{\mathsf{Fr}}|\,\mathcal{F}_C\right)
\end{aligned}
$$

44

$$= \sum_{[x]\in[X(\mathsf{k})]} \frac{1}{\#A(x)} \cdot (\mathsf{Tr}_{\mathsf{Fr}}\,|\,\mathcal{E}_x) \cdot \Big(\sum_{f\in A(x)} \mathsf{Tr}_{\mathsf{Fr}}\,|\,\mathcal{F}_{p_G(x,f)} \Big) = \mathrm{RHS}\,(5.3.2). \qquad \square$$

6. Factorization property of inertia stacks

6.1. **Moduli stacks.** The goal of this subsection is to formalize the concept of moduli stack of objects of a given k-linear category. The approach suggested below is, perhaps, neither the most optimal nor the most general one. In any case, it will be sufficient for the limited purposes of the present paper.

It is well known that the adequate setting for studying moduli problems is that of *stacks* since, usually, moduli spaces do not exist as schemes. Let k be an arbitrary field and $\mathscr{C}_\mathsf{k}$ a k-linear category. We would like to consider a moduli stack of objects of $\mathscr{C}_\mathsf{k}$. For such a stack to be an Artin stack requires, in particular, being able to define the notion of a flat family of objects of $\mathscr{C}_\mathsf{k}$ over a scheme. For the purposes of this paper, we formalize this as follows.

Let AffSch be the category of affine schemes over k. Given a fibered category $\pi_{\mathscr{C}} : \mathscr{C} \to$ AffSch, let $\mathfrak{X}(\mathscr{C})$ be a not necessarily full subcategory of $\mathscr{C}$ that has the same objects as $\mathscr{C}$ and such that the morphisms in $\mathfrak{X}(\mathscr{C})$ are defined to be the cartesian arrows, cf. [Ols]. The category $\mathfrak{X}(\mathscr{C})$ is a prestack, i.e. it is fibered in groupoids over AffSch.

as1

Definition 6.1.1. A *sheaf-category* over k is a fibered category $\pi_{\mathscr{C}} : \mathscr{C} \to$ AffSch equipped, for each $S \in$ AffSch, with the structure of a k$[S]$-linear category on the fiber $\mathscr{C}_S := \pi_{\mathscr{C}}^{-1}(S)$, the fiber of $\mathscr{C}$ over S, such that for every morphism $f : S' \to S$, in AffSch, we have

(1) The pull-back functor $f^* : \mathscr{C}_S \to \mathscr{C}_{S'}$ (defined up to a canonical isomorphism) is k$[S]$-linear.
(2) The natural morphism k$[S'] \otimes_{\mathsf{k}[S]} \mathrm{Hom}(x,y) \to \mathrm{Hom}(f^*x, f^*y)$ is an isomorphism of k$[S']$-modules, for all objects x, y of $\mathscr{C}_S$.

In addition, we require the prestack $\mathfrak{X}(\mathscr{C})$ be a stack, to be called *the moduli stack* of objects of $\mathscr{C}$.

We let $\mathscr{C}_\mathsf{k} := \mathscr{C}(\mathrm{Spec}\,\mathsf{k})$, a k-linear category, and refer to $\mathscr{C}$ as a *sheaf-category enhancement* of $\mathscr{C}_\mathsf{k}$.

The descent property of $\mathfrak{X}(\mathscr{C})$, which is part of the definition of a stack, guarantees that the presheaf $\mathscr{C} : S \mapsto \mathscr{C}_S$ is, in fact, a sheaf in fppf topology. Thus, the name 'sheaf-category' is legitimate.

toen

Remark 6.1.2. (i) In the case of an abelian category, the notion of flat family of objects has been studied by Lowen and Van den Bergh [LvdB], and also by Joyce [J]. Unfortunately, it is not sufficient for applications we are interested in to restrict to the case of abelian categories (there is a more general notion of *quasi-abelian category*, cf. [An], [BvdB, Appendix B], that is sufficient for most applications). In any case, all examples of k-linear categories considered in this paper come equipped with a natural enhancement to a sheaf-category. So, in the rest of the paper we simply work with sheaf-categories and will not discuss whether or not a given k-linear category comes from some sheaf-category.

(ii) A much better, and a more correct approach would be to use formalism of derived algebraic geometry as follows. Let $\mathcal{D}$ be a finite-type dg category, i.e. a dg-category such that its category of modules is equivalent to the category modules over some homotopically finitely presented dg-algebra. For any t-structure on $\mathcal{D}$, Toën and Vaquie [TV] constructed a moduli space of objects of the heart of the t-structure. This moduli space is an ∞-stack, which is a direct limit of derived n-stacks for all n. Using [TV, Corollary 3.21], one can find a substack which is in fact a derived Artin 1-stack. Finally, one can consider an underlying classical stack of that Artin 1-stack.

(iii) A finite type category is saturated in the sense of Kontsevich. In such case, an analogue of (2) in Definition 6.1.1 would says that the natural functor k$[S'] \otimes_{\mathsf{k}[S]} \mathscr{C}_S \to \mathscr{C}_{S'}$ is an equivalence. $\Diamond$

45

The sheaf-category $\mathscr{C}$ is said to be *Karoubian* if, for any affine scheme U of finite type over k, the category $\mathscr{C}(U)$ is Karoubian, i.e. for any $M \in \mathscr{C}(U)$ every idempotent morphism $p \in \mathrm{Hom}_{\mathscr{C}(U)}(M, M)$, $p^2 = p$, is the projection to a direct summand of M.

<div style="border:1px solid; display:inline-block; padding:2px">clear</div> **Lemma 6.1.3.** *Let $\mathscr{C}$ be a Karoubian sheaf-category. Then, for any separable field extension $K/$k, there is a natural equivalence $K \otimes_{\mathsf{k}} \mathscr{C}_{\mathsf{k}} \cong \mathscr{C}(\mathrm{Spec}\, K)$ and a natural bijection of the set of isomorphism classes of objects of category $K \otimes_{\mathsf{k}} \mathscr{C}$ with the set $\mathfrak{X}(\mathscr{C})(K)$.*

Proof. It will be convenient to consider a more general setting. Fix a finite *étale* morphism $S' \to S$. Given a finite étale morphism $U \to S$, let $\mathscr{C}'(U) := \mathscr{C}_{U \times_S S'}$ ($= \mathcal{O}_{U \times_S S'} \otimes_{\mathcal{O}_U} \mathscr{C}(U)$), resp. $\mathscr{C}''(U) := \mathscr{C}_{U \times_S S'}$. We claim that the fibers over S of the fibered categories $\mathscr{C}'$ and $\mathscr{C}''$ are equivalent.

To prove the claim, we introduce a category T of 'trivialized' morphisms $U \to S$. That is, an object of T is an étale morphism $U \to S$ equipped with an isomorphism $U \times \Xi \to U$ for some finite set Ξ. Let $\mathscr{C}^\Xi$ be a fibered category over T such that its fiber over an object $U \in T$ as above is $\prod_{\xi \in \Xi} \mathscr{C}(U)$. Then, the Karoubi property of $\mathscr{C}$ implies that the pullback of the category $\mathscr{C}'$ to such an $U \in T$ is canonically equivalent to $\mathscr{C}^\Xi$. Therefore, applying descent to the sheaf of categories $U \mapsto \mathscr{C}(U)$, we deduce that $\mathscr{C}'(U)$ is equivalent to the category of sections of fiber products of categories $\mathscr{C}^\Xi$ over the the diagram $U \rightrightarrows U \otimes_S U \rrbracket U \otimes_S U \otimes_S U \dots$. This yields our claim since all the above clearly applies to the sheaf of categories $U \mapsto \mathscr{C}''(U)$ as well.

The statement of the lemma follows as a special case where the morphism $S' \to S$ is taken to be $\mathrm{Spec}\, K \to \mathrm{Spec}\, \mathsf{k}$. $\qquad\square$

<div style="border:1px solid; display:inline-block; padding:2px">bun</div> *Example* 6.1.4. An I-graded coherent sheaf on a scheme U is, by definition, a direct sum $\mathcal{V} = \oplus_{i \in I} \mathcal{V}_i$ where V_i is a coherent sheaf on U. Equivalently, one can view an I-graded coherent sheaf as a coherent sheaf on a disjoint union of I copies of U. Let $\mathcal{V}ect^I(U)$, the category of I-graded algebraic vector bundles on U, be a full subcategory of the category of I-graded coherent sheaves whose objects are locally free sheaves. The category $\mathcal{V}ect^I(U)$ has a natural enhancement to a sheaf-category defined by $\mathcal{V}ect^I_U : S \mapsto \mathcal{V}ect^I(U \times S)$. The corresponding moduli stack $\mathfrak{X}(\mathcal{V}ect^I_U)$ is a disjoint union $\mathfrak{X}(\mathcal{V}ect^I_U) = \sqcup_{\mathbf{v} \geq 0} Bun_{\mathbf{v}}(U)$ where $Bun_{\mathbf{v}}(U)$, $\mathbf{v} = (v_i)_{i \in I}$, is the stack that parametrizes vector bundles $\mathcal{V} = \oplus_{i \in I} \mathcal{V}_i$, on U, such that $\mathrm{rk}\, \mathcal{V}_i = v_i$, $\forall i \in I$.

In the special case where I is a one element set we use simplified notation $\mathcal{V}ect(U) = \mathcal{V}ect^I(U)$, resp. $\mathcal{V}ect_U = \mathcal{V}ect^I_U$, etc.

<div style="border:1px solid; display:inline-block; padding:2px">fun-def</div> **Definition 6.1.5.** A functor of sheaf-categories is a functor $F : \mathscr{C} \to \mathscr{C}'$ such that $\pi_{\mathscr{C}'} \circ F = \pi_{\mathscr{C}}$ strictly, and the induced functor $\mathscr{C}_S \to \mathscr{C}'_S$ is $\mathsf{k}[S]$-linar for all $S \in \mathsf{AffSch}$.

We will use the notation $\mathcal{V}ect^I = \mathcal{V}ect(\mathrm{pt})$, a sheaf enhancement of the categry of finite dimensional vector spaces. By definition, one has $\mathfrak{X}_{\mathbf{v}}(\mathcal{V}ect^I) \cong \mathrm{pt}/\mathrm{GL}_{\mathbf{v}}$. This stack comes equipped with a 'tautological' universal I-graded vector bundle of I-graded rank $\mathbf{v}$.

Let $\mathscr{C}$ be a sheaf category and $F : \mathscr{C} \to \mathcal{V}ect^I$ a sheaf functor . Giving such a functor amounts to giving, for every affine scheme S, a $\mathsf{k}[S]$-linear functor $F(S) : \mathscr{C}(S) \to \mathrm{Vect}(S)$ and, for every morphism $j : S' \to S$, of affine schemes over k, an isomorphism of functors $F(S') \circ j^*_{\mathscr{C}} \cong j^* \circ F(S)$, where $\mathscr{C}(j) : \mathscr{C}(S) \to \mathscr{C}(S')$ is a functor induced by the presheaf structure.

The sheaf-functor F induces a morphism $\mathfrak{X}(\mathscr{C}) \to \mathfrak{X}(\mathcal{V}ect^I)$, of moduli stacks. For any dimension vector $\mathbf{v}$, one has a substack $\mathfrak{X}_{\mathbf{v}}(\mathscr{C}) = \mathfrak{X}(\mathscr{C}) \times_{\mathfrak{X}(\mathcal{V}ect^I)} \mathfrak{X}(\mathcal{V}ect^I_{\mathbf{v}})$ parametrizing all objects $x \in \mathscr{C}$ such that $\mathbf{rk}\, F(x) = \mathbf{v}$. The substack $\mathfrak{X}_{\mathbf{v}}(\mathscr{C})$ is both open and closed in $\mathfrak{X}(\mathscr{C})$, so $\mathfrak{X}(\mathscr{C}) = \sqcup_{\mathbf{v}} \mathfrak{X}_{\mathbf{v}}(\mathscr{C})$ is a disjoint union. The universal vector bundle on $\mathfrak{X}(\mathcal{V}ect^I_{\mathbf{v}})$ gives, by pull-back, a natural I-graded rank $\mathbf{v}$ vector bundle $\mathfrak{F}_{\mathbf{v}}$ on $\mathfrak{X}_{\mathbf{v}}(\mathscr{C})$. Let $\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F) \to \mathfrak{X}_{\mathbf{v}}(\mathscr{C})$ be the 'frame bundle' of the vector bundle $\mathfrak{F}_{\mathbf{v}}$, cf. §5.1. Thus, $\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)$ is a $\mathrm{GL}_{\mathbf{v}}$-torsor over $X_{\mathbf{v}}(\mathscr{C})$ and $\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)/\mathrm{GL}_{\mathbf{v}} \cong \mathfrak{X}_{\mathbf{v}}(\mathscr{C})$. An object of the category $\mathrm{Spec}\, \mathsf{k} \to \mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)$ is a pair $y = (x, b)$ where x is an object of $\mathscr{C}_{\mathsf{k}}$ and b is a

frame, i.e. an isomorphism $F(x) \cong \oplus_{i\in I}\mathsf{k}^{v_i}$, of I-graded vector spaces. We will refer to $\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)$ as the 'framed stack' for $\mathfrak{X}_{\mathbf{v}}(\mathscr{C})$.

Example 6.1.6. Let S be a scheme over k and $D = \sqcup_{i\in I} D_i \subset S$ a disjoint union of closed sub-schemes. Given a dimension vector $\mathbf{m} \in \mathbb{Z}_{\geq 0}^I$ one has a k-linear category $\mathcal{V}ect(S, D, \mathbf{m})$ of parabolic bundles, cf. §1.4. An object of $\mathcal{V}ect(S, D, \mathbf{m})$ is a locally free sheaf $\mathcal{V}$ on S equipped, for each $i \in I$, with a partial flag of subbundles $\mathcal{V}|_{D_i} = \mathcal{V}_i^{(0)} \supseteq \mathcal{V}_i^{(1)} \supseteq \ldots \supseteq \mathcal{V}_i^{(m_i)} = 0$, i.e. a chain of coherent subsheaves such that each quotient $\mathcal{V}_i^{(j)}/\mathcal{V}_i^{(j+1)}$ is a locally free sheaf on D_i. Category $\mathcal{V}ect(S, D, \mathbf{m})$ has an enhancement to a sheaf category. Specifically, for any test scheme U let $D \times U := \sqcup_{i\in I} D_i \times U$, a disjoint union of closed subschemes of $S \times U$. Then, the sheaf category is given by the presheaf $U \mapsto \mathcal{V}ect(U \times S, D \times S, \mathbf{m})$. We will often abuse the notation and write $\mathcal{V}ect(S, D, \mathbf{m})$ for the sheaf enhancement of the category of parabolic bundles denoted by the same symbol.

Below, we are mostly be interested in the special case where S is a smooth curve and $D = \{c_i \in S(\mathsf{k}), \ i \in I\}$ a collection of marked points. In this case, we have the natural identification $\{c_i\} \times U \cong U$, so we may (and will) view $V_i^{(j)}/V_i^{(j+1)}$ as a locally free sheaf on U. The assignment

$$\mathcal{V} \mapsto F(\mathcal{V}) = \bigoplus_{i\in I} \bigoplus_{1\leq j\leq m_i} V_i^{(j)}/V_i^{(j+1)}.$$

yields a sheaf-functor $F : \mathcal{V}ect(C, D, \mathbf{m}) \to \mathcal{V}ect^{\Xi(\mathbf{m})}(\mathrm{pt}) = \mathcal{V}ect^{\Xi(\mathbf{m})}$. For a type $\mathbf{r} \in \Xi(\mathbf{m})$, the corresponding framed stack $\mathfrak{X}_{\mathbf{r}}(\mathcal{V}ect(C, D, \mathbf{m}), F)$ classifies pairs $(\mathcal{V}, b)$ where $\mathcal{V}$ is a parabolic bundle on S of type $\mathbf{r}$ and $b = (b_i^{(j)})_{(i,j)\in\Xi(\mathbf{m})}$ is a collection of bases, where $b_i^{(j)}$ is a k-basis of the vector space $V_i^{(j)}/V_i^{(j+1)}$. The forgetful map $\mathfrak{X}_{\mathbf{r}}(\mathcal{V}ect(S, D, \mathbf{m}), F) \to \mathfrak{X}_{\mathbf{r}}(\mathcal{V}ect(S, D, \mathbf{m}))$ is a $G_{\mathbf{r}}$-torsor.

6.2. Endomorphisms and automorphisms.

Fix a sheaf-category $\mathscr{C}$ and write $\mathfrak{X} = \mathfrak{X}(\mathscr{C})$. Let S be an affine scheme, AffSch_S the category of affine schemes over S, and $\mathfrak{X}(S)$ the groupoid corresponding to S. We write $\mathrm{Aut}(x)$ for the stabilizer group scheme of an object $x \in \mathfrak{X}(S)$.

For any objects $x, y \in \mathfrak{X}(S)$, let $\mathrm{Hom}_{x,y}$ be a functor $\mathsf{AffSch}_S \to \mathsf{Sets}$ defined by the assignment $(p : S' \to S) \mapsto \mathrm{Hom}_{\mathscr{C}_{S'}}(p^*x, p^*y)$.

Lemma 6.2.1. (i) *The functor $\mathrm{Hom}_{x,y}$ is represented by an affine scheme $\underline{\mathrm{Hom}}(x, y)$ that has the natural structure of a generalized vector bundle on S in the sense of Definition 7.1.1. Furthermore, there is a canonical $\mathsf{k}[S]$-module isomorphism of $\mathrm{Hom}_{\mathscr{C}_S}(x, y)$ and $\Gamma(S, \underline{\mathrm{Hom}}(x, x))$, the space of sections of this generalized vector bundle.*

(ii) The stack $\mathfrak{X}$ has an affine diagonal. In particular, the projection $I(\mathfrak{X}(\mathscr{C})) \to \mathfrak{X}(\mathscr{C})$ is an affine morphism and $\mathfrak{X}$ is a stack with affine stabilizers.

Proof. For any $f \in \mathrm{Hom}_{\mathscr{C}_{S'}}(p^*x, p^*y)$, the matrix $\begin{vmatrix} \mathrm{Id}_x & 0 \\ f & \mathrm{Id}_y \end{vmatrix}$ gives an automorphism of $p^*x \oplus p^*y$, so one has an imbedding of the set $\mathrm{Hom}_{\mathscr{C}_{S'}}(p^*x, p^*y)$ into the set $\mathrm{Aut}(p^*(x \oplus y))$. The functor $(p : S' \to S) \mapsto \mathrm{Aut}(p^*x \oplus p^*y)$ is represented by the group scheme $\mathrm{Aut}(x \oplus y) \in \mathsf{AffSch}_S$. Hence, the presheaf of sets over AffSch_S associated with the functor $\mathrm{Hom}_{x,y}$ is a sub-presheaf of the presheaf associated with the functor represented by $\mathrm{Aut}(x \oplus y)$. Therefore, the functor $\mathrm{Hom}_{x,y}$ is represented by a sub-groupscheme of $\mathrm{Aut}(x \oplus y)$. It may be characterized as follows. We have a natural homomorphism $\gamma : \mathbb{G} \to \mathrm{Aut}(x \oplus y)$ given by $t \mapsto \mathrm{diag}(\mathrm{Id}_x, t \cdot \mathrm{Id}_y)$. Let $\mathbb{G}$ act on $\mathrm{Aut}(x \oplus y)$ by group scheme automorphisms via conjugation. Then, we have $\underline{\mathrm{Hom}}(x, y) = \{g \in \mathrm{Aut}(x \oplus y) \mid \lim_{t\to 0} \gamma(t)g\gamma(t)^{-1} \, \mathrm{Id}\}$. This can be reformulated scheme-theoretically by identifying $\underline{\mathrm{Hom}}(x, y)$ with the scheme of $\mathbb{G}$-equivariant maps $u : \mathbb{A} \to \mathrm{Aut}(x \oplus y)$ such that $u(0) = 0$. It follows from this last

47

reformulation that the scheme $\underline{\mathrm{Hom}}(x, y)$ is affine. The rest of the proof of part (i) are left to the reader.

To prove (ii) observe first that, for any $x, y, z \in \mathfrak{X}(S)$, the composition map $\mathrm{Hom}_{\mathscr{C}_S}(x, y) \times \mathrm{Hom}_{\mathscr{C}_S}(y, z) \to \mathrm{Hom}_{\mathscr{C}_S}(x, z)$ induces a morphism $\circ : \underline{\mathrm{Hom}}(x, y) \times \underline{\mathrm{Hom}}(y, z) \to \underline{\mathrm{Hom}}(x, z)$, of affine schemes over S.

The pair (x, y) gives a morphism $S \to \mathfrak{X} \times \mathfrak{X}$ and we also have a morphism $\{(\mathrm{Id}_x, \mathrm{Id}_y)\}_S : S \to \underline{\mathrm{Hom}}(x, y) \times_S \underline{\mathrm{Hom}}(y, x)$, of schemes over S. Further, let $\underline{\mathrm{Hom}}(x, y) \times_S \underline{\mathrm{Hom}}(y, x) \to S$ be a morphism given by $(f, g) \mapsto (g \circ f, f \circ g))$. With this notation, one has a natural isomorphism

$$S \to \mathfrak{X} \times \mathfrak{X} = (\underline{\mathrm{Hom}}(x, y) \times_S \underline{\mathrm{Hom}}(y, x)) \times_{\underline{\mathrm{Hom}}(x,x) \times_S \underline{\mathrm{Hom}}(y,y)} \{(\mathrm{Id}_x, \mathrm{Id}_y)\}_S$$

Part (i) implies that the RHS of this formula is affine over S. It follows that $S \times_{\mathfrak{X} \times \mathfrak{X}} \mathfrak{X}$ is an affine schemes over S, proving (ii). $\qquad\square$

In the special case $x = y$, we put $\mathrm{End}(x) := \mathrm{Hom}_{\mathscr{C}_S}(x, x)$. By the above lemma, we have $\mathrm{End}(x) = \Gamma(S, \underline{\mathrm{Hom}}(x, x))$. Let $\underline{\mathrm{Aut}}(x)$ be a closed subscheme of $\underline{\mathrm{Hom}}(x, x) \times_S \underline{\mathrm{Hom}}(x, x)$ defined by the equations $g' \circ g = g \circ g' = \mathrm{Id}$. It is easy to show that $\underline{\mathrm{Aut}}(x)$ is canonically isomorphic to $\mathrm{Aut}\, x$ as a group scheme over S.

Now, we fix a sheaf functor $F : \mathscr{C} \to \mathcal{V}ect^I$. For $x \in \mathfrak{X}(S)$, let $F_x : \mathrm{End}(x) \to \mathrm{End}\, F(x)$ be the $\mathsf{k}[S]$-algebra homomorphism induced by F.

end_lem | **Lemma 6.2.2.** *For any $x \in \mathfrak{X}(S)$, the ideal $\mathrm{Ker}\, F_x$ is nilpotent, i.e. we have $(\mathrm{Ker}\, F_x)^N = 0$ for $N \gg 0$.*

Proof. Put $B = \mathsf{k}[S]$. First, we consider the case where $B = \mathsf{k}$. Let J be the kernel of the map F_x and let $A := \mathsf{k} + J$. Thus, A is a finite dimensional unital subalgebra of $\mathrm{End}(x)$ and J is a codimension one two-sided ideal of A. The lemma would follow provided we show that J is contained in $\mathfrak{N}$, the nilradical of A (which is the same as the Jacobson radical).

Assume that $J \not\subseteq \mathfrak{N}$. Let $\bar{J} \cong J/(J \cap \mathfrak{N})$ be the image of J in $A/\mathfrak{N}$. Thus, $\bar{J}$ is a nonzero codimension one two-sided ideal of $A/\mathfrak{N}$. Any two-sided ideal of $A/\mathfrak{N}$, a semisimple finite dimensional algebra, is generated by a central idempotent.

We deduce that the ideal $\bar{J}$ contains a nonzero idempotent $\bar{e}$. The ideal $\mathfrak{N}$ being nilpotent, the idempotent $\bar{e}$ can be lifted to an idempotent $e \in A$ such that $e \pmod{\mathfrak{N}} = \bar{e} \in \bar{J}$. Since $A = \mathsf{k} + J$ this implies that $e \in J$. Thus, we have constructed a nonzero idempotent $e \in \mathrm{End}(x)$ that belongs to the kernel of the homomorphism $F_x : \mathrm{End}(x) \to \mathrm{End}_{\mathsf{k}}\, F(x)$.

Associated with the decomposition $1 = e + (1 - e)$, in $\mathrm{End}(x)$, there is a direct sum decomposition $x = x' \oplus x''$, in $\mathscr{C}$. This gives a vector space decomposition $F(x) = F(x') \oplus F(x'')$. It is clear, by functoriality, that the map $F_x(e) : F(x) \to F(x)$ is equal to the composition $F(x) \twoheadrightarrow F(x') \hookrightarrow F(x)$, of the first projection and an imbedding of the first direct summand. We deduce

$$e \neq 0 \;\Rightarrow\; x' \neq 0 \;\Rightarrow\; F(x') \neq 0 \;\Rightarrow\; F_x(e) \neq 0,$$

On the other hand, by construction we have $e \in \mathrm{Ker}(F_x)$, hence $F_x(e) = 0$. The contradiction implies that $J \subset \mathfrak{N}$. This completes the proof in the special case $B = \mathsf{k}$.

We now consider the general case, and put $S = \mathrm{Spec}\, B$. Thus, $\mathrm{End}(x)$ is a B-algebra which is finite as a B-module. For every closed point $s : \mathrm{Spec}\, \mathsf{k} \hookrightarrow S$, one has the composite map $x_s : \mathrm{Spec}\, \mathsf{k} \to X$ and the corresponding finite dimensional k-algebra $\mathrm{End}(x_s)$. The algebra $\mathrm{End}(x)$ being finite over B, there is an integer N such that $\dim_{\mathsf{k}} \mathrm{End}(x_s) < N$ for all closed points $s \in S$. Observe further that given an arbitrary k-algebra R such that $\dim_{\mathsf{k}} R < N$ and a two-sided ideal $\mathfrak{J}$ of R such that $\mathfrak{J}^m = 0$ for some $m \gg 0$, one automatically has that $\mathfrak{J}^N = 0$. Therefore, from the special case $B = \mathsf{k}$ of the lemma, we deduce that $(\mathrm{Ker}\, F_{x_s})^N = 0$ holds for all s.

We claim that $(\mathrm{Ker}\, F_x)^N = 0$. Indeed, if $a_1 \cdots a_N \neq 0$ for some elements $a_1, \dots, a_N \in \mathrm{Ker}\, F_x$ then one can find a closed point $s \in S$ such that $a_1|_s \cdots a_N|_s \neq 0$. That would contradict the equation $(\mathrm{Ker}\, F_{x_s})^N = 0$, since $a_j|_s \in \mathrm{Ker}\, F_{x_s}$, $j = 1, \dots, N$. $\qquad\square$

Fix a sheaf-functor $F : \mathscr{C} \to \mathcal{V}ect^I$. We will use the notation $X = \mathfrak{X}_{\mathbf{v}}(\mathscr{C})$, $Y = \mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)$, resp. $\mathfrak{g} = \mathfrak{gl}_{\mathbf{v}}$, $G = \mathrm{GL}_{\mathbf{v}}$ and $\mathfrak{g}(S) = \mathsf{k}[S]$, $G(S)$ for S-poits of G. For $y = (x, b) \in Y(S)$ and $f \in \mathrm{End}(x)$ there is a unique element $p_{\mathfrak{g},y}(f) \in \mathfrak{g}(S)$ such that $F_x(f)(b) = p_{\mathfrak{g},y}(f)(b)$. Let $\mathfrak{g}_y$ be the image of the map $f \mapsto p_{\mathfrak{g},y}(f)$. Thus, we have an exact sequence

$$0 \longrightarrow \mathrm{Ker}\, F_x \longrightarrow \mathrm{End}(x) \xrightarrow{\;p_{\mathfrak{g},y}\;} \mathfrak{g}_y. \qquad (6.2.3) \quad \boxed{\texttt{algalg}}$$

Observe that the map $p_{\mathfrak{g},y}$ is a homomorphism of associative subalgebras.

The group $\mathrm{Aut}(y)$, resp. $\mathrm{Aut}(x)$, is the group of units of the associative algebra $\mathrm{End}(y)$, resp. $\mathrm{End}(x)$. Furthermore, the restriction of this map to $\mathrm{Aut}(x)$ is nothing but the map p_y in (5.1.1). Thus, the exact sequence in (5.1.1) is obtained from (6.2.3), an exact sequence of associative algebras, by restriction to the groups of units. On the other hand, clearly, one has $\mathrm{Lie}\,\mathrm{Aut}(x) = \mathrm{End}(x)$, where $\mathrm{End}(x)$ is viewed as a Lie algebra with the commutator bracket. Therefore, (6.2.3) may be identified with the exact sequence of Lie algebras induced by (5.1.1). In particular, the group G_y is the group of units of the associative algebra $\mathfrak{g}_y$ and we have $\mathrm{Lie}\,G_y = \mathfrak{g}_y$.

$\boxed{\texttt{unip}}$ **Corollary 6.2.4.** *Let $y \in Y(\mathsf{k})$ and $x \in X(\mathsf{k})$ be the image of x. Then,*
(i) *The groups $\mathrm{Aut}(x)$ and G_y are connected.*
(ii) *The group $\mathrm{Aut}(y)$ is unipotent and we have $\mathrm{Aut}(y) = \mathrm{Ker}\,\big[F_x : \mathrm{Aut}(x) \to \mathrm{GL}(F(x))\big]$.*

Proof. It is clear that the group of units of a finite dimensional algebra is connected. It follows that the group $\mathrm{Aut}(x)$, hence also G_y, is connected, proving (i). The equation of part (ii) follows from the exact sequence (5.1.1). Hence, we deduce $\mathrm{Aut}(y) = \mathrm{Id} + \mathrm{Ker}[F_x : \mathrm{End}(x) \to \mathrm{End}\,F(x)]$. Lemma 6.2.2 insures that this group is unipotent, and (ii) follows. $\qquad\square$

$\boxed{\texttt{unicor}}$ **Corollary 6.2.5.** *If the functor F is faithful then, for any $\mathbf{v}$, the stack Y is an algebraic space.*

Proof. This is an immediate consequence of Corollary 6.2.4(ii) and [LM, Corollaire (8.1.1)(iii)]. $\quad\square$

$\boxed{\texttt{rational}}$ **Corollary 6.2.6.** *If k is a finite field then each fiber of the natural map $[Y(\mathsf{k})] \to [X(\mathsf{k})]$ is a $G(\mathsf{k})$-orbit.*

Proof. The fiber of the map $[Y(\mathsf{k})] \to [(Y/G)(\mathsf{k})]$ over $x \in X(\mathsf{k})$ is isomorphic to the set of k-rational points of a G-orbit which is defined over k. Our claim is a consequence of a combination of two well known results. The first result says that any such orbit contains a k-rational point, say $y \in Y(\mathsf{k})$. Then, G_y is a connected subgroup of $G = \mathrm{GL}_{\mathbf{v}}$ defined over k, and we can identify the fiber with $(G/G_y)(\mathsf{k})$. The second result says that, the group G_y being connected, the $G(\mathsf{k})$-action on $(G/G_y)(\mathsf{k})$ is transitive. $\qquad\square$

Note for any $y = (x, b) \in Y(\mathsf{k})$ the group $\mathrm{Aut}(x)$, resp. $\mathrm{Aut}(y)$, contains the subgroup $\mathbb{G}_{\mathrm{scalars}}$ of scalar automorphisms and one has $p_{\mathfrak{g},y}(\mathbb{G}_{\mathrm{scalars}}) = \mathbb{G}^{\mathbf{v}}_{\Delta} \subset G_y$.

$\boxed{\texttt{fitting}}$ **Corollary 6.2.7.** *For any $y = (x, b) \in Y(\mathsf{k})$ we have*
(i) *If x is not absolutely indecomposable then the group G_y contains a torus H defined over k which is a $G(\bar{\mathsf{k}})$-conjugate of a diagonal torus $\mathbb{G}^{\mathbf{v}}_{\mathfrak{A}} \neq \mathbb{G}^{\mathbf{v}}_{\Delta}$.*
(ii) *The following conditions are equivalent:*

x *is absolutely indecomposable* $\;\Leftrightarrow\;$ $\mathrm{Aut}(x)/\mathbb{G}_{\mathrm{scalars}}$ *is unipotent* $\;\Leftrightarrow\;$ $G_y/\mathbb{G}^{\mathbf{v}}_{\Delta}$ *is unipotent.*

Proof. Let J be the (nil)radical of the algebra $\mathrm{End}(x)(\mathsf{k})$ and $A = \mathrm{End}(x)(\mathsf{k})/J$, a finite dimensional semisimple k algebra. By the Wedderburn theorem, we have A is isomorphic to a finite direct sum of matrix algebras over some division algebras of finite dimension over k. Let $B \subset A$ be a direct sum of the subalgebras of diagonal matrices of all these matrix algebras. Thus, we have $B \cong \oplus_{\alpha} Q_{\alpha}$, where each Q_{α} is a division algebra over k. For every α, we choose a maximal subfield $K_{\alpha} \subset Q_{\alpha}$. Then, K_{α} is a finite extension of k. Therefore, the group $\prod_{\alpha} K_{\alpha}^{\times}$ defines,

by Weil restriction, an algebraic subgroup H of $\mathrm{Aut}(x)$, which is defined over k and is such that $H(\mathsf{k}) = \prod_\alpha K_\alpha^\times$. Moreover, H is a torus.

For each α we have $\bar{\mathsf{k}} \otimes_\mathsf{k} K_\alpha = \oplus_{\gamma \in \Gamma_\alpha} \bar{\mathsf{k}} 1_{\alpha,\gamma}$, a direct sum of several copies of the field $\bar{\mathsf{k}}$ with the corresponding unit element being denoted by $1_{\alpha,\gamma}$. The elements $\{1_{\alpha,\gamma}\}$ form a collection of orthogonal idempotents of the algebra $\bar{\mathsf{k}} \otimes_\mathsf{k} A$ which can be lifted to a collection $\{e_{\alpha,\gamma}\}$ of orthogonal idempotents of the algebra $\mathrm{End}(x)(\bar{\mathsf{k}})$. Let $\bar{\mathsf{k}} \otimes_\mathsf{k} x = \oplus_{\alpha,\gamma} x_{\alpha,\gamma}$ be the corresponding direct sum decomposition, where $x_{\alpha,\gamma} \in \mathscr{C}_{\bar{\mathsf{k}}}$. It is clear from the construction that we have $\mathrm{End}(x_{\alpha,\gamma})(\bar{\mathsf{k}}) = e_{\alpha,\gamma} \mathrm{End}(x)(\bar{\mathsf{k}}) e_{\alpha,\gamma} = \bar{\mathsf{k}} \oplus J_{\alpha,\gamma}$, where $J_{\alpha,\gamma}$ is a nilpotent ideal. Thus, $\mathrm{End}(x_{\alpha,\gamma})(\bar{\mathsf{k}})$ is a local $\bar{\mathsf{k}}$-algebra. Hence, $x_{\alpha,\gamma}$ is an indecomposable object, by the Fitting lemma.

We have $F(x) = \oplus F(x_{\alpha,\gamma})$, a direct sum of $\bar{\mathsf{k}} I$-modules. The homomorphism $p_y : \mathrm{Aut}(x) \to \mathrm{GL}$ induces an isomorphism $H(\bar{\mathsf{k}}) = \prod \mathbb{G}(\bar{\mathsf{k}}) \, \mathrm{Id}_{x_{\alpha,\gamma}} \xrightarrow{\sim} p_y(H)(\bar{\mathsf{k}}) = \prod \mathbb{G}(\bar{\mathsf{k}}) \, \mathrm{Id}_{F(x_{\alpha,\gamma})}$ Let $b_{\alpha,\gamma}$ be a $\bar{\mathsf{k}}$-basis of $F(x_{\alpha,\gamma})$ which is compatible with the I-grading. Then, there is a unique element $g \in G$ that takes the given basis b of $F(x)$ to the basis $\sqcup_{\alpha,\gamma} b_{\alpha,\gamma}$. It is clear that $\mathrm{Ad}\, g(p_y(H)(\bar{\mathsf{k}}))$ is a diagonal torus that contains $\mathbb{G}_\Delta^\vee$, proving (i).

To prove (ii) observe that the object x is absolutely indecomposable if and only if $x = x_{\alpha,\gamma}$ for a single (α,γ). The latter holds if and only if $\dim H = \sum_\alpha \dim_\mathsf{k} K_\alpha = 1$, that is, if and only if one has $p_y(H) = \mathbb{G}_\Delta^\vee$. $\qquad\square$

6.3. Factorization of fixed point loci.

Let $G_\mathbf{v} = \mathrm{GL}_\mathbf{v}$, resp. $\mathfrak{g}_\mathbf{v} = \mathfrak{gl}_\mathbf{v}$ and $X_\mathbf{v} = \mathfrak{X}_\mathbf{v}(\mathscr{C})$, $Y_\mathbf{v} = \mathfrak{X}_\mathbf{v}(\mathscr{C}, F)$. We have the stack $I(Y_\mathbf{v}, G_\mathbf{v})$, resp. $I(Y_\mathbf{v}, \mathfrak{g}_\mathbf{v})$, whose objects are triples (x, b, f) where $(x, b) \in Y$ and $f \in \mathrm{Aut}(x)$, resp. $f \in \mathrm{End}(x)$. We recall the isomorphism $X_\mathbf{v} = Y_\mathbf{v}/G_\mathbf{v}$ and observe that the above definition of the stack $I^+(X_\mathbf{v})$ agrees with the definition $I^+(Y_\mathbf{v}/G_\mathbf{v}) = I(Y_\mathbf{v}, \mathfrak{g}_\mathbf{v})/G_\mathbf{v}$, given at the end of section 5.2. Thus, there are natural commutative diagrams of open imbeddings

$$
\begin{array}{ccc}
I(Y_\mathbf{v}, G_\mathbf{v}) & \lhook\joinrel\longrightarrow & I(Y_\mathbf{v}, \mathfrak{g}_\mathbf{v}) \\
\downarrow{\scriptstyle \mathrm{pr}_G} & & \downarrow{\scriptstyle \mathrm{pr}_\mathfrak{g}} \\
G_\mathbf{v} & \overset{\varepsilon_G}{\lhook\joinrel\longrightarrow} & \mathfrak{g}
\end{array}
\qquad\qquad
\begin{array}{ccc}
I(X_\mathbf{v}) & \lhook\joinrel\longrightarrow & I^+(X_\mathbf{v}) \\
\downarrow{\scriptstyle p_G} & & \downarrow{\scriptstyle p_\mathfrak{g}} \\
G_\mathbf{v}/_{\mathrm{Ad}} G_\mathbf{v} & \overset{\varepsilon_{G/G}}{\lhook\joinrel\longrightarrow} & \mathfrak{g}_\mathbf{v}/G_\mathbf{v}
\end{array}
\qquad (6.3.1) \quad \boxed{\texttt{GGAA}}
$$

$\boxed{\texttt{Zcart}}$ **Corollary 6.3.2.** *Each of the diagrams (6.3.1) is cartesian.*

Proof. The map $\mathrm{pr}_\mathfrak{g}$, resp. pr_G, sends (x, b, f) to $p_y(f)$, where $y = (x, b)$. Thus, at the level of objects, the statement amounts to the claim that an endomorphism $f \in \mathrm{End}(x)$ is invertible if and only if the element $p_y(f) \in \mathfrak{g}$ is invertible. The latter claim is a consequence of Lemma 6.2.2. Indeed, we have $p_y(f) \in \mathrm{Im}\, p_y \cong \mathrm{End}(x)/\mathrm{Ker}\, p_y$ and, the ideal $\mathrm{Ker}\, p_y$ being nilpotent, an element of the algebra $\mathrm{End}(x)$ is invertible iff so is its image in $\mathrm{End}(x)/\mathrm{Ker}\, p_y$.

The proof for morphisms is similar and is left for the reader. $\qquad\square$

Let $\mathbf{v}_1, \mathbf{v}_2$ be a pair of dimension vectors. Recall the notation of §4.2 and consider a natural commutative diagram

$$
\begin{array}{ccc}
I^+(X_{\mathbf{v}_1}) \times I^+(X_{\mathbf{v}_2}) & \xrightarrow{\ i_{\mathbf{v}_1,\mathbf{v}_2}\ } & I^+(X_{\mathbf{v}_1+\mathbf{v}_2}) \\
\downarrow{\scriptstyle p_{\mathfrak{g}_{\mathbf{v}_1}} \times p_{\mathfrak{g}_{\mathbf{v}_2}}} & & \downarrow{\scriptstyle p_{\mathfrak{g}_{\mathbf{v}_1+\mathbf{v}_2}}} \\
U^{\mathrm{disj}} := \mathfrak{g}_{\mathbf{v}_1,\mathbf{v}_2}^{\mathrm{disj}}/(G_{\mathbf{v}_1} \times G_{\mathbf{v}_2}) \overset{J_{\mathbf{v}_1,\mathbf{v}_2}}{\lhook\joinrel\longrightarrow} & \mathfrak{g}_{\mathbf{v}_1}/G_{\mathbf{v}_1} \times \mathfrak{g}_{\mathbf{v}_2}/G_{\mathbf{v}_2} & \xrightarrow{\ \imath_{\mathbf{v}_1,\mathbf{v}_2}\ } \mathfrak{g}_{\mathbf{v}_1+\mathbf{v}_2}/G_{\mathbf{v}_1+\mathbf{v}_2},
\end{array}
\qquad (6.3.3) \quad \boxed{\texttt{d12}}
$$

where the map $\imath_{\mathbf{v}_1,\mathbf{v}_2}$, resp. $i_{\mathbf{v}_1,\mathbf{v}_2}$, sends a pair of objects to their direct sum.

Proposition 6.3.4. *The morphism*

$$U^{disj} \underset{\mathfrak{g}_{\mathbf{v}_1}/G_{\mathbf{v}_1} \times \mathfrak{g}_{\mathbf{v}_2}/G_{\mathbf{v}_2}}{\times} \left(I^+(X_{\mathbf{v}_1}) \times I^+(X_{\mathbf{v}_2}) \right) \xrightarrow{\mathrm{Id}\, \times i_{\mathbf{v}_1, \mathbf{v}_2}} U^{disj} \underset{\mathfrak{g}_{\mathbf{v}_1+\mathbf{v}_2}/G_{\mathbf{v}_1+\mathbf{v}_2}}{\times} I^+(X_{\mathbf{v}_1+\mathbf{v}_2}),$$

induced by (6.3.3), *is an isomorphism of stacks over* $\mathfrak{g}^{disj}_{\mathbf{v}_1,\mathbf{v}_2}/(G_{\mathbf{v}_1} \times G_{\mathbf{v}_2})$.

Proof. We introduce simplified notation $\mathfrak{g}_j = \mathfrak{g}_{\mathbf{v}_j}, G_j = G_{\mathbf{v}_j}, j = 1, 2$, resp. $\mathfrak{g} = \mathfrak{g}_{\mathbf{v}_1+\mathbf{v}_2}, G = G_{\mathbf{v}_1+\mathbf{v}_2}$.

It suffices to show that, for any affine scheme S and a morphism $g : S \to \mathfrak{g}^{disj}_{\mathbf{v}_1, \mathbf{v}_2}$, the corresponding functor

$$\{g\} \times_{(\mathfrak{g}_1/G_1 \times \mathfrak{g}_2/G_2)(S)} \left(I^+(X_{\mathbf{v}_j})(S) \times I^+(X_{\mathbf{v}_j})(S) \right) \longrightarrow \{g\} \times_{(\mathfrak{g}/G)(S)} I^+(X_{\mathbf{v}_1+\mathbf{v}_2})(S) \qquad (6.3.5)$$

is an equivalence of categories. We first show that this functor is full. An object of the category on the right is a triple (x, f, h) where $x \in X(S)$, $f \in \mathrm{End}(x)$, and $h \in G(S)$ is such that $p_{\mathfrak{g}}(f) = \mathrm{Ad}\, h(g)$, cf. (6.2.3). Without loss of generality we may (and will) assume that $h = 1$, so $p_{\mathfrak{g}}(f) = g$.

The map g is given by a pair of elements $g_j \in \mathfrak{g}_j \otimes \mathsf{k}[S]$, $j = 1, 2$. The vector space $\mathrm{End}(x)$, resp. $\mathfrak{g}_j \otimes \mathsf{k}[S]$, has the natural structure of an associative $\mathsf{k}[S]$-algebra. Further, there is a natural evaluation homomorphism $\mathrm{ev}_f : \mathsf{k}[t] \otimes \mathsf{k}[S] = \mathsf{k}[\mathbb{A} \times S] \to \mathrm{End}\, F(x)$, resp. $\mathrm{ev}_j : \mathsf{k}[t] \otimes \mathsf{k}[S] \to \mathfrak{g}_j \otimes \mathsf{k}[S]$, an algebra homomorphism defined by the assignment $t \otimes 1 \mapsto f$, resp. $t \otimes 1 \mapsto g_j$. We put $J := \mathrm{Ker}(\mathrm{ev}_f)$, resp. $J_j := \mathrm{Ker}(\mathrm{ev}_j)$. Let $A_f \subset \mathrm{End}(x)$, resp. $A_g \subset \mathfrak{g} \otimes \mathsf{k}[S]$ and $A_j \subset \mathfrak{g}_j \otimes \mathsf{k}[S]$, be a $\mathsf{k}[S]$-subalgebra generated by f, resp. g and g_j. Thus, we get the following homomorphisms:

$$A_f = \mathsf{k}[\mathbb{A} \times S]/J \overset{u}{\twoheadrightarrow} A_g = \mathsf{k}[\mathbb{A} \times S]/(J_1 \cap J_2) \overset{v}{\hookrightarrow} A_1 \bigoplus A_2 = \mathsf{k}[\mathbb{A} \times S]/J_1 \bigoplus \mathsf{k}[\mathbb{A} \times S]/J_2,$$

where the second map is induced by the diagonal imbedding $\mathsf{k}[t] \otimes \mathsf{k}[S] \to (\mathsf{k}[t] \otimes \mathsf{k}[S]) \oplus (\mathsf{k}[t] \otimes \mathsf{k}[S])$.

Further, let $p_j(t, s) = \det(t \, \mathrm{Id}_{\mathsf{k}^{\mathbf{v}_j}} - g_j(s))$ be the characteristic polynomial of g_j. Thus, we have $p_j \in \mathsf{k}[\mathbb{A} \times S]$ and, moreover, the Hamilton-Cayley theorem implies that $p_j(t, s) \in J_j$. The polynomials p_1 and p_2 have no common zeros since $g \in \mathfrak{g}_{\mathbf{v}_1, \mathbf{v}_2}$. Therefore, the ideals J_1 and J_2 have no common zeros and, hence, $J_1 + J_2 = \mathsf{k}[\mathbb{A} \times S]$, by the Nullstellensatz. This implies that v, the second map in the diagram above, is an isomorphism. We deduce that there is a canonical direct sum decomposition $A_g \cong A_1 \oplus A_2$. Thus, the element $(1_{A_1}, 0) \in A_1 \oplus A_2$ produces a nontrivial idempotent of the algebra A_g.

Observe next that $\mathrm{Ker}(u)$ is a nilpotent ideal in A_f, thanks to Lemma 6.2.2. The algebra A_f being commutative, it follows that the idempotent $(1_{A_1}, 0)$ can be lifted *uniquely* to an idempotent $e = e_{x,f,g} \in A_f$. By construction, e is an element of $\mathrm{End}(x)$ that commutes with f. We put $x_1 := ex$ and $f_1 = e \circ f$, resp. $x_2 = (1 - e)x$, and $f_2 = (1 - e) \circ f$. Thus, we have a direct sum $(x, f) = (x_1, f_1) \oplus (x_2, f_2)$ and, moreover, $p_G(f_j) = g_j$. This proves that the functor in (6.3.5) is full.

Now, let $(x, f, g) \to (x', f', g')$ be a morphism in the category on the right of (6.3.5). This means that we have a morphism $\varphi : x \to x'$ such that $\varphi \circ f = f' \circ \varphi$, and also $p_{\mathfrak{g}}(f') = g' \in \mathrm{Ad}\, h(g)$ for some $h \in G(S)$. Therefore, the map $\widetilde{\varphi} : A_f \overset{\sim}{\to} A_{f'}$, $u \mapsto \varphi \circ u \circ \varphi^{-1}$, is an algebra isomorphism. Furthermore, we have $\widetilde{\varphi}(e_{x,f,g}) = e_{x',f',g'}$, since the elements g_j and $\mathrm{Ad}\, h(g_j)$ have equal characteristic polynomials and the idempotent are determined by the corresponding triples uniquely. It follows that the morphism φ takes the direct sum decomposition $(x, f) = (x_1, f_1) \oplus (x_2, f_2)$ to the corresponding decomposition $(x', f') = (x'_1, f'_1) \oplus (x'_2, f'_2)$, resulting from $e_{x',f',g'}$. This implies that the functor (6.3.5) is fully faithful. Hence, this functor is an equivalence, as required. $\square$

Proposition 6.3.4 and its proof have an immediate generalization to the case where the stack $I^+(\mathfrak{X}_{\mathbf{v}}(\mathscr{C}))$ is replaced by the stack $\mathfrak{X}_{\mathbf{v}}(\mathsf{k}[C] \otimes \mathscr{C})$, cf. §1.6, where $C \subset \mathbb{A}$ is a fixed Zariski open nonempty subset. In particular, for $C = \mathbb{G}$ from Proposition 6.3.4, using Corollary 6.3.2, we obtain

Corollary 6.3.6. *The natural morphism below is an isomorphism:*

$$G^{disj}_{\mathbf{v}_1,\mathbf{v}_2} \times_{(G_{\mathbf{v}_1}/_{\mathrm{Ad}} G_{\mathbf{v}_1}) \times (G_{\mathbf{v}_2}/_{\mathrm{Ad}} G_{\mathbf{v}_2})} \left(I(X_{\mathbf{v}_1}) \times I(X_{\mathbf{v}_2}) \right) \longrightarrow G^{disj}_{\mathbf{v}_1,\mathbf{v}_2} \times_{G_{\mathbf{v}_1+\mathbf{v}_2}/_{\mathrm{Ad}} G_{\mathbf{v}_1+\mathbf{v}_2}} I(X_{\mathbf{v}_1+\mathbf{v}_2}).$$

Remark 6.3.7. Proposition $\overset{\text{Zfix}}{6.3.4}$ can also be extended to the case of stacks $\mathfrak{X}_{\mathbf{v}}(\mathsf{k}[C] \otimes \mathscr{C})$, where C is a smooth affine curve which is not necessarily contained in $\mathbb{A}$. We will neither use nor prove such a generalization.

6.4. We now introduce the factirization sheaf $\mathcal{R}^{F,\phi}$ associated to a triple $(\mathscr{C}, F, \phi)$, as has been outlined in §$\overset{\text{h-intro}}{1.7}$ in a more general setting of an smooth scheme C. Thus, we fix a potential, a morphism $\phi : \mathfrak{X}(\mathscr{C}) \to \mathbb{A}$, of stacks over k. For each $\mathbf{v}$, put $\phi_{\mathbf{v}} = \phi|_{X_{\mathbf{v}}}$. According to $(\overset{\text{liie}}{5.2.5})$, one has a diagram

$$\mathbb{A} \xleftarrow{\phi_{\mathbf{v}}} \mathfrak{X}_{\mathbf{v}}(\mathscr{C}) = \mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)/\mathrm{GL}_{\mathbf{v}} \xleftarrow{p_{\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)/\mathrm{GL}_{\mathbf{v}}}} I^+(\mathfrak{X}_{\mathbf{v}}(\mathscr{C}) \xrightarrow{p_{\mathfrak{gl}_{\mathbf{v}}}} \mathfrak{gl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}.$$

We define a sheaf $\mathcal{R}^{F,\phi}_{\mathbf{v}}$, on $\mathfrak{gl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}$, by $\mathcal{R}^{F,\phi}_{\mathbf{v}} := (p_{\mathfrak{gl}_{\mathbf{v}}})_! \, p^*_{\mathfrak{X}_{\mathbf{v}}(\mathscr{C},F)/\mathrm{GL}_{\mathbf{v}}} \, \wp^{\phi_{\mathbf{v}}}$, where $\wp^{\phi_{\mathbf{v}}} = \phi^*_{\mathbf{v}} \wp_\psi$ is a pull-back of the Artin-Schreier local system.

Theorem 6.4.1. *If the morphism ϕ is additive in the sense of §$\overset{\text{exp-sec}}{1.1}$, then the collection $\mathcal{R}^{F,\phi} = (\mathcal{R}^{F,\phi}_{\mathbf{v}})_{\mathbf{v} \in \mathbb{Z}^I_{\geq 0}}$ has the natural structure of an Aff-equivariant factorization sheaf on $\mathfrak{gl}$.*

Proof. Fix dimension vectors $\mathbf{v}_1, \mathbf{v}_2$. We use simplified notation from the proof of Proposition $\overset{\text{Zfix}}{6.3.4}$. Thus, for $j = 1, 2$, we write $X_j = \mathfrak{X}_{\mathbf{v}_j}(\mathscr{C})$, resp. $Y_{\mathbf{v}_j} = \mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)$, $\phi_j = \phi_{\mathbf{v}_j}$, $G_j = \mathrm{GL}_{\mathbf{v}}$, $\mathfrak{g}_j = \mathfrak{gl}_{\mathbf{v}_j}$, $I^+_j = I^+_1(X_{\mathbf{v}_j})$, and $I^+ = I^+(X_{\mathbf{v}_1+\mathbf{v}_2}), \phi = \phi_{Y_{\mathbf{v}_1+\mathbf{v}_2}}$. Also, $U^{disj} := \mathfrak{g}^{disj}_{\mathbf{v}_1,\mathbf{v}_2}/(G_1 \times G_2), \imath = \imath_{\mathbf{v}_1,\mathbf{v}_2}, \jmath = \jmath_{\mathbf{v}_1,\mathbf{v}_2}, \mathbf{i} = \mathbf{i}_{\mathbf{v}_1,\mathbf{v}_2}$. Then, using diagrams $(\overset{\text{liie}}{5.2.5})$ and $(\overset{\text{d12}}{6.3.3})$, one obtains the following commutative diagram

$$\begin{array}{ccccccc}
\mathbb{A} \times \mathbb{A} & \xleftarrow{\phi_1 \times \phi_2} & X_{\mathbf{v}_1} \times X_{\mathbf{v}_2} & \xleftarrow{(p_{X_1} \times p_{X_2}) \circ pr} & U^{disj} \times_{\mathfrak{g}_1/G_1 \times \mathfrak{g}_2/G_2} (I^+_1 \times I^+_2) & \xrightarrow{\mathrm{Id} \times (p_{\mathfrak{g}_1} \times p_{\mathfrak{g}_2})} & U^{disj} \\
\downarrow{+} & & \downarrow{\oplus} & & \cong \downarrow{\mathrm{Id} \times \mathbf{i}} & & \| \mathrm{Id} \times \imath \circ \jmath \\
\mathbb{A} & \xleftarrow{\phi} & X_{\mathbf{v}_1+\mathbf{v}_2} & \xleftarrow{p_X \circ pr} & U^{disj} \times_{\mathfrak{g}/G} I^+ & \xrightarrow{\mathrm{Id} \times p_{\mathfrak{g}}} & U^{disj} \times_{\mathfrak{g}/G} \mathfrak{g}/G,
\end{array}$$

where pr denotes the projection to the second factor.

$$\begin{array}{ccccccc}
\mathbb{A} \times \mathbb{A} & \xleftarrow{\phi_1 \times \phi_2} & X_{\mathbf{v}_1} \times X_{\mathbf{v}_2} & \xleftarrow{p_{X_1} \times p_{X_2}} & I^+_1 \times I^+_2 & \longrightarrow (\mathfrak{g}_1/G_1 \times \mathfrak{g}_2/G_2) \times_{\mathfrak{g}/G} I^+ & \xrightarrow{p_{\mathfrak{g}_1} \times p_{\mathfrak{g}_2}} \mathfrak{g}_1/G_1 \times \mathfrak{g}_2/G_2 \\
\downarrow{+} & & \downarrow{\oplus} & & \downarrow{\mathbf{i}} & & \downarrow{\imath} \\
\mathbb{A} & \xleftarrow{\phi} & X_{\mathbf{v}_1+\mathbf{v}_2} & \xleftarrow{p_X} & I^+ & \xrightarrow{\hspace{5em} p_{\mathfrak{g}} \hspace{5em}} & \mathfrak{g}/G,
\end{array}$$

By the additivity of the morphism ϕ, using the canonical isomorphism $(+)^* \wp_\psi \cong \wp_\psi \boxtimes \wp_\psi$ where the map '$+$' is given by addition $(a, a') \mapsto a + a'$, we obtain an isomorphism

$$(\oplus)^* \phi^* \wp_\psi = (\phi_1 \times \phi_2)^* (+)^* \wp_\psi = (\phi_1 \times \phi_2)^* (\wp_\psi \boxtimes \wp_\psi) = \phi_1^* \wp_\psi \boxtimes \phi_2^* \wp_\psi. \qquad (6.4.2) \quad \boxed{\texttt{fact\_phi}}$$

We compute

$$\begin{aligned}
\imath^* \mathcal{R}^{F,\phi}_{\mathbf{v}_1+\mathbf{v}_2} &= \imath^* (p_{\mathfrak{g}})_! \, p^*_X \phi^* \wp_\psi = (\mathrm{Id} \times \imath)^* (\mathrm{Id} \times p_{\mathfrak{g}})_! \, p^*_X \, \phi^* \wp_\psi \quad \text{(Base change)} \\
&= \left(\mathrm{Id} \times (p_{\mathfrak{g}_1} \times p_{\mathfrak{g}_2}) \right)_! \, \mathbf{i}^* \, p^*_X \, \phi^* \wp_\psi \\
&= \left(\mathrm{Id} \times (p_{\mathfrak{g}_1} \times p_{\mathfrak{g}_2}) \right)_! \, (p_{X_1} \times p_{X_2})^* (\oplus)^* \phi^* \wp_\psi \quad \text{(by } (\overset{\text{fact\_phi}}{6.4.2})) \\
&= \left(\mathrm{Id} \times (p_{\mathfrak{g}_1} \times p_{\mathfrak{g}_2}) \right)_! \, (p_{X_1} \times p_{X_2})^* (\phi_1^* \wp_\psi \boxtimes \phi_2^* \wp_\psi) \quad \text{(Base change)}
\end{aligned}$$

$$= \jmath^* \, (p_{\mathfrak{g}_1} \times p_{\mathfrak{g}_2})_! \, (p_{X_1} \times p_{X_2})^* \, (\phi_1^* \wp_\psi \boxtimes \phi_2^* \wp_\psi)$$
$$= \jmath^* \, \big((p_{\mathfrak{g}_1})_! \, p_{X_1}^* \, \phi_1^* \wp_\psi \; \boxtimes \; (p_{\mathfrak{g}_2})_! \, p_{X_2}^* \, \phi_2^* \wp_\psi\big) \; = \; \jmath^* (\mathcal{R}_{\mathbf{v}_1}^{F,\phi} \boxtimes \mathcal{R}_{\mathbf{v}_2}^{F,\phi}).$$

$$\mathcal{R}_{\mathbf{v}_1}^{F,\phi} \boxtimes \mathcal{R}_{\mathbf{v}_2}^{F,\phi} \; = \; (p_{\mathfrak{g}_1})_! \, p_{X_1}^* \, \phi_1^* \wp_\psi \; \boxtimes \; (p_{\mathfrak{g}_2})_! \, p_{X_2}^* \, \phi_2^* \wp_\psi \; = \; (p_{\mathfrak{g}_1} \times p_{\mathfrak{g}_2})_! \, (p_{X_1} \times p_{X_2})^* \, (\phi_1^* \wp_\psi \boxtimes \phi_2^* \wp_\psi)$$
$$= \; (p_{\mathfrak{g}_1} \times p_{\mathfrak{g}_2})_! \, (p_{X_1} \times p_{X_2})^* \, (\oplus)^* \, \phi^* \, \wp_\psi \; = \; (p_{\mathfrak{g}_1} \times p_{\mathfrak{g}_2})_! \, \imath^* \, p_X^* \, \phi^* \, \wp_\psi$$

Therefore, the base change morphism $\imath^*(p_{\mathfrak{g}})_! \to (p_{\mathfrak{g}_1} \times p_{\mathfrak{g}_2})_! \imath^*$ yields a canonical morphism

$$\imath^* \mathcal{R}_{\mathbf{v}_1 + \mathbf{v}_2}^{F,\phi} \; = \; \imath^* \, (p_{\mathfrak{g}})_! \, p_X^* \phi^* \, \wp_\psi \; \xrightarrow{\varphi} \; (p_{\mathfrak{g}_1} \times p_{\mathfrak{g}_2})_! \, \imath^* \, p_X^* \phi^* \, \wp_\psi \; = \; \mathcal{R}_{\mathbf{v}_1}^{F,\phi} \boxtimes \mathcal{R}_{\mathbf{v}_2}^{F,\phi}$$

To complete the proof observe that for any affine scheme S and an object $x \in \mathscr{C}(S)$, there is an *Aff*-action on on $\mathrm{End}(x)$, resp. $\mathbb{G}$-action on $\mathrm{Aut}(x)$, defined by "$az + b$" : $g \mapsto a \cdot g + b \cdot \mathrm{Id}_x$, resp. "$a$" : $g \mapsto a \cdot g$, similarly to §4.2 This gives the stack $I^+(X_{\mathbf{v}})$, resp. $I(X_{\mathbf{v}})$, the structure of an *Aff*-stack, resp. $\mathbb{G}$-stack. Furthermore, the morphism $p_{\mathfrak{g}}$ in diagram (5.2.5) is clearly *Aff*-equivariant, resp. the morphism p_G in diagram (5.2.3) is $\mathbb{G}$-equivariant. It follows $\mathcal{R}_{\mathbf{v}}^{F,\phi}$ is an *Aff*-equivariant sheaf on $\mathfrak{gl}_{\mathbf{v}}/G_{\mathbf{v}}$. It is straightforward to see that the factorization sheaf structure on $\mathcal{R}^{F,\phi} = (\mathcal{R}_{\mathbf{v}}^{F,\phi})$ constructed in the proof of Lemma 6.4.1 is compatible with the *Aff*-equivariant structure. This gives $\mathcal{R}_{\mathrm{Lie}}^{F,\phi}$ the structure of an *Aff*-equivariant factorization sheaf. $\qquad\square$

7. INERTIA STACKS AND FOURIER TRANSFORM

7.1. Fourier transform for stacks. Let Y be a stack, viewed as a $\mathbb{G}$-stack with a trivial action.

Definition 7.1.1. A morphism $E \to Y$, of $\mathbb{G}$-stacks, is called
- a *vector bundle* on Y if for any test scheme S and a morphism $S \to Y$, the first projection $S \times_X Y \to S$ is a vector bundle on S and the induced $\mathbb{G}$-action on $S \times_X Y$ is by dilations.
- a *generalized vector bundle* on Y if E is a representable stack over Y of the form $\mathrm{Spec}_Y(\mathrm{Sym}\, M)$ where M is a coherent sheaf on Y.
- a *vector space stack* if there exists an open covering $\{u_\alpha : Y_\alpha \to Y\}$ in fppf-topology and, for each α, a morphism $f_\alpha : E'_\alpha \to E''_\alpha$ of vector bundles on Y_α and a $\mathbb{G}$-equivariant isomorphism $u_\alpha^*(E) \cong \mathrm{Coker}(f_\alpha)$. In this case, we put $\mathrm{rk}\, E := \mathrm{rk}\, E''_\alpha - \mathrm{rk}\, E'_\alpha$. Morphisms of vector space stacks on Y are defined to be the $\mathbb{G}$-equivariant morphisms of stacks over Y.

Given a morphism $f : E' \to E''$ of vector bundles on a stack Y one may view E' as a flat group scheme over Y and E'' as a E'-stack, where E' acts on E'' by translation. We write $\mathrm{Coker}(f)$ for the corresponding quotient stack. The map f is equivariant under the dilation action of $\mathbb{G}$ on E' and E''. Thus, $\mathrm{Coker}(f)$ is a vector space stack to be denoted E''/E'. By definition, any vector space stack has an fppf-local presentation of the form E''/E'.

Let $f : Y_1 \to Y_2$ be a morphism of stacks. Given a generalized vector bundle $E = \mathrm{Spec}_{Y_2}(\mathrm{Sym}\, M)$, resp. a vector space stack $E = E''/E'$ on Y_1, one defines its pull-back by the formula $f^*E = \mathrm{Spec}_{Y_1}(\mathrm{Sym}\, f^*M)$, resp. $f^*E = f^*E''/f^*E'$. It is immediate to check that the latter formula is independent of the choice of a presentation of E in the form E''/E'. There is a canonical morphism $f_\sharp : f^*E \to E$, of stacks, such that $p_2 \circ f_\sharp = f \circ p_1$, where $p_1 : f^*E \to Y_1$ and $p_2 : E \to Y_2$ denote the projections.

The dual of a vector space stack E on Y is , by definition, a *generalized vector bundle* E^* on Y defined as follows. Given a test scheme S let $E^*(S, y)$ be category of morphisms $E \times_Y S \to \mathbb{A}_S = \mathbb{A} \times S$ of vector space stacks on S. It is easy to see that the category $E^*(S)$ is in fact a groupoid and we let E^* be the stack defined by the assignment $S \mapsto E^*(S)$. In a special case where $E = E''/E'$ this definition is equivalent to the formula $E^* = E_2^* \times_{E_1^*} Y = \mathrm{Spec}_Y \mathrm{Sym}(\mathrm{Coker}_Y(E_1 \to E_2))$, where we have used the the morphism $Y \to E_1^*$ given by the zero section.

Example 7.1.2. Let Y be a smooth stack. Then, the tangent complex of Y is a short complex $d :$ $T'_X \to T''_X$. In this case, we have $\mathrm{T}Y \cong \mathrm{Coker}(d)$, so the tangent stack is a vector space stack over Y. Further, we have $\mathrm{T}^*Y = \mathrm{Spec}_Y(\mathrm{Coker}(d))$. Thus, the cotangent stack of Y is a generalized vector bundle, the dual of the vector space stack $\mathrm{T}Y$.

Remark 7.1.3. Associated with a vector space stack E, there is also a dual vector space stack $E^*[1]$ defined by $\mathrm{Hom}_Y(S, E^*[1]) = \mathrm{Hom}(E \times_Y S, B\mathbb{G}_a \times S)$ and a generalized vector bundle $\Omega E(= E[-1]) := Y \times_E Y$. One has a canonical isomorphism $\Omega(E^*[1]) \cong E^*$. $\diamond$

We proceed to define the Fourier-Deligne transform in this setting, cf. [BG, §7.3] for a special case. Thus, we assume k to be a finite field and let $\wp_\psi$ be the Artin-Schreier sheaf associated with a fixed additive character $\psi : \mathsf{k} \to \mathbb{C}^\times$. Let Y be a stack over k and E a vector space stack on Y. We have a natural evaluation morphism $ev_E : E \times_Y E^* \to \mathbb{A}$. For $\mathcal{F} \in D_{\mathrm{abs}}(E)$, define $\mathsf{F}_E(\mathcal{F}) = \mathrm{pr}_{2!}(\mathrm{pr}_1^* \mathcal{F} \otimes ev_E^*(\wp_\psi)) \in D_{\mathrm{abs}}(E^*)$, where pr_i is the projection from $E \times_Y E^*$ to the i-th factor. It will be more convenient to use a renormalized Fourier-Deligne transform defined by $\mathrm{F}_E(\mathcal{F}) = \mathsf{F}_E(\mathcal{F})[\,\mathrm{rk}E](\frac{1}{2}\mathrm{rk}E)$. The latter commutes with the Verdier duality.

Proposition 7.1.4. (i) *The Fourier-Deligne transform* F_E *is an equivalence of categories;*

(ii) *Let* $h : E_1 \to E_2$ *be a morphism of vector space stacks and* $h^\vee : E_2^* \to E_1^*$ *the dual morphism of generalized vector bundles. Put* $d = \mathrm{rk}\, E_2 - \mathrm{rk}\, E_1$. *Then, there are isomorphisms of functors*

$$\mathrm{F}_{E_2} \circ h_! = h^{\vee,*} \circ \mathrm{F}_{E_1}[d](\tfrac{1}{2}d), \qquad \mathrm{F}_{E_1} \circ h^* = h_!^\vee \circ \mathrm{F}_{E_2}[d](\tfrac{1}{2}d). \qquad (7.1.5) \quad \boxed{\text{four-iso}}$$

(iii) *Let* $f : Y_1 \to Y_2$ *be a morphism of stacks, E a vector space stack on Y_2, and* $f_E^\sharp : f^*E \to E$, *resp.* $f_{E^*}^\sharp : f^*E^* \to E^*$, *the canonical morphisms. Then, there are isomorphisms of functors*

$$(f_{E^*}^\sharp)_! \circ \mathrm{F}_{f^*E} = \mathrm{F}_E \circ (f_E^\sharp)_!, \qquad (f_{E^*}^\sharp)^* \circ \mathrm{F}_E = \mathrm{F}_{f^*E} \circ (f_E^\sharp)^*.$$

Example 7.1.6. Any vector space stack on $Y = \mathrm{pt}$ is isomorphic to a stack of the form $E = \mathbb{A}^m \times (\mathrm{pt}/\mathbb{A}^n)$. For the dual generalized vector bundle E^* one finds that $E^* = (\mathbb{A}^m)^* \times (\mathrm{pt}/\mathbb{A}^n)^* = (\mathbb{A}^m)^*$. Let $pr : \mathbb{A}^m \times \mathrm{pt}/\mathbb{A}^n \to \mathbb{A}^m$ be the first projection. Then the functor $f^* : D_{\mathrm{abs}}(\mathbb{A}^m) \to D_{\mathrm{abs}}(\mathbb{A}^m \times \mathrm{pt}/\mathbb{A}^n)$ is an equivalence since the additive group $\mathbb{A}^n$ is a unipotent group. Thus, the equivalence of part (i) of the Proposition reduces, in this case, to an analogous equivalence in the case of $E = \mathbb{A}^m$.

Sketch of proof of the Proposition. The first, resp. second, isomorphism in (ii) amounts to functorial isomorphisms $\mathrm{F}_{E_2}(h_!A) = h^{\vee,*}\mathrm{F}_{E_1}(A)$, resp. $\mathrm{F}_{E_1}(h^*B) = h_!^\vee \mathrm{F}_{E_2}(B)[2d](d)$ for any $A \in D_{\mathrm{abs}}(E_1)$ and $B \in D_{\mathrm{abs}}(E_2)$. We only prove the first isomorphism. To this end, consider the following commutative diagram

The squares in the diagram are Cartesian. The required isomorphism is a consequence of the following chain of isomorphisms

$$h^{\vee,*}\mathrm{F}_{E_1}(A) = h^{\vee,*}\mathrm{pr}_{2,!}^{E_1}(\mathrm{pr}_1^{E_1,*}(A) \otimes ev_{E_1}^*(\wp_\chi))$$

$$= \widetilde{\mathrm{pr}}_{2,!}(id \times h^\vee)^*(\mathrm{pr}_1^{E_1,*}(A) \otimes ev_{E_1}^*(\wp_\chi)) \qquad \text{(base change)}$$

54

$$= \widetilde{\mathrm{pr}}_{2,!}(\widetilde{\mathrm{pr}}_1^*(A) \otimes (id \times h^\vee)^* ev_{E_1}^*(\wp_\chi))$$
$$= \widetilde{\mathrm{pr}}_{2,!}(\widetilde{\mathrm{pr}}_1^*(A) \otimes (h \times id)^* ev_{E_2}^*(\wp_\chi)) \qquad (\text{since } ev_{E_1} \circ (id \times h^\vee) = ev_{E_2} \circ (h \times id))$$
$$= \mathrm{pr}_{2,!}^{E_2}(h \times id)_!(\widetilde{\mathrm{pr}}_1^*(A) \otimes (h \times id)^* ev_{E_2}^*(\wp_\chi)) \qquad (\text{projection formula})$$
$$= \mathrm{pr}_{2,!}^{E_2}((h \times id)_! \widetilde{\mathrm{pr}}_1^*(A) \otimes ev_{E_2}^*(\wp_\chi)) \qquad (\text{base change})$$
$$= \mathrm{pr}_{2,!}^{E_2}(\mathrm{pr}_1^{E_2,*} h_!(A) \otimes ev_{E_2}^*(\wp_\chi)) = \mathsf{F}_{E_2}(h_! A).$$

The proof of the equivalence in (i) mimics the standard proof of a similar equivalence in the case of vector bundles on a scheme, cf. [KL]. Specifically, define a functor $\mathsf{F}_{E^*} : D_{\mathrm{abs}}(E^*) \to D_{\mathrm{abs}}(E)$ by the formula $\mathsf{F}_{E^*}(\mathcal{F}) = \mathrm{pr}_{1!}(\mathrm{pr}_2^* \mathcal{F} \otimes ev_E^*(\wp_{-\psi}))[\mathrm{rk}E](\frac{1}{2}\mathrm{rk}E)$. Then, one establishes the Fourier inversion formula saying that there are isomorphisms of functors $\mathsf{F}_{E^*} \circ \mathsf{F}_E \cong \mathrm{Id}_{D_{\mathrm{abs}}(E)}$ and $\mathsf{F}_E \circ \mathsf{F}_{E^*} \cong \mathrm{Id}_{D_{\mathrm{abs}}(E^*)}$, respectively.

To prove the isomorphisms, let $i : E \times_Y E^* \hookrightarrow E \times E^*$, resp. $i^\vee : E^* \times_Y E \hookrightarrow E^* \times E$, be the imbedding and put $K_E := i_! ev_E^*(\wp_\psi)[\mathrm{rk}E](\frac{1}{2}\mathrm{rk}E)$, resp. $K_{E^*} = i_!^\vee ev_{E^*}^*(\wp_{-\psi})[\mathrm{rk}E](\frac{1}{2}\mathrm{rk}E)$. The functors F_E, resp. F_{E^*}, can be written as a convolution functor $\mathsf{F}_E(\mathcal{F}) = \mathcal{F} * K_E$, resp. $\mathsf{F}_{E^*}(\mathcal{E}) = \mathcal{E} * K_{E^*}$. Let $\Delta_E : E \to E \times_Y E$, resp. $\Delta_{E^*} : E^* \times_Y E^*$, denote the diagonal imbedding. Using adjunction, one constructs natural morphisms $(\Delta_E)_! \underline{\mathbf{C}}_E \to K_E * K_{E^*}$, resp. $(\Delta_{E^*})_! \underline{\mathbf{C}}_{E^*} \to K_{E^*} * K_E$. The Fourier inversion formula is equivalent to a statement that these morphisms are isomorphisms. The latter statement is local with respect to Y. Therefore, we may (and will) assume that $E = E''/E'$. Thus, the category $D_{\mathrm{abs}}(E)$ may be identified with the E'-equivariant derived category of E''. Further, the group E' being unipotent, the forgetful functor $D_{\mathrm{abs}}(E) \to D_{\mathrm{abs}}(E'')$ is an imbedding. The essential image of this imbedding is a full subcategory of $D_{\mathrm{abs}}(E'')$ whose objects are sheaves $\mathcal{F}$ such that there exists an isomorphism $a^* \mathcal{F} \cong p^* \mathcal{F}$, where a, resp. p, is the action, resp. second projection, $E' \times_Y E'' \to E''$. This way, we are reduced to proving the result in the case where E is a vector bundle on Y. The latter case follows from the corresponding result for schemes, which is known.

The proof of part (iii) is similar. We omit it. $\qquad \square$

7.2. Infinitesimal inertia and Fourier transform.

Let G be an algebraic group and $\mathfrak{g}$ its Lie algebra. Let Y be a smooth G-stack, $i_Y : Y \to TY$ the zero section, $a_{\mathfrak{g}} : \mathfrak{g} \times Y \to TY$ the infinitesimal action, and $f : Y/G \to \mathrm{pt}/G$ the map induced by a constant map $Y \to \mathrm{pt}$. We have a commutative diagram

$$
\begin{array}{ccccccc}
Y/G & \xrightarrow{i} & TY/G & \xleftarrow{a} & (\mathfrak{g} \times Y)/G & \xrightarrow{f_{\mathfrak{g}/G}^\sharp = pr_1} & \mathfrak{g}/G \\
\downarrow{\scriptstyle Id} & & \downarrow & & \downarrow & \square & \downarrow \\
Y/G & = & Y/G & = & Y/G & \xrightarrow{f} & \mathrm{pt}/G
\end{array}
\qquad (7.2.1) \quad \boxed{\texttt{ddiag}}
$$

Here, the first 3 vertical maps are vector space stacks, the map i, resp. a, induced by i_Y, resp. $a_{\mathfrak{g}}$, is a morphism of vector space stacks on Y/G. The square on the right is cartesian, so we have $pr_1 = f_{\mathfrak{g}/G}^\sharp$.

Let $\mu : \mathrm{T}^* Y \to \mathfrak{g}^*$ be the moment map and $\pi : \mathrm{T}^* Y \to Y$ the projection. We have a commutative diagram

$$
\begin{array}{ccccccc}
Y/G & \xleftarrow{\pi'} & T^* Y/G & \xrightarrow{\nu} & (\mathfrak{g}^* \times Y)/G & \xrightarrow{f_{\mathfrak{g}^*/G}^\sharp = pr_1} & \mathfrak{g}^*/G \\
\downarrow{\scriptstyle Id} & & \downarrow & & \downarrow & \square & \downarrow \\
Y/G & = & Y/G & = & Y/G & \xrightarrow{f} & \mathrm{pt}/G
\end{array}
$$

where the maps ν and π' are induced by $\mu \times \pi$ and π, respectively. The map ν is the dual of the map a' by the definition of the moment map. Hence, the above diagram is a diagram of morphisms of generalized vector bundles which is dual to diagram (7.2.1).

Recall the notation of diagram (5.2.5) and let $p : I^+(Y/G) \to (\mathfrak{g} \times Y)/G$ be a composition of the map $\mathrm{pr}_{\mathfrak{g}} \times \mathrm{pr}_Y : I^+(Y/G) \to \mathfrak{g} \times Y$ and the quotient map $\mathfrak{g} \times Y \to (\mathfrak{g} \times Y)/G$. Let $\mu' : \mathrm{T}^*Y/G \to Y/G$ be the map induced by μ.

Proposition 7.2.2. *For any morphism $\phi : Y/G \to \mathbb{A}$, there is an isomorphism*

$$\mathrm{F}_{\mathfrak{g}/G \to \mathbb{B}G}((p_{\mathfrak{g}})_! p_{Y/G}^* \wp^\phi) = \mu'_!(\pi')^* \wp^\phi \{2 \dim Y - \dim G\}.$$

Proof. We have the following commutative diagram where the square is cartesian, cf. (5.2.5):

$$
\begin{array}{ccccc}
\mathrm{T}Y/G & \xleftarrow{\;a\;} & (\mathfrak{g} \times Y)/G & \xrightarrow{\;pr_1\;} & \mathfrak{g}/G \\
{\scriptstyle i}\uparrow & & {\scriptstyle p}\uparrow & \nearrow{\scriptstyle p_{\mathfrak{g}}} & \\
Y/G & \xleftarrow{\;p_{Y/G}\;} & I^+(Y/G) & &
\end{array}
$$

By base change we obtain

$$(p_{\mathfrak{g}/G})_! p_{Y/G}^* \wp^\phi = (pr_1)_! p_! p_{Y/G}^* \wp^\phi = (pr_1)_! a_{\mathfrak{g}/G}^* i_! \wp^\phi = (f_{\mathfrak{g}/G}^\sharp)_! a_{\mathfrak{g}/G}^* i_! \wp^\phi.$$

Thus, we compute

$$
\begin{aligned}
\mathrm{F}_{\mathfrak{g}/G \to \mathbb{B}G}((p_{\mathfrak{g}})_! p_{Y/G}^* \wp^\phi) &= \mathrm{F}_{\mathfrak{g}/G \to \mathbb{B}G}((f_{\mathfrak{g}/G}^\sharp)_! a^* i_! \wp^\phi) && \text{(by Prop. 7.1.4(iii))} \\
&= (f_{\mathfrak{g}^*/G}^\sharp)_! \mathrm{F}_{(\mathfrak{g}^* \times Y)/G \to Y/G}(a^* i_! \wp^\phi) && \text{(by Prop. 7.1.4(ii)).}
\end{aligned}
$$

The rank of the vector bundle $\mathrm{T}Y/G \to Y/G$, resp. $(\mathfrak{g} \times Y)/G \to Y/G$, equals $\dim Y$, resp. $\dim \mathfrak{g}$. Hence, we have

$$
\begin{aligned}
\mathrm{F}_{(\mathfrak{g} \times Y)/G \to Y/G}(a^* i_! \wp^\phi) &= \mu'_! \mathrm{F}_{(\mathrm{T}Y/G \to Y/G)}(i_! \wp^\phi)\{\dim Y\} \\
&= \mu'_!(\pi')^* \wp^\phi \{\dim Y - \dim \mathfrak{g}\}\{\dim Y\} = \mu'_!(\pi')^* \wp^\phi \{2 \dim Y - \dim \mathfrak{g}\}.
\end{aligned}
$$

The result follows using that $f_{\mathfrak{g}^*/G}^\sharp \circ \nu = \mu'$. $\qquad\square$

Remark 7.2.3. There is a more general construction where $\mathbb{B}G$ in $Y/G \to \mathbb{B}G$ above is replaced by a more general stack. Suppose that we have a smooth map $q : Y \to W$ ('a family of smooth stacks'). Then there is a map $\mathrm{T}^*(Y/W) \to q^*(\mathrm{T}Y)^*[1]$ and the 'dual' $q^* \Omega(\mathrm{T}Y) \to \mathrm{T}(Y/W)$ which specialize to μ and a above for $Y/G \to \mathbb{B}G$. However these are maps from generalized vector bundles to vector space stacks. The functoriality with respect to such maps is not covered by Proposition 7.1.4, though it could be stated separately. $\qquad\diamond$

7.3. **Proof of theorem 1.1.5.** For each $\mathbf{v}$ we have a diagram, cf. (5.2.5),

$$
\begin{array}{ccccc}
\mathfrak{X}_{\mathbf{v}}(\mathscr{C}) & \xleftarrow{\;p_{\mathfrak{X}}^+\;} & I^+(\mathfrak{X}_{\mathbf{v}}(\mathscr{C})) & \xleftarrow{\;\tilde{\varepsilon}\;} & I(\mathfrak{X}_{\mathbf{v}}(\mathscr{C})) \\
\Big\| {\scriptstyle \mathrm{Id}} & & \downarrow {\scriptstyle p_{\mathfrak{g}}} & \square & \downarrow {\scriptstyle p_G} \\
\mathfrak{X}_{\mathbf{v}}(\mathscr{C}) & \xleftarrow{\;p_{\mathfrak{X}}\;} & \mathfrak{gl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}} & \xleftarrow{\;\varepsilon\;} & \mathrm{GL}_{\mathbf{v}}/_{\mathrm{Ad}} \mathrm{GL}_{\mathbf{v}}
\end{array}
$$

where we have used the notation $p_G = p_{\mathrm{GL}_{\mathbf{v}}}$, resp. $p_{\mathfrak{g}} = p_{\mathfrak{gl}_{\mathbf{v}}}$, $p_{\mathfrak{X}}^+ = p_{\mathfrak{X}_{\mathbf{v}}(\mathscr{C},F)/\mathrm{GL}_{\mathbf{v}}}^+$, and $p_{\mathfrak{X}} = p_{\mathfrak{X}_{\mathbf{v}}(\mathscr{C},F)/\mathrm{GL}_{\mathbf{v}}}$. Let $\phi : \mathfrak{X}_{\mathbf{v}}(\mathscr{C}) \to \mathbb{A}$ be an additive potential and $\mathcal{R}_{\mathbf{v}} := \mathcal{R}_{\mathbf{v}}^{F,\phi} = (p_{\mathfrak{g}})_! (p_{\mathfrak{X}}^+)^* \wp^\phi$.

We apply Proposition $\overset{\texttt{tr\_prop}}{5.3.1}$ in the special case where $\mathcal{F} = \underline{\mathbf{C}}_{\mathrm{GL}_{\mathbf{v}}/_{\mathrm{Ad}}\mathrm{GL}_{\mathbf{v}}} \in D_{\mathrm{abs}}(\mathrm{GL}_{\mathbf{v}}/_{\mathrm{Ad}}\mathrm{GL}_{\mathbf{v}})$ and $\mathcal{E} = \wp^{\phi} \in D_{\mathrm{abs}}(X_{\mathbf{v}})$. Clearly, we have $\mathsf{Tr}_{\mathsf{Fr}}|\,\mathcal{E}_x = \mathsf{Tr}_{\mathsf{Fr}}|\,(\wp^{\phi}|_x) = \psi(\phi(x))$ and $\mathsf{Tr}_{\mathsf{Fr}}|\,(\mathcal{F}_{p_G(x,f)} = 1,$ for all $[x] \in [X_{\mathbf{v}}(\mathsf{k})]$, $f \in \mathrm{Aut}(x)(\mathsf{k})$. Further, by base change, from the above diagram we get $(p_G)_! p_{\mathfrak{X}}^* \mathcal{E} = (p_G)_! \, \tilde{\varepsilon}^*(p_{\mathfrak{X}}^+)^* \wp^{\phi} = \varepsilon^* \mathcal{R}_{\mathbf{v}}^{F,\phi}$. Thus, formula $\overset{\texttt{tr2}}{(5.3.2)}$ yields

$$\mathsf{Tr}_{\mathsf{Fr}}|\,\mathrm{R\Gamma}_c(\mathrm{GL}_{\mathbf{v}}/_{\mathrm{Ad}}\mathrm{GL}_{\mathbf{v}},\, \varepsilon^* \mathcal{R}_{\mathbf{v}}) = \mathsf{Tr}_{\mathsf{Fr}}|\,\mathrm{R\Gamma}_c(\mathrm{GL}_{\mathbf{v}}/_{\mathrm{Ad}}\mathrm{GL}_{\mathbf{v}},\, (p_G)_! p_{\mathfrak{X}}^* \mathcal{E})$$

$$= \sum_{[x] \in [X_{\mathbf{v}}(\mathsf{k})]} \left(\sum_{f \in \mathrm{Aut}(x)(\mathsf{k})} 1 \right) \cdot \frac{1}{\#\,\mathrm{Aut}(x)} \cdot (\mathsf{Tr}_{\mathsf{Fr}}|\,\mathcal{E}_x)$$

$$= \sum_{[x] \in [X_{\mathbf{v}}(\mathsf{k})]} (\mathsf{Tr}_{\mathsf{Fr}}|\,\mathcal{E}_x) = \sum_{[x] \in [X_{\mathbf{v}}(\mathsf{k})]} \psi(\phi(x)). \qquad (7.3.1) \quad \boxed{\texttt{tr3}}$$

Recall next that the $\mathrm{GL}_{\mathbf{v}}$-action $\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)$ factors through a $\mathrm{PGL}_{\mathbf{v}}$-action. It follows that one has a natural isomorphism $I^+(\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F), \mathfrak{gl}_{\mathbf{v}}) = I^+(\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F), \mathfrak{pgl}_{\mathbf{v}}) \times_{\mathfrak{pgl}_{\mathbf{v}}} \mathfrak{gl}_{\mathbf{v}}$. Further, the group Aff acts naturally on $I^+(\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F), \mathfrak{gl}_{\mathbf{v}})$. Write $I^+(\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F), \mathfrak{gl}_{\mathbf{v}})/\mathbb{A}$ for the quotient of $I^+(\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F), \mathfrak{gl}_{\mathbf{v}})$ by the additive group. From the above isomorphism we deduce $I^+(\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F), \mathfrak{gl}_{\mathbf{v}})/\mathbb{A} \cong I^+(\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F), \mathfrak{pgl}_{\mathbf{v}})$. Let $p_{\mathfrak{pgl}} : I^+(\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F), \mathfrak{pgl}_{\mathbf{v}})/\mathrm{GL}_{\mathbf{v}} \to \mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}$, resp. $\bar{p}_{\mathfrak{X}}^+ : I^+(\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F), \mathfrak{pgl}_{\mathbf{v}})/\mathrm{GL}_{\mathbf{v}} \to \mathfrak{X}_{\mathbf{v}}(\mathscr{C})$, be the natural morphism. We deduce that the sheaf $\mathcal{R}_{\mathbf{v}} = (p_{\mathfrak{gl}_{\mathbf{v}}})_!(p_{\mathfrak{X}}^+)^* \wp^{\phi}$, on $\mathfrak{gl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}$, descends to a sheaf $\bar{\mathcal{R}}_{\mathbf{v}}$ on $\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}$ and, moreover, we have $\bar{\mathcal{R}}_{\mathbf{v}} = (p_{\mathfrak{pgl}})_! \bar{p}_{\mathfrak{X}}^+ \wp^{\phi}$.

Let $\pi' : \mathrm{T}^* \mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)/\mathrm{GL}_{\mathbf{v}} \to \mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)/\mathrm{GL}_{\mathbf{v}} = \mathfrak{X}_{\mathbf{v}}(\mathscr{C})$ be the projection and $\mu'_{\mathbf{v}} : \mathrm{T}^* \mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)/\mathrm{GL}_{\mathbf{v}} \to \mathfrak{pgl}_{\mathbf{v}}^*/\mathrm{GL}_{\mathbf{v}}$ the map induced by the moment map for the $\mathrm{PGL}_{\mathbf{v}}$-action on $\mathrm{T}^* \mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)$. A slight modification of Proposition $\overset{\texttt{Zfour}}{7.2.2}$ yields $\mathrm{F}_{\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}}((p_{\mathfrak{pgl}})_! \bar{p}_{\mathfrak{X}}^* \wp^{\phi}) = \mu'_!(\pi')^* \wp^{\phi}\{2\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F) - \dim \mathfrak{pgl}_{\mathbf{v}}\}$. Writing $d_{\mathbf{v}} = \dim \mathfrak{X}_{\mathbf{v}}(\mathscr{C})$, we compute

$$2\dim \mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F) - \dim \mathfrak{pgl}_{\mathbf{v}} = 2d_{\mathbf{v}} + 2\dim \mathrm{GL}_{\mathbf{v}} - \dim \mathfrak{pgl}_{\mathbf{v}} = 2d_{\mathbf{v}} + \mathbf{v} \cdot \mathbf{v} + 1.$$

By Theorem $\overset{\texttt{Rfact}}{6.4.1}$, the collection $(\mathcal{R}_{\mathbf{v}})$ is an Aff-equivariant factorization sheaf on $\mathfrak{gl}$. Thus, we obtain

$$\Phi_{\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}}(\mathcal{R}_{\mathbf{v}})_{\eta} = \mathrm{F}_{\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}}(\bar{\mathcal{R}}_{\mathbf{v}})_{\eta} = \mathrm{F}_{\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}}((p_{\mathfrak{pgl}})_! \bar{p}_{\mathfrak{X}}^+ \wp^{\phi}) \qquad (7.3.2) \quad \boxed{\texttt{phi-form}}$$

$$= (\mu'_{\mathbf{v}})_!(\pi')^* \wp^{\phi}\{2\dim \mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F) - \dim \mathrm{PGL}_{\mathbf{v}}\} = (\mu'_{\mathbf{v}})_!(\pi')^* \wp^{\phi}\{2d_{\mathbf{v}} + \mathbf{v} \cdot \mathbf{v} + 1\}.$$

Finally, applying $\overset{\texttt{3eqsG}}{(4.3.4)}$ and using $\overset{\texttt{tr3}}{(7.3.1)}$, we compute

$$\sum_{\mathbf{v}} z^{\mathbf{v}} \cdot \left(\sum_{[x] \in [X_{\mathbf{v}}(\mathsf{k})]} \psi(\phi(x)) \right) = \sum_{\mathbf{v}} z^{\mathbf{v}} \cdot [\mathrm{R\Gamma}_c(\mathrm{GL}_{\mathbf{v}}/_{\mathrm{Ad}}\mathrm{GL}_{\mathbf{v}},\, \varepsilon_{\mathbf{v}}^* \mathcal{R}_{\mathbf{v}})]$$

$$= \mathrm{Sym}\left((\mathbb{L}^{-\frac{1}{2}} - \mathbb{L}^{\frac{1}{2}}) \cdot \sum_{\mathbf{v} > 0} (-1)^{|\mathbf{v}|} \cdot z^{\mathbf{v}} \cdot \mathbb{L}^{\frac{|\mathbf{v}|}{2}} \cdot \left[\Phi_{\mathfrak{pgl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}}(\mathcal{R}_{\mathbf{v}})^{\langle \mathrm{sign} \rangle}|_{\eta_{\mathbf{v}}} \right] \right)$$

$$= \mathrm{Sym}\left((\mathbb{L}^{-\frac{1}{2}} - \mathbb{L}^{\frac{1}{2}}) \cdot \sum_{\mathbf{v} > 0} (-1)^{|\mathbf{v}|} \cdot z^{\mathbf{v}} \cdot \mathbb{L}^{\frac{|\mathbf{v}|}{2}} \cdot \left[\mu'_{\mathbf{v}})_!(\pi')^* \wp^{\phi} \right]^{\langle \mathrm{triv} \rangle}|_{\eta} \{2d_{\mathbf{v}} + \mathbf{v} \cdot \mathbf{v} + 1\} \right)$$

$$= \mathrm{Sym}\left((\mathbb{L}^{-\frac{1}{2}} - \mathbb{L}^{\frac{1}{2}}) \cdot \sum_{\mathbf{v} > 0} (-1)^{|\mathbf{v}|} \cdot z^{\mathbf{v}} \cdot \mathbb{L}^{\frac{|\mathbf{v}|}{2}} \cdot (-1)^{2d_{\mathbf{v}}+\mathbf{v}\cdot\mathbf{v}+1} \cdot \mathbb{L}^{-d_{\mathbf{v}} - \frac{\mathbf{v}\cdot\mathbf{v}}{2} - \frac{1}{2}} \cdot \left[(\mu'_{\mathbf{v}})_!(\pi')^* \wp^{\phi} \right]^{\langle \mathrm{triv} \rangle}|_{\eta} \right)$$

$$= \mathrm{Sym}\left((1 - \mathbb{L}^{-1}) \cdot \sum_{\mathbf{v} > 0} z^{\mathbf{v}} \cdot \mathbb{L}^{-(d_{\mathbf{v}} + \frac{\mathbf{v}\cdot\mathbf{v} - |\mathbf{v}|}{2})} \cdot \left[(\mu'_{\mathbf{v}})_!(\pi')^* \wp^{\phi} \right]^{\langle \mathrm{triv} \rangle}|_{\eta} \right),$$

where in the last equality we have used that $(-1)^{|\mathbf{v}|} \cdot (-1)^{2d_{\mathbf{v}}+\mathbf{v}\cdot\mathbf{v}+1} = -1$ since $2d_{\mathbf{v}}\mathbf{v} + \cdot\mathbf{v} + |\mathbf{v}|$ is an even integer.

By definition, we have $\phi_{\mathbf{v}} = (\pi')^* \wp^{\phi}$ and $\left[(\mu'_{\mathbf{v}})_!(\pi')^* \wp^{\phi} \right]^{\langle \mathrm{triv} \rangle}|_{\eta} = H^{\bullet}(\mathcal{M}_{Q,\mathbf{v}}^{\overset{\texttt{thm1}}{1}} \wp^{\phi_{\mathbf{v}}})$. Thus, taking the trace of Frobenius in the RHS side of the last equation yields formula $\overset{\texttt{thm1}}{(1.1.6)}$.

To prove formula (1.1.7), we consider the following cartesian diagram

$$\begin{array}{ccc}
\mathfrak{X}_{\mathbf{v}}(\mathscr{C}) & \xrightarrow{\ \widetilde{i}\ } & I^+(\mathfrak{X}_{\mathbf{v}}(\mathscr{C})) \\
{\scriptstyle p_0}\Big\downarrow & \square & \Big\downarrow{\scriptstyle p_{\mathfrak{g}}} \\
0/\mathrm{GL}_{\mathbf{v}} & \xrightarrow{\ i\ } & \mathfrak{gl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}
\end{array} \qquad\qquad (7.3.3) \quad \boxed{\texttt{in-f}}$$

where we have used the notation $p_0 = p_{\mathfrak{g}}|_{\mathfrak{X}_{\mathbf{v}}(\mathscr{C})}$. Applying base change for this diagram, we find

$$\mathrm{R}\Gamma_c(\mathfrak{X}_{\mathbf{v}}(\mathscr{C}), \wp^\phi) = \mathrm{R}\Gamma_c(\mathfrak{X}_{\mathbf{v}}(\mathscr{C}), \widetilde{i}^* p_{\mathfrak{X}}^* \wp^\phi) = \mathrm{R}\Gamma_c(\mathfrak{gl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}, (p_{\mathfrak{g}})_! \mathrm{R}\Gamma_c(0/\mathrm{GL}_{\mathbf{v}}, (p_0)_! \widetilde{i}^* p_{\mathfrak{X}}^* \wp^\phi)$$
$$= \mathrm{R}\Gamma_c(\mathfrak{gl}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}, i^*(p_{\mathfrak{g}})_! p_{\mathfrak{X}}^* \wp^\phi) \mathrm{R}\Gamma_c(0/\mathrm{GL}_{\mathbf{v}}, i^* \mathcal{R}_{\mathbf{v}}).$$

Thus, using (7.3.2) and (4.3.5) we compute

$$\sum_{\mathbf{v}} z^{\mathbf{v}} \cdot \mathbb{L}^{-\frac{\mathbf{v}\cdot\mathbf{v}}{2}} \cdot [\mathrm{R}\Gamma_c(X_{\mathbf{v}}, \wp^\phi)] = \mathrm{Sym}\left(\sum_{\mathbf{v}>0} z^{\mathbf{v}} \cdot \mathbb{L}^{-\frac{|\mathbf{v}|}{2}} \cdot [\Phi(\mathcal{R}_{\mathbf{v}})|_\eta]^{\langle\mathrm{triv}\rangle}\right)$$

$$= \mathrm{Sym}\left(\sum_{\mathbf{v}>0} z^{\mathbf{v}} \cdot \mathbb{L}^{-\frac{|\mathbf{v}|}{2}} \cdot (-1)^{2d_{\mathbf{v}}+\mathbf{v}\cdot\mathbf{v}+1} \cdot \mathbb{L}^{-d_{\mathbf{v}}-\frac{\mathbf{v}\cdot\mathbf{v}}{2}-\frac{1}{2}} \cdot \left[(\mu'_{\mathbf{v}})_!(\pi')^* \wp^\phi\right]^{\langle\mathrm{triv}\rangle}|_\eta\right)$$

$$= \mathrm{Sym}\left(-\mathbb{L}^{-\frac{1}{2}} \cdot \sum_{\mathbf{v}>0} (-1)^{|\mathbf{v}|} \cdot z^{\mathbf{v}} \cdot \mathbb{L}^{-(d_{\mathbf{v}}+\frac{\mathbf{v}\cdot\mathbf{v}+|\mathbf{v}|}{2})} \cdot \left[H^\bullet(\mathcal{M}_O, \wp^{\phi_{\mathbf{v}}})\right]^{\langle\mathrm{triv}\rangle}\right).$$

Equation (1.1.7) follows from the above by the Lefschetz trace formula applied to the sheaf $\wp^\phi$ on the stack $\mathfrak{X}_{\mathbf{v}}(\mathscr{C})$. This completes the proof of Theorem 1.1.5.

8. ANOTHER PROOF OF THEOREM 1.2.2

<div style="text-align:left">$\boxed{\texttt{pf\_sec}}$</div>

This section is devoted to an alternative proof of Theorem 1.2.2 that bypasses the use of Hua's formula. A key difference between this proof and the one based on Hua's formula is that the machinery of factorization sheaves will noow be applied to $\mathcal{R} \otimes \mathcal{S}$ rather than $\mathcal{R}$, where $\mathcal{S} = (\mathcal{S}_{\mathbf{v}})$ is a suitably chosen factorization sheaf formed by sufficiently generic rank one local systems $\mathcal{S}_{\mathbf{v}}$. A disadvantage of the new proof is that it only works under certain (mild) restrictions on the characteristic of the ground field.

8.1. **Generic characters.** Let $\mathbf{v} = (v_i)_{i\in I}$ be a dimension vector. For any integer vector $\boldsymbol{\lambda} = (\lambda_i)_{i\in I} \in \mathbb{Z}^I$ we consider a multiplicative character of the group $\mathrm{GL}_{\mathbf{v}}$ defined by

$$\chi_{\boldsymbol{\lambda},\mathbf{v}}: \ \mathrm{GL}_{\mathbf{v}} = \prod_{i\in I} \mathrm{GL}_{v_i} \longrightarrow \mathbb{G}, \quad (g_i)_{i\in I} \mapsto \prod_{i\in I} (\det g_i)^{\lambda_i}.$$

We identify $\mathbb{G}^{\mathbf{v}}$ with the maximal torus of diagonal matrices in $\mathrm{GL}_{\mathbf{v}}$. The restriction of $\chi_{\boldsymbol{\lambda},\mathbf{v}}$ to $\mathbb{G}^{\mathbf{v}}$ is a $\mathfrak{S}$-invariant character given by the formula

$$\mathbb{G}^{\mathbf{v}} = \prod_{i\in I} \mathbb{G}^{v_i} \longrightarrow \mathbb{G}, \quad (t_i^\beta)_{i\in I, \beta\in[v_i]} \longmapsto \prod_{i\in I} \left(\prod_{\beta\in[v_i]} t_i^\beta\right)^{\lambda_i}, \qquad (8.1.1) \quad \boxed{\texttt{chila}}$$

From now until the end of section 8, we let k be a finite field.

Definition 8.1.2. We say that $\boldsymbol{\lambda} \in \mathbb{Z}^I$ is *generic for* (k, $\mathbf{v}$) if the function $\chi_{\boldsymbol{\lambda}}|_{\mathbb{G}^{\mathbf{v}}_{\mathfrak{A}}(k)}$ is not identically equal to 1 for any decomposition $\mathfrak{A} \in \mathfrak{P}(\mathbf{v})$ such that $\mathfrak{A} \neq \Delta$ and it is identically equal to 1 for $\mathfrak{A} = \Delta$.

Our first goal is to show that there are 'sufficiently many' generic elements $\boldsymbol{\lambda} \in \mathbb{Z}^I$.

We write $\gcd(\mathbf{v}) = \gcd(v_i)_{i\in I}$. The following proposition, and Lemma 8.1.4 below, is very similar to [HLV2, Lemma 2.1.2].

Proposition 8.1.3. *Fix a dimension vector* $\mathbf{v} = (v_i)$ *and a prime* $p > 0$ *that does not divide* $\gcd(\mathbf{v})$. *Then, there exists an integer* $n_0 > 0$ *and* $\boldsymbol{\lambda} \in \mathbb{Z}^I$ *such that* $\boldsymbol{\lambda}$ *is generic for* $(\mathbb{F}_{p^n}, \mathbf{v})$ *for all* $n \geq n_0$.

Given $\mathbf{u} = (u_i)_{i \in I}$, $\mathbf{v} = (v_i)_{i \in I} \in \mathbb{Z}^I$, we write $\mathbf{u} < \mathbf{v}$ whenever $v_i - u_i \geq 0$, $\forall i$, and, moreover, $\mathbf{u} \neq \mathbf{v}$. Also, put $\mathbf{u} \cdot \mathbf{v} := \sum_{i \in I} u_i v_i$.

The proof of the proposition is based on the following number-theoretic result.

Lemma 8.1.4. *Given a dimension vector* $\mathbf{v}$ *and a prime* p *such that* p *does not divide* $\gcd(\mathbf{v})$, *there exists* $n_0 > 0$ *and* $\boldsymbol{\lambda} \in \mathbb{Z}^I$ *such that, for all* $n \geq n_0$, *we have* $\boldsymbol{\lambda} \cdot \mathbf{v} = 0 \bmod (p^n - 1)$ *and*

$$\boldsymbol{\lambda} \cdot \mathbf{u} \neq 0 \ \bmod (p^n - 1) \quad \text{for any} \quad 0 < \mathbf{u} < \mathbf{v}. \tag{8.1.5}$$

Proof of Lemma. Fix an integer $n > 0$ and put $q = p^n$, resp. $d := \gcd(\mathbf{v})$. Then, we have $\mathbf{v} = d \cdot \mathbf{w}$ where $\mathbf{w}$ is a dimension vector such that $\gcd(\mathbf{w}) = 1$. Let f_n be the composition of the following maps

$$\mathbb{Z}^I \xrightarrow{\ g: \ \boldsymbol{\lambda} \mapsto \mathbf{w} \cdot \boldsymbol{\lambda}\ } \mathbb{Z} \xrightarrow{\ h: \ m \mapsto d \cdot m\ } \mathbb{Z} \xrightarrow{\ r_n\ } \mathbb{Z}/(q-1),$$

where the last map is the natural projection.

Observe that if $\boldsymbol{\lambda} = (\lambda_i)_{i \in I}$ satisfies the requirements of the lemma then so does $\boldsymbol{\lambda}' = (\lambda_i + (q-1)m_i)_{i \in I}$, where $(m_i)_{i \in I}$ is any I-tuple of integers. Therefore, Put $s := \#I$ and $d := \gcd(\mathbf{v})$We will analize the dependence of the need to solve the equation $\sum_{i \in I} \lambda_i v_i = 0$ in the cyclic group $\mathbb{Z}/(q-1)$ simultaneously with the set of inequalities of the form $\sum_i \lambda_i u_i \neq 0$ for an I-tuple $\mathbf{u} = (u_i)_{i \in I}$ as above. Applying the Smith normal form for systems of linear equations over the principal ideal domain $\mathbb{Z}/(q-1)$ to one equation $\sum_i \lambda_i v_i = 0$, we see that this equation gives a subgroup isomorphic to $(\mathbb{Z}/(q-1))^{s-1} \times \mathbb{Z}/(d)$ where $d = \gcd(\mathbf{v})$. Fix a vector $\mathbf{u}$ as above which is not collinear to $\mathbf{v}$, i.e. there is no c such that $v_i = cu_i$ for all i. Then the Smith normal form applied to the system of two equations $\sum_i \lambda_i v_i = 0$ and $\sum_i \lambda_i u_i = 0$ tells us that the index of the subgroup given by these equations in the above subgroup $(\mathbb{Z}/(q-1))^{s-1} \times \mathbb{Z}/(d)$ increases as N increases, and in fact tends to infinity as N tends to infinity. Note that furthermore the number of tuples of the u_i as above does not change as N increases. Therefore the ratio of the total number of elements of the above group which satisfy one of the equations $\sum_i \lambda_i u_i = 0$ above with the u_i noncollinear to the v_i to the order of the group $(\mathbb{Z}/(q-1))^{s-1} \times \mathbb{Z}/(d)$ tends to zero as N increases.

We conclude the proof by noticing that any tuple $\lambda_i i$ that does not satisfy the equations $\sum_i \lambda_i u_i = 0$ above with the u_i noncollinear to the v_i will suit us as long as it does not satisfy $\sum_i \lambda_i u_i = 0$ with the u_i collinear to the v_i. But the last requirement means that in the group $(\mathbb{Z}/(q-1))^{s-1} \times \mathbb{Z}/(d)$ the element given by the λ_i projects onto a primitive element of $\mathbb{Z}/d$. Combining the conclusion of the last sentence of the previous paragraph with the fact that the ratio of the number of elements of $(\mathbb{Z}/(q-1))^{s-1} \times \mathbb{Z}/(d)$ to the order of $(\mathbb{Z}/(q-1))^{s-1} \times \mathbb{Z}/(d)$ is positive and constant as N increases, we will be able to select a tuple λ_i with the needed properties for N which is big enough. $\qquad \square$

Below, we will use the following well known result, see e.g. [DL, §5.3].

Lemma 8.1.6. *Let* H *be a torus defined over a finite field* k *and let* K *be a finite field extention of* k. *Then the norm map* $\mathsf{N}_{H(K)/H(\mathsf{k})} : H(K) \to H(\mathsf{k})$ *is surjective.* $\qquad \square$

Proof of Proposition. We first obtain an explicit formula for the restriction of the character $\chi_{\boldsymbol{\lambda}}$ to the diagonal torus $T_{\mathfrak{A}}^{\mathbf{v}}$, where $\mathfrak{A} = \{J^\alpha, \ \alpha \in [r]\}$ is a decomposition of $[\mathbf{v}]$. The torus $T_{\mathfrak{A}}^{\mathbf{v}}$ is, by definition, the image of a torus imbedding $\Delta(\mathfrak{A}) : \mathbb{G}^r \to \mathbb{G}^{\mathbf{v}} = \prod_{i \in I} \mathbb{G}^{v_i}$, given by

$$z = (z^1, \ldots, z^r) \longmapsto \Delta(\mathfrak{A})(z) = (\Delta(\mathfrak{A})(z)_i^\beta)_{i \in I, \beta \in [v_i]}, \quad \text{where} \quad \Delta(\mathfrak{A})(z)_i^\beta := z^\alpha \ \ \forall \beta \in [v_i] \cap J^\alpha.$$

Recall the notation $v_i^\alpha := \#([v_i] \cap J^\alpha)$, $\mathbf{v}^\alpha := (v_i^\alpha)_{i \in I}$. Then, one easily finds the formula

$$\chi_{\boldsymbol{\lambda}} \circ \Delta(\mathfrak{A})(z) = \prod_{\alpha=1}^r z_\alpha^{\Sigma_i \ \lambda_i v_i^\alpha} = \prod_{\alpha=1}^r z_\alpha^{\boldsymbol{\lambda} \cdot \mathbf{v}^\alpha}, \tag{8.1.7}$$

where we have used

Now, the group $\mathbb{F}_q^\times$ is a cyclic group of order $q-1$. Hence, the above formula shows that

$$\chi|_{\mathbb{G}_{\mathfrak{A}}^{\mathsf{v}}(\mathbb{F}_q)} = 1 \quad \Leftrightarrow \quad (q-1)|(\boldsymbol{\lambda} \cdot \mathbf{v}^\alpha), \quad \forall\, \alpha = 1, \ldots, r.$$

Note that $\mathbb{G}_{\mathfrak{A}}^{\mathsf{v}} = \mathbb{G}_{\Delta}^{\mathsf{v}}$ is the principal diagonal if and only if $r=1$. In this case, we have $\mathbf{v}^\alpha = \mathbf{v}$ and the above conditions reduce to a single condition $(q-1)|(\boldsymbol{\lambda} \cdot \mathbf{v})$. Further, for any decomposition $\mathfrak{A}$ with $r > 1$ one clearly has $\mathbf{v}^\alpha < \mathbf{v}$ for all $\alpha \in [r]$. Therefore, Lemma 8.1.4 insures the existence of $n \gg 0$ and $\boldsymbol{\lambda}$ such that the pair $(\mathbf{v}, \boldsymbol{\lambda})$ is generic over $\mathbb{F}_{p^n}$.

To complete the proof, we use Lemma 8.1.6. Thanks to the lemma, for any $m > n$ and any decomposition $\mathfrak{A}$, the norm map $\mathsf{N}_{\mathbb{G}_{\mathfrak{A}}^{\mathsf{v}}(\mathbb{F}_{p^m})/\mathbb{G}_{\mathfrak{A}}^{\mathsf{v}}(\mathbb{F}_{p^n})} : \mathbb{G}_{\mathfrak{A}}^{\mathsf{v}}(\mathbb{F}_{p^m}) \twoheadrightarrow \mathbb{G}_{\mathfrak{A}}^{\mathsf{v}}(\mathbb{F}_{p^n})$ is a surjective group homomorphism. We have shown that there exists $g \in \mathbb{G}_{\mathfrak{A}}^{\mathsf{v}}(\mathbb{F}_{p^n})$ such that $\chi_{\boldsymbol{\lambda}}(g) \neq 1$. Let $g' \in \mathbb{G}_{\mathfrak{A}}^{\mathsf{v}}(\mathbb{F}_{p^m})$ be a preimage of g. The multiplicative character $\chi_{\boldsymbol{\lambda}}$ being defined over $\mathbb{Z}$, it commutes with the Frobenius and the norm map. We deduce

$$\mathsf{N}_{\mathbb{G}(\mathbb{F}_{p^m})/\mathbb{G}(\mathbb{F}_{p^n})}(\chi_{\boldsymbol{\lambda}}(g')) = \chi_{\boldsymbol{\lambda}}(\mathsf{N}_{\mathbb{G}_{\mathfrak{A}}^{\mathsf{v}}(\mathbb{F}_{p^m})/\mathbb{G}_{\mathfrak{A}}^{\mathsf{v}}(\mathbb{F}_{p^n})}(g')) = \chi_{\boldsymbol{\lambda}}(g) \neq 1.$$

We conclude that $\chi_{\boldsymbol{\lambda}}(g') \neq 1$, so the pair $(\mathbf{v}, \boldsymbol{\lambda})$ is generic over $\mathbb{F}_{p^m}$ for all $m \geq n$, and the proposition follows. $\qquad\square$

8.2. Detecting absolutely indecomposable objects via generic characters.

We return to the setting of §6.2. Let $y = (x, b) \in Y_{\mathbf{v}}(\mathsf{k})$, where x is an object of $\mathscr{C}_{\mathsf{k}}$, and write $A(x, \mathsf{k}) := \mathrm{Aut}(x)(\mathsf{k})$. This is a finite group We have a homomorphism $p_y : A(x, \mathsf{k}) \to \mathrm{GL}_{\mathbf{v}}(\mathsf{k})$, cf. (5.1.1), of finite groups. Note that replacing the pair $y = (x, b)$ by a pair of the form $y' = (x, g(b))$ for some $g \in G_{\mathbf{v}}$ has the effect of replacing the map p_y by the map $p_{y'} = \mathrm{Ad}\, g \circ p_y$. Clearly, we have $\chi_{\boldsymbol{\lambda}, \mathbf{v}} \circ p_{y'} = \chi_{\boldsymbol{\lambda}, \mathbf{v}} \circ \mathrm{Ad}\, g \circ p_y = \chi_{\boldsymbol{\lambda}, \mathbf{v}} \circ p_y$. Therefore, the composite map $\chi_{\boldsymbol{\lambda}, \mathbf{v}} \circ p_y$ does not depend on the choice of an object y over x.

Notation 8.2.1. We reserve the notation θ for an *injective* multiplicative character $\theta : \mathbb{G}(\mathsf{k}) \hookrightarrow \underline{\mathbf{C}}^\times$, and we write $\theta_{\boldsymbol{\lambda}, \mathbf{v}, x} := \theta \circ \chi_{\boldsymbol{\lambda}, \mathbf{v}} \circ p_y$ for the composition:

$$A(x, \mathsf{k}) \xrightarrow{p_y} \mathrm{GL}_{\mathbf{v}}(\mathsf{k}) \xrightarrow{\chi_{\boldsymbol{\lambda}, \mathbf{v}}} \mathsf{k}^\times \xrightarrow{\theta} \underline{\mathbf{C}}^\times.$$

(As has been explained above, this composition depends on x but not on the choice of y over x.)

Proposition 8.2.2. *Let $\boldsymbol{\lambda}$ be generic for $(\mathsf{k}, \mathbf{v})$. Then, we have*

$$\frac{1}{\#A(x, \mathsf{k})} \sum_{f \in A(x, \mathsf{k})} \theta_{\boldsymbol{\lambda}, \mathbf{v}, x}(f) = \begin{cases} 1 & \text{if } x \text{ is absolutely indecomposable;} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Assume that x is absolutely indecomposable. Thus, $\bar{\mathsf{k}} \otimes_{\mathsf{k}} x \in \mathscr{C}_{\bar{\mathsf{k}}}$ is an indecomposable object and Corollary 6.2.7(ii) implies that $G_y = \mathbb{G}_{\Delta}^{\mathsf{v}} \cdot U$, where U is a unipotent subgroup of $\mathrm{GL}_{\mathbf{v}}$. It follows that $\chi_{\boldsymbol{\lambda}, \mathbf{v}}(U(\mathsf{k})) = 1$. Also, by our assumptions on $\boldsymbol{\lambda}$, we have that $\chi_{\boldsymbol{\lambda}, \mathbf{v}}(\mathbb{G}_{\Delta}^{\mathsf{v}}(\mathsf{k})) = 1$. We deduce that $\chi_{\boldsymbol{\lambda}, \mathbf{v}}(p_y(\mathrm{Aut}(x)) = \chi_{\boldsymbol{\lambda}, \mathbf{v}}(G_y) = \chi_{\boldsymbol{\lambda}, \mathbf{v}}(\mathbb{G}_{\Delta}^{\mathsf{v}}) = 1$, hence $\theta_{\boldsymbol{\lambda}, \mathbf{v}, x}(f) = 1$ for all $f \in A(x, \mathsf{k})$. This yields the required equality in the case where x is absolutely indecomposable.

Assume next that $\bar{\mathsf{k}} \otimes_{\mathsf{k}} x$ is a decomposable object of $\mathscr{C}_{\bar{\mathsf{k}}}$. Then, Corollary 6.2.7(i) says that the group G_y contains a torus H which is defined over k and is a $\mathrm{GL}_{\mathbf{v}}(\bar{\mathsf{k}})$-conjugate of a diagonal torus $T \neq \mathbb{G}_{\Delta}^{\mathsf{v}}$. If $\chi_{\boldsymbol{\lambda}, \mathbf{v}}|_{H(\mathsf{k})} \neq 1$, then $\theta_{\boldsymbol{\lambda}, \mathbf{v}, x}$ is a nontrivial multiplicative character of the finite group $A(x, \mathsf{k})$. In such a case, by the general character theory of finite groups it follows that one has $\sum_{f \in A(x, \mathsf{k})} \theta_{\boldsymbol{\lambda}, \mathbf{v}, x}(f) = 0$. Thus, to complete the proof of the proposition it suffices to prove the following result

Claim 8.2.3. For any $\boldsymbol{\lambda} \in \mathbb{Z}^I$ which is generic for $(\mathsf{k}, \mathbf{v})$ there exists a k-rational point $h \in H(\mathsf{k})$ such that $\chi_{\boldsymbol{\lambda}, \mathbf{v}}(h) \neq 1$.

Proof of Claim For any finite field extention K/k, we have a composite map $\mathsf{N}_{K/\mathsf{k}} \circ \chi_{\boldsymbol{\lambda},\mathbf{v}} :$ $H(K) \to \mathsf{k}^\times$ given by $g \mapsto \mathsf{N}_{K/\mathsf{k}}(\chi_{\boldsymbol{\lambda},\mathbf{v}}(g))$. The character $\chi_{\boldsymbol{\lambda},\mathbf{v}}$ is defined over $\mathbb{Z}$, hence, the homomorphism $\chi_{\boldsymbol{\lambda},\mathbf{v}}|_H$ commutes with the norm map. Therefore, using Lemma 8.1.6 we deduce the following chain of equalities of sets

$$\mathsf{N}_{K/\mathsf{k}}\big(\chi_{\boldsymbol{\lambda},\mathbf{v}}(H(K))\big) \;=\; \chi_{\boldsymbol{\lambda},\mathbf{v}}\big(\mathsf{N}_{H(K)/H(\mathsf{k})}(H(K))\big) \;=\; \chi_{\boldsymbol{\lambda},\mathbf{v}}(H(\mathsf{k})). \tag{8.2.4} \quad \boxed{\texttt{norm}}$$

Now, according to the assumptions of the claim, there is a diagonal torus $T \neq \mathbb{G}_\Delta^\vee$, in $G_\mathbf{v}$, and an element $g \in \mathrm{GL}_n(\bar{\mathsf{k}})$ such that we have $H = \mathrm{Ad}\, g(T)$. Let K be a finite field extension of k such that $g \in \mathrm{GL}_n(K)$. Then, we clearly have $\chi_{\boldsymbol{\lambda},\mathbf{v}}(H(K)) = \chi_{\boldsymbol{\lambda},\mathbf{v}}(T(K))$, an equality of subsets of $K^\times$. On the other hand, analogues of equations (8.2.4) hold for the torus T as well. Hence, we deduce

$$\chi_{\boldsymbol{\lambda},\mathbf{v}}(H(\mathsf{k})) \;=\; \mathsf{N}_{K/\mathsf{k}}\big(\chi_{\boldsymbol{\lambda},\mathbf{v}}(H(K))\big) \;=\; \mathsf{N}_{K/\mathsf{k}}\big(\chi_{\boldsymbol{\lambda},\mathbf{v}}(T(K))\big) \;=\; \chi_{\boldsymbol{\lambda},\mathbf{v}}(T(\mathsf{k})).$$

Since $T \neq \mathbb{G}_\Delta^\vee$ and $\boldsymbol{\lambda}$ is generic for $(\mathsf{k}, \mathbf{v})$, we have $\chi_{\boldsymbol{\lambda},\mathbf{v}}(T(\mathsf{k})) \neq \{1\}$. It follows that $\chi_{\boldsymbol{\lambda},\mathbf{v}}(H(\mathsf{k})) \neq \{1\}$, which is what we wanted. $\qquad\square$

8.3. **Factorization sheaves and generic local systems.** Let $\theta : \mathsf{k}^\times \to \underline{\mathbf{C}}^\times$ be a multiplicative character, and $\wp = \wp_\theta$ an associated rank one local system on $\mathbb{G}$. For any $\mathbf{v}$ and $\boldsymbol{\lambda}$, we have a local system $\chi_{\boldsymbol{\lambda},\mathbf{v}}^* \wp_\theta$, the pull-back of $\wp_\theta$ to the group $G_\mathbf{v}$. This local system comes equipped with a natural $\mathrm{Ad}\, G_\mathbf{v}$-equivariant structure and therefore descends to a local system $\wp_{\theta,\boldsymbol{\lambda},\mathbf{v}}^G$ on the quotient stack $G_\mathbf{v}/_{\mathrm{Ad}} G_\mathbf{v}$. Similarly, the character $\chi_{\boldsymbol{\lambda},\mathbf{v}}$ on $\mathbb{G}^\vee$ being symmetric, the corresponding local system on $\mathbb{G}^\vee$ is $\mathfrak{S}_\mathbf{v}$-equivariant and descends to a local system $\wp_{\theta,\boldsymbol{\lambda},\mathbf{v}}^T$ on the quotient stack $\mathbb{G}^\vee/\mathfrak{S}_\mathbf{v}$.

Definition 8.3.1. A rank one local system $\mathcal{S}$ on an algebraic group G is called *multiplicative* if there is an isomorphism $m_G^* \mathcal{S} \cong \mathcal{S} \boxtimes \mathcal{S}$, where $m_G : G \times G \to G$ is the multiplication map, satisfying an apropriate associativity constraint.

Lemma 8.3.2. (i) *For any $\boldsymbol{\lambda}$ and $\mathbf{v}$, the local system $\wp_{\theta,\boldsymbol{\lambda},\mathbf{v}}^G$, resp. $\wp_{\theta,\boldsymbol{\lambda},\mathbf{v}}^T$, is multiplicative.*

(ii) *For any fixed $\boldsymbol{\lambda}$, the collection $\wp_{\theta,\boldsymbol{\lambda}}^G = (\wp_{\theta,\boldsymbol{\lambda},\mathbf{v}}^G)_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^I}$, resp. $\wp_{\theta,\boldsymbol{\lambda}}^T = (\wp_{\theta,\boldsymbol{\lambda},\mathbf{v}}^T)_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^I}$, has the natural structure of a factorization sheaf on GL, resp. $\mathbb{G}$.*

Proof. We only prove (ii), since the proof of (i) is similar.

Let $\mathbf{v}_1, \mathbf{v}_2$ be a pair of dimension vectors and fix an imbedding $i_{\mathbf{v}_1,\mathbf{v}_2} : G_{\mathbf{v}_1} \times G_{\mathbf{v}_2} \hookrightarrow G_{\mathbf{v}_1+\mathbf{v}_2}$. Note that each of the maps $i_{\mathbf{v}_1,\mathbf{v}_2}$ and $m_\mathbb{G}$ is a group homomorphism and one has an equality $m_\mathbb{G} \circ (\chi_{\boldsymbol{\lambda},\mathbf{v}_1} \times \chi_{\boldsymbol{\lambda},\mathbf{v}_2}) = \chi_{\boldsymbol{\lambda},\mathbf{v}_1+\mathbf{v}_2} \circ i_{\mathbf{v}_1,\mathbf{v}_2}$. Hence, writing $\wp = \wp_\theta$ and using the canonical isomorphism $m_\mathbb{G}^* \wp \cong \wp \boxtimes \wp$, we obtain

$$i_{\mathbf{v}_1,\mathbf{v}_2}^* \chi_{\boldsymbol{\lambda},\mathbf{v}_1+\mathbf{v}_2}^* \wp \cong (\chi_{\boldsymbol{\lambda},\mathbf{v}_1} \times \chi_{\boldsymbol{\lambda},\mathbf{v}_2})^* m_\mathbb{G}^* \wp = (\chi_{\boldsymbol{\lambda},\mathbf{v}_1} \times \chi_{\boldsymbol{\lambda},\mathbf{v}_2})^* (\wp \boxtimes \wp) = \chi_{\boldsymbol{\lambda},\mathbf{v}_1}^* \wp \boxtimes \chi_{\boldsymbol{\lambda},\mathbf{v}_2}^* \wp.$$

This provides, for every pair $\mathbf{v}_1, \mathbf{v}_2$, a factorization isomorphism $\bar{\varphi}_{\mathbf{v}_1,\mathbf{v}_2}$, see Definition 4.2.2. We leave to the reader to check that the isomorphisms $\bar{\varphi}_{\mathbf{v}_1,\mathbf{v}_2}$ defined in this way satisfy all the necessary requirements. $\qquad\square$

Recall the notation $\bar{\iota}_\mathfrak{A} : \mathbb{G}_\mathfrak{A}^\vee \hookrightarrow \mathbb{G}^\vee$ for the closed imbedding of the diagonal torus associated with a decomposition $\mathfrak{A} \in \mathfrak{P}(\mathbf{v})$.

Definition 8.3.3. A rank one local system $\mathcal{S}$ on $\mathbb{G}^\vee$ is called *generic* if, for any $\mathfrak{A} \in \mathfrak{P}(\mathbf{v})$, $\mathfrak{A} \neq \Delta$, one has $\mathrm{R\Gamma}(\mathbb{G}_\mathfrak{A}^\vee, \bar{\iota}_\mathfrak{A}^* \mathcal{S}) = 0$ and, moreover, we have $\bar{\iota}_\Delta^* \mathcal{S} \cong \underline{\mathbf{C}}_{\mathbb{G}_\Delta^\vee}$.

Let $\wp_\theta$ be the local system associated with our *injective* multiplicative character θ.

Lemma 8.3.4. *If $\boldsymbol{\lambda}$ is generic for $(\mathsf{k}, \mathbf{v})$ then $\chi_{\boldsymbol{\lambda},\mathbf{v}}^* \wp_\theta$ is a generic local system on $\mathbb{G}^\vee$.*

Proof. Let $\boldsymbol{\lambda}$ be generic for $(\mathsf{k}, \mathbf{v})$. The restriction of the character $\chi_{\boldsymbol{\lambda},\mathbf{v}}$ to $\mathbb{G}_\Delta^{\vee}(\mathsf{k})$ being the trivial character, it follows that $(\chi_{\boldsymbol{\lambda},\mathbf{v}}^* \wp_\theta)|_{\mathbb{G}_\Delta^{\vee}} \cong \underline{\mathbf{C}}_{\mathbb{G}_\Delta^{\vee}}$. Next, let $\mathfrak{A} \neq \Delta$. To simplify notation, put $T = \mathbb{G}_{\mathfrak{A}}^{\vee}$ and $\mathcal{S} := \chi_{\boldsymbol{\lambda},\mathbf{v}}^* \wp_\theta$.

Assume first that $\mathrm{Hom}(\underline{\mathbf{C}}_T, \mathcal{S}) = H^0(T, \mathcal{S}) \neq 0$. This means that there exists a nonzero morphism $\underline{\mathbf{C}}_T \to \mathcal{S}$, of rank one local systems on T. Any such morphism must be an isomorphism. But the local system $\mathcal{S}$ can not be constant since the function $T(\mathsf{k}) \to \underline{\mathbf{C}}^{\times}$, $t \mapsto \mathsf{Tr}_{\mathsf{Fr}} | \mathcal{S}_t = \theta(\chi_{\boldsymbol{\lambda},\mathbf{v}}(t))$, is not a constant function, by the assumptions of the lemma. Thus, we conclude that $H^0(T, \mathcal{S}) = 0$.

Assume next that $H^{\bullet}(T, \mathcal{S}) \neq 0$ and let $n > 0$ be the minimal integer such that $H^n(T, \mathcal{S}) \neq 0$. We have

$$
\begin{aligned}
H^{\bullet}(T, \mathcal{S}) \otimes H^{\bullet}(T, \mathcal{S}) \; = \; H^{\bullet}(T \times T, \; \mathcal{S} \boxtimes \mathcal{S}) \; &= \; H^{\bullet}(T \times T, \; m^* \mathcal{S}) \; = \; H^{\bullet}(T, \; m_* m^* \mathcal{S}) \\
&= \; H^{\bullet}(T, \; \mathcal{S} \otimes m_* m^* \underline{\mathbf{C}}_T) \; = \; H^{\bullet}(T, \mathcal{S}) \otimes H^{\bullet}(T, \underline{\mathbf{C}}_T),
\end{aligned}
$$

where the first isomorphism holds since the local system $\mathcal{S}$ is multiplicative (Lemma 8.3.2) and the third isomorphism is the projection formula. The minimal degree of a nonvanishing homogeneous component of the tensor product on the left, resp. right, of the above chain of isomorphisms equals $2n$, resp. n. Hence, we must have $2n = n$, which can not be the case since $n > 0$. Thus, we deduce that $H^{\bullet}(T, \mathcal{S}) = 0$, as required. $\qquad\square$

We return to the setting of §6.2. Thus, given a dimension vector $\mathbf{v}$, we have the morphisms $p_X : I(X_{\mathbf{v}}) \to X_{\mathbf{v}} = \mathfrak{X}_{\mathbf{v}}(\mathscr{C}_{\mathsf{k}})$, $(x, f) \mapsto x$, resp. $p_G : I(X_{\mathbf{v}}) \to G_{\mathbf{v}} /_{\mathrm{Ad}} G_{\mathbf{v}}$, $(x, f) \mapsto \mathrm{Ad}\, G_{\mathbf{v}}(p_y(f))$, where $y = (x, b) \in Y_{\mathbf{v}}$ for any choice of base b of $F(x)$.

Proposition 8.3.5. *Let $\mathcal{E} \in D_{\mathrm{abs}}(X_{\mathbf{v}})$. Then, for any $\boldsymbol{\lambda} \in \mathbb{Z}^I$ which is generic for $(\mathsf{k}, \mathbf{v})$, we have*

$$
\mathsf{Tr}_{\mathsf{Fr}} | \, H_c^{\bullet}(G_{\mathbf{v}} /_{\mathrm{Ad}} G_{\mathbf{v}}, \; \wp_{\theta,\boldsymbol{\lambda},\mathbf{v}} \otimes (p_{G_{\mathbf{v}}/G_{\mathbf{v}}})_! p_{X_{\mathbf{v}}}^* \mathcal{E}) \; = \; \sum_{[x] \in \mathrm{AI}_{\mathbf{v}}(\mathscr{C}_{\mathsf{k}})} \mathsf{Tr}_{\mathsf{Fr}} | \, \mathcal{E}_x.
$$

Proof. Recall that $\mathsf{Tr}_{\mathsf{Fr}} | (\wp_{\theta,\boldsymbol{\lambda},\mathbf{v}})_{G_{\mathbf{v}}(g)} = \mathsf{Tr}_{\mathsf{Fr}} | (\wp_{\theta} \circ \chi_{\boldsymbol{\lambda},\mathbf{v}})_g = \theta(\chi_{\boldsymbol{\lambda},\mathbf{v}}(g))$, for any $g \in G_{\mathbf{v}}(\mathsf{k})$. Hence, applying formula (5.3.2) in the case where $\mathcal{F} = \wp_{\theta,\boldsymbol{\lambda},\mathbf{v}}$, we find

$$
\begin{aligned}
\mathsf{Tr}_{\mathsf{Fr}} | \, &H_c^{\bullet}(G /_{\mathrm{Ad}} G, \; \wp_{\theta,\boldsymbol{\lambda},\mathbf{v}} \otimes (p_G)_! p_X^* \mathcal{E}) \\
&= \; \sum_{[x] \in [X(\mathsf{k})]} \; \sum_{f \in A(x,\mathsf{k})} \frac{1}{\# A(x, \mathsf{k})} (\mathsf{Tr}_{\mathsf{Fr}} | (\wp_{\theta,\boldsymbol{\lambda},\mathbf{v}})_{p_G(x,f)}) \cdot (\mathsf{Tr}_{\mathsf{Fr}} | \mathcal{E}_x) \\
&= \; \sum_{[x] \in [X(\mathsf{k})]} \mathsf{Tr}_{\mathsf{Fr}} | \mathcal{E}_x \cdot \frac{1}{\# A(x, \mathsf{k})} \left(\sum_{f \in A(x,\mathsf{k})} \theta_{\boldsymbol{\lambda},\mathbf{v},x}(f) \right) \; = \; \sum_{[x] \in \mathrm{AI}(\mathscr{C}_{\mathsf{k}})} \mathsf{Tr}_{\mathsf{Fr}} | \mathcal{E}_x,
\end{aligned}
$$

where the last equality follows from Proposition 8.2.2. $\qquad\square$

8.4. We are now ready to prove the following, slightly weaker version of Theorem 1.2.2.

Theorem 8.4.1. *Let $(\mathscr{C}, F, \phi)$ be as in Corollary 1.2.2. Further write $q = p^r$, where p is a prime, and let $\mathbf{v}$ be a dimension vector such that p does not divide $\gcd(\{v_i\}_{i \in I})$. Then, there exists an integer $n \geq r$ such that for any $m \geq n$ the following holds:* For any multiplicative character $\psi : \mathbb{F}_{p^m} \to \mathbb{C}^{\times}$ and a sufficiently general regular semisimple $\mathrm{GL}_{\mathbf{v}}$-orbit $O_{\mathbf{v}} \subset \mathfrak{g}_{\mathbf{v}}^{\diamond}$, defined over $\mathbb{F}_{p^m}$, one has

$$
\sum_{x \in \mathrm{AI}_{\mathbf{v}}(\mathscr{C}_{\mathbb{F}_{p^m}})} \psi(\phi(x)) \; = \; q^{\frac{1}{2} \dim \mathcal{M}_{\eta\mathbf{v}}} \cdot \mathsf{Tr}_{\mathsf{Fr}} | \left(H_c^{\bullet}(\mathcal{M}_{\eta\mathbf{v}}, \; \mathrm{pr}^* \wp^{\phi})^{\langle \mathrm{sign} \rangle} \right).
$$

Proof. Fix a prime p and a dimension vector $\mathbf{v}$ as in the statement of the theorem. Thanks to Proposition 8.1.3, one can find an integer $n \geq r$ and an element $\boldsymbol{\lambda} \in \mathbb{Z}^I$ such that $\boldsymbol{\lambda}$ is generic for $(\mathbb{F}_{p^m}, \mathbf{v})$ for all $m \geq n$. We fix such a $\boldsymbol{\lambda}$ and given $m \geq n$ put $\mathsf{k} := \mathbb{F}_{p^m}$.

Now, in addition to the multiplicative character $\psi : \mathbb{A}(\mathsf{k}) \to \underline{\mathbf{C}}^{\times}$ that appears in the formula of the theorem, we choose (as we may) an injective multiplicative character $\theta : \mathbb{G}(\mathsf{k}) \to \underline{\mathbf{C}}^{\times}$. Thus, we have a local system $\wp_{\theta,\boldsymbol{\lambda},\mathbf{v}} \in D_{\mathrm{abs}}(G_{\mathbf{v}}/_{\mathrm{Ad}}G_{\mathbf{v}})$, and a local system $\wp^{\phi} = \phi^*\wp_{\psi} \in D_{\mathrm{abs}}(X_{\mathbf{v}})$.

The element $\boldsymbol{\lambda}$ being generic for $(\mathsf{k}, \mathbf{v})$, we are in a position to apply Proposition 8.3.5. Recall first that for $\mathcal{E} = \wp^{\phi}$ we have $(p_{G_{\mathbf{v}}/G_{\mathbf{v}}})_! p^*_{X_{\mathbf{v}}}\mathcal{E} = (p_{G_{\mathbf{v}}/G_{\mathbf{v}}})_! p^*_{X_{\mathbf{v}}}\wp^{\phi} = \mathcal{R}^{F,\phi}_{\mathbf{v}}$, by definition. Further, one has $\mathsf{Tr}_{\mathsf{Fr}}|\,\mathcal{E}_x = \mathsf{Tr}_{\mathsf{Fr}}|\,(\wp^{\phi})_x = \psi(\phi(x))$. Thus, in the case $\mathcal{E} = \wp^{\phi}$ the formula of the proposition reads

$$\mathsf{Tr}_{\mathsf{Fr}}|\,\mathrm{R}\Gamma_c(G_{\mathbf{v}}/_{\mathrm{Ad}}G_{\mathbf{v}},\ \wp_{\theta,\boldsymbol{\lambda},\mathbf{v}} \otimes \mathcal{R}^{F,\phi}_{\mathbf{v}}) \ = \sum_{[x] \in \mathrm{AI}_{\mathbf{v}}(\mathscr{C}_{\mathsf{k}})} \psi(\phi(x)). \tag{8.4.2}$$

Remark 8.4.3. The above equation may be compared with equation (7.3.1). Note that the summation in the RHS of (7.3.1) goes over the set of isomorphism classes of *all* objects of $X_{\mathbf{v}}$ while in the RHS of (8.4.2) only absolutely indecomposable objects are involved. $\diamondsuit$

We have a chain of isomorphisms

$$(\mathsf{F}_{\mathfrak{t}}\,\mathfrak{R}_{\mathfrak{t}}\,\mathcal{R}^{\mathrm{Lie}}_{\mathbf{v}})^{\mathfrak{S}_{\mathbf{v}}}_{\eta_{\mathbf{v}}} = (\mathrm{sign} \otimes \mathfrak{R}_{\mathfrak{t}*}\,\mathsf{F}_{\mathfrak{g}}\mathcal{R}^{\mathrm{Lie}}_{\mathbf{v}})^{\mathfrak{S}_{\mathbf{v}}}_{\eta_{\mathbf{v}}} = (\mathfrak{R}_{\mathfrak{t}*}\,\mathsf{F}_{\mathfrak{g}}\mathcal{R}^{\mathrm{Lie}}_{\mathbf{v}})^{\mathrm{sign}}_{\eta_{\mathbf{v}}} \qquad \text{(by (4.1.9))}$$

$$= ((\mathfrak{w}_{\mathfrak{t}*})_!\,\mathfrak{R}_{\mathfrak{t}*}(\mu_{\mathbf{v}})_! \wp_{\mathbf{v},\psi})^{\mathrm{sign}}_{\eta_{\mathbf{v}}}[2d_{\mathbf{v}}](d_{\mathbf{v}}) \qquad \text{(Prop. 7.2.2)}$$

$$= ((\mathfrak{w}_{\mathfrak{g}*})_!(\mu_{\mathbf{v}})_! \wp_{\mathbf{v},\psi})^{\mathrm{sign}}_{\eta_{\mathbf{v}}} \qquad \text{(by (4.1.3), cf. Lemma 4.1.8)}$$

$$= \mathrm{R}\Gamma_c((\mathfrak{w}_{\mathfrak{g}*} \circ \mu_{\mathbf{v}})^{-1}(\eta_{\mathbf{v}}),\ \wp^{\phi})^{\mathrm{sign}}$$

$$= \mathrm{R}\Gamma_c(\mathcal{M}_{\eta_{\mathbf{v}}}, \mathrm{pr}^*_{\mathbf{v}}\phi^*_{\mathbf{v}}\wp_{\psi})^{\mathrm{sign}}. \tag{8.4.4}$$

Next, we use isomorphism (4.1.12) in the case where $\wp = \wp_{\theta}$, $\chi = \chi_{\boldsymbol{\lambda},\mathbf{v}}$, and $\mathcal{F} = \mathcal{R}^{F,\phi}_{\mathrm{Lie},\mathbf{v}}$. Since $\varepsilon^*_G\mathcal{R}^{F,\phi}_{\mathrm{Lie},\mathbf{v}} = \mathcal{R}^{F,\phi}_{\mathbf{v}}$, we get

$$\mathrm{R}\Gamma_c(G_{\mathbf{v}}/_{\mathrm{Ad}}G_{\mathbf{v}},\ \wp_{\theta,\boldsymbol{\lambda},\mathbf{v}} \otimes \mathcal{R}^{F,\phi}_{\mathbf{v}}) \ = \ \mathrm{R}\Gamma_c(\mathbb{G}^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}},\ \wp_{\theta,\boldsymbol{\lambda},\mathbf{v}} \otimes (\varepsilon^*_T\,\mathfrak{R}_{\mathfrak{t}}\,\mathcal{R}^{F,\phi}_{\mathrm{Lie},\mathbf{v}})).$$

Combining this with (8.4.2), and comparing the resulting expression with (8.4.4), we see that we have reduced the proof to showing that in the Grothendieck group $\mathsf{K}^I_{\mathrm{abs}}$, one has

$$[\mathrm{R}\Gamma_c(\mathbb{G}^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}},\ \wp_{\theta,\boldsymbol{\lambda},\mathbf{v}} \otimes (\varepsilon^*_T\,\mathfrak{R}_{\mathfrak{t}}\,\mathcal{R}^{F,\phi}_{\mathrm{Lie},\mathbf{v}}))] \ = \ [(\mathsf{F}_{\mathfrak{t}}\,\mathfrak{R}_{\mathfrak{t}}\,\mathcal{R}^{F,\phi}_{\mathrm{Lie},\mathbf{v}})^{\mathfrak{S}_{\mathbf{v}}}_{\eta_{\mathbf{v}}}] \otimes \mathbb{M}. \tag{8.4.5}$$

We know that the collection $\mathcal{R}^{F,\phi}_{\mathrm{Lie}} = (\mathcal{R}^{F,\phi}_{\mathrm{Lie},\mathbf{w}})_{\mathbf{w} \in \in \mathbb{Z}^I_{\geq 0}}$ is an *Aff*-equivariant factorization sheaf on $\mathbb{A}$. We also know that $\wp_{\theta,\boldsymbol{\lambda},\mathbf{v}}$ is a generic local system on $\mathbb{G}^{\mathbf{v}}$, by Lemma 8.3.4. Therefore, to show that (8.4.5) holds, it is sufficient to prove the following more general result.

Claim 8.4.6. For any *Aff*-equivariant weak factorization sheaf $\mathcal{F}$, on $\mathbb{A}$, and any generic local system $\mathcal{S}$, on $\mathbb{G}^{\mathbf{v}}$, one has

$$[\mathrm{R}\Gamma_c(\mathbb{G}^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}},\ (\varepsilon^*_T\mathcal{F}_{\mathbf{v}}) \otimes \mathcal{S})] \ = \ [(\mathsf{F}_{\mathfrak{t}}\mathcal{F}_{\mathbf{v}})^{\mathfrak{S}_{\mathbf{v}}}_{\eta_{\mathbf{v}}}] \otimes \mathbb{M}. \tag{8.4.7}$$

Proof of Claim. Equation (8.4.7) is an equation in the the Grothendieck group $\mathsf{K}^I_{\mathrm{abs}}$. Therefore, as has been explained in the course of the proof of Theorem 3.3.1, it suffices to prove (8.4.7) assuming that our weak factorization sheaf $\mathcal{F} = (\mathcal{F}_{\mathbf{v}})$ is semisimple. We assume this from now on. Then, using the notation of the proof of Theorem 3.3.1, one has a decomposition $\widetilde{\mathcal{F}}_{\mathbf{v}} = \bigoplus_{\mathfrak{A} \in \mathfrak{P}(\mathbf{v})} \widetilde{\mathcal{F}}^{\mathfrak{A}}_{\mathbf{v}}$, where $\widetilde{\mathcal{F}}^{\mathfrak{A}}_{\mathbf{v}} \cong L_{\mathfrak{A}} \otimes \underline{\mathbf{C}}_{\mathfrak{A}}$, cf. §3.7. The local system $\mathcal{S}$ being generic, for any $\mathfrak{A} \neq \Delta$, we obtain

$$\mathrm{R}\Gamma_c(\mathbb{G}^{\mathbf{v}},\ \varepsilon^*\widetilde{\mathcal{F}}^{\mathfrak{A}}_{\mathbf{v}} \otimes \mathcal{S}) \ \cong \ L_{\mathfrak{A}} \otimes \mathrm{R}\Gamma_c(\mathbb{G}^{\mathbf{v}},\ \varepsilon^*\underline{\mathbf{C}}_{\mathfrak{A}} \otimes \mathcal{S}) \ \cong \ L_{\mathfrak{A}} \otimes \mathrm{R}\Gamma_c(\mathbb{G}^{\mathbf{v}}_{\mathfrak{A}},\ \mathcal{S}|_{\mathbb{G}^{\mathbf{v}}_{\mathfrak{A}}}) = 0,$$

where we have used simplified notation $\varepsilon = \varepsilon_T$. Further, for $\mathfrak{A} = \Delta$, we have

$$\mathrm{R}\Gamma_c(\mathbb{G}^{\mathbf{v}},\ \varepsilon^*\widetilde{\mathcal{F}}^{\Delta}_{\mathbf{v}} \otimes \mathcal{S}) = \mathrm{R}\Gamma_c(\mathbb{G}^{\mathbf{v}},\ \varepsilon^*\widetilde{\mathcal{F}}^{\Delta}_{\mathbf{v}}) = \mathrm{R}\Gamma_c(\mathbb{A}^{\mathbf{v}},\ \widetilde{\mathcal{F}}^{\Delta}_{\mathbf{v}}) \otimes \mathbb{M} = \mathsf{F}(\widetilde{\mathcal{F}}_{\mathbf{v}})_{\eta'_{\mathbf{v}}} \otimes \mathbb{M}.$$

We conclude that

$$R\Gamma_c(\mathbb{G}^{\mathbf{v}}/\mathfrak{S}_{\mathbf{v}},\ \varepsilon^*\mathcal{F}_{\mathbf{v}}\otimes\mathcal{S}) \ =\ R\Gamma_c(\mathbb{G}^{\mathbf{v}},\ \varepsilon^*\widetilde{\mathcal{F}}_{\mathbf{v}}\otimes\mathcal{S})^{\mathfrak{S}_{\mathbf{v}}} \ =\ R\Gamma_c(\mathbb{G}^{\mathbf{v}},\ \varepsilon^*\widetilde{\mathcal{F}}_{\mathbf{v}}^{\Delta}\otimes\mathcal{S})^{\mathfrak{S}_{\mathbf{v}}} \ =\ (\mathcal{F}_{\hat{\mathbf{v}}}|_{\eta_{\mathbf{v}}})^{\langle\mathfrak{S}_{\mathbf{v}}\rangle}\otimes\mathbb{M},$$

proving (8.4.7). $\hspace{3mm}$ [kabs-eq] $\hspace{10mm}$ $\square$

This completes the proof of Theorem 8.4.1. [weak-main]

<h2 style="text-align:center">9. PARABOLIC BUNDLES ON A CURVE</h2>

[higgs]

The goal of this section is to prove Theorem 1.4.5. [ind-bundles]

9.1. Approximation categories for vector bundles.
In this subsection we consider the case of vector bundles without parabolic structure.

Fix a smooth geometrically connected curve C. Recall that the slope of a vector bundle $\mathcal{V}$, on C, is defined by the formula *slope* $\mathcal{V} = \deg\mathcal{V}/\operatorname{rk}\mathcal{V}$. One has the corresponding notion of semistable, resp. stable, vector bundle in the sense of Mumford. For any vector bundle $\mathcal{V}$ there is a unique ascending filtration $0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \ldots \subset \mathcal{V}_n = \mathcal{V}$, by vector sub-bundles, called the *Harder-Narasimhan filtration*, such that each of the vector bundles $\mathrm{HN}_j(\mathcal{V}) := \mathcal{V}_j/\mathcal{V}_{j-1}$, $j = 1,\ldots,n$, is semistable and the following inequalities hold

$$slope\,\mathrm{HN}_1(\mathcal{V}) > slope\,\mathrm{HN}_2(\mathcal{V}) > \ldots > slope\,\mathrm{HN}_n(\mathcal{V}).$$

It is known that one has

$$slope\,\mathrm{HN}_1(\mathcal{V}) \geq slope\,\mathcal{V} \geq slope\,\mathrm{HN}_n(\mathcal{V}). \hspace{2cm} (9.1.1) \quad \boxed{\text{hn}}$$

Also, the Harder-Narasimhan filtration is functorial in the sense that any vector bundle morphism $f:\mathcal{V}\to\mathcal{V}'$ maps $\mathcal{V}_i$ to $\mathcal{V}'_j$ whenever *slope* $\mathcal{V}_i \leq$ *slope* $\mathcal{V}'_j$, cf. e.g. [An].

Recall that $\mathcal{V}ect(U)$ denotes the category of vector bundles on a scheme U, cf. Example 6.1.4. For any real number $\gamma > 0$, we let $\mathcal{V}ect_\gamma(C)$ be a full subcategory of $\mathcal{V}ect(C)$ whose objects are vector bundles $\mathcal{V}$ such that one has $-\gamma \leq$ *slope* $\mathrm{HN}_j(\mathcal{V}) \geq \gamma$ for all j. The functoriality of the Harder-Narasimhan filtration insures that $\mathcal{V}ect_\gamma$ is stable under finite direct sums and direct summands, so it is a k-linear Karoubian category. Moreover, the category $\mathcal{V}ect_\gamma(C)$ is quasi-abelian, [An]. Clearly, for any $\gamma < \delta$ there is a full imbedding $\mathcal{V}ect_\gamma(C) \hookrightarrow \mathcal{V}ect_\delta(C)$, so one has $\varinjlim_\gamma \mathcal{V}ect_\gamma(C) = \mathcal{V}ect(C)$.

[prestack] **Lemma 9.1.2.** *Let S be an affine connected scheme and $\mathcal{V}$ a vector bundle on $C \times S$. Then, the set formed by the (closed) points $s \in S$ such that $\mathcal{V}|_{C\times\{s\}} \in \mathcal{V}ect_\gamma(C)$ is a Zariski open subset of S.*

Proof. By a result of Shatz [Sh], the scheme S has a canonical partition $S = \sqcup_P S_P$ parametrized by convex polygons P in the plane $\mathbb{R}^2$, called Harder-Narasimhan polygons. For each P, the set S_P is a locally-closed subscheme of S, see [Sh, p. 183], all vector bundles $\mathcal{V}_s := \mathcal{V}|_{C\times\{s\}}$, $s \in S_P$, have the Harder-Narasimhan filtration of the same length $n(P)$, the number of vertices of P and, moreover, the coordinates of the j-th vertex are $(\operatorname{rk}\mathrm{HN}_j(\mathcal{V}_s),\ \deg\mathrm{HN}_j(\mathcal{V}_s))$.

Proving the lemma amounts to showing that the set

$$S_\gamma = \big\{s \in S \ \big|\ \exists j \text{ such that } slope\,\mathrm{HN}_j(\mathcal{V}_s) \notin [-\gamma,\gamma]\big\}$$

is a closed subset of S. It is sufficient to show that, for any Harder-Narasimhan polygon P, we have $\overline{S_\gamma \cap S_P} \subset S_\gamma$.

Let $s \in S_\gamma \cap S_P$, $s' \in \overline{S_\gamma \cap S_P}$, and let Q be the polygon such that $\mathcal{V}_{s'} \in S_Q$ where one may have $P = Q$, in general. The polygons P and Q share the same vertex $(0,0)$, that corresponds to the 0-th term of the Harder-Narasimhan filtration, and the same vertex $(\operatorname{rk}\mathcal{V}_s, \deg\mathcal{V}_s) = (\operatorname{rk}\mathcal{V}_{s'}, \deg\mathcal{V}_{s'})$, that corresponds to the last term of the Harder-Narasimhan filtration. By the semi-continuity theorem [Sh, Theorem 3], the polygon Q is on or above the polygon P. Further, since $s \in S_\gamma$

<div style="text-align:center">64</div>

we have that either $slope\,\mathrm{HN}_1(\mathcal{V}_s) > \gamma$ or $slope\,\mathrm{HN}_{n(P)}(\mathcal{V}_s) < -\gamma$. Therefore, in the first case one has $slope\,\mathrm{HN}_1(\mathcal{V}_{s'}) \geq slope\,\mathrm{HN}_1(\mathcal{V}_s) > \gamma$ and in the second case one has $slope\,\mathrm{HN}_{n(Q)}(\mathcal{V}_{s'}) \leq slope\,\mathrm{HN}_{n(P)}(\mathcal{V}_s) < -\gamma$. Thus, in either case, we deduce that $s' \in S_\gamma$, and the lemma follows. $\qquad\square$

Fix $\gamma > 0$. Given an affine test scheme S, let $\mathcal{V}ect_\gamma(C \times S)$ be a full subcategory of $\mathcal{V}ect(C \times S)$ whose objects are vector bundles $\mathcal{V}$ on $C \times S$ such that $\mathcal{V}|_{C \times \{s\}} \in \mathcal{V}ect_\gamma(C)$ for any closed point $s \in S$. The assignment $S \mapsto \mathcal{V}ect_\gamma(C \times S)$ defines a sheaf subcategory, $\mathcal{V}ect_{C,\gamma}$, of the sheaf category $\mathcal{V}ect_C$, cf. Example 6.1.4. The corresponding moduli stack $\mathfrak{X}(\mathcal{V}ect_\gamma(C))$ is a substack of the stack $\mathfrak{X}(\mathcal{V}ect(C))$. Specifically, we have $\mathfrak{X}(\mathcal{V}ect_\gamma(C)) = \sqcup_r \mathfrak{X}_r(\mathcal{V}ect_\gamma(C))$, where $\mathfrak{X}_r(\mathcal{V}ect_\gamma(C))$ is the stack parametrizing the objects of category $\mathcal{V}ect_\gamma(C)$ of rank r. Lemma 9.1.2 implies that $\mathfrak{X}_r(\mathcal{V}ect_\gamma(C))$ is an open substack of the stack $Bun_r(C)$. Hence, $\mathfrak{X}_r(\mathcal{V}ect_\gamma(C))$ is a smooth Artin stack.

Given r and γ, let $\mathcal{P}_{r,\gamma}$ be the set of Harder-Narasimhan polygons P with vertices contained in the set $\mathcal{P}_{r,\gamma} = \{(r', d') \in \mathbb{Z}^2 \mid 0 \leq r' \leq r,\ -\gamma \cdot r' \leq d' \leq \gamma \cdot r'\}$. The latter is finite. Hence, $\mathcal{P}_{r,\gamma}$ is a finite set. The result of Shatz [Sh, Proposition 11] yields a decomposition $\mathfrak{X}_r(\mathcal{V}ect_\gamma(C)) = \sqcup_{P \in \mathcal{P}_{r,\gamma}} Bun_P$, where Bun_P is a locally closed substack of $Bun_r(C)$ that parametrizes vector bundles with Harder-Narasimhan polygon P. Furthermore, each stack Bun_P has finite type. We deduce that for any fixed r and γ the stack $\mathfrak{X}_r(\mathcal{V}ect_\gamma(C))$ has finite type. We obtain the following result

Corollary 9.1.3. *For any $\gamma > 0$ and $r = 0, 1, \ldots$, the stack $\mathfrak{X}_r(\mathcal{V}ect_\gamma(C))$ is an open substack of $\mathfrak{X}_r(\mathcal{V}ect(C))$ of finite type and we have $\mathfrak{X}_r(\mathcal{V}ect(C)) = \varinjlim_\gamma \mathfrak{X}_r(\mathcal{V}ect_\gamma(C))$.*

Our next result shows that the category $\mathcal{V}ect_\gamma(C)$ 'captures' all indecomposable vector bundles provided γ is large enough.

Lemma 9.1.4. *Let $\mathcal{V}$ be an indecomposable vector bundle of rank $r \geq 1$ and degree d. Put*

$$\gamma_0 = \frac{|d|}{r} + 2(g-1)(r-1)\ \text{ if }\ genus(C) \geq 1, \quad \text{resp.} \quad \gamma_0 = \frac{|d|}{r}\ \text{ if }\ genus(C) = 0.$$

Then, $\mathcal{V}$ is an object of the category $\mathcal{V}ect_\gamma(C)$ for any $\gamma > \gamma_0$.

Proof. For $C = \mathbb{P}^1$ any indecomposable vector bundle has rank 1 and the result is clear.

Assume next that $genus(C) \geq 1$ and write $0 = \mathcal{V}_0 \subset \ldots \subset \mathcal{V}_n = \mathcal{V}$ for the Harder-Narasimhan filtration. By [Sh, Proposition 8], for an indecomposable vector bundle, we have $slope\,\mathrm{HN}_1(\mathcal{V}) - slope\,\mathrm{HN}_n(\mathcal{V}) \leq (n-1)(2g-2)$. Hence, using (9.1.1), for any $j = 1, \ldots, n$, we deduce

$$|slope\,\mathrm{HN}_j(\mathcal{V}) - slope\,\mathcal{V}| \leq slope\,\mathrm{HN}_1(\mathcal{V}) - slope\,\mathrm{HN}_n(\mathcal{V}) \leq (n-1)(2g-2) \leq 2(r-1)(g-1).$$

The result follows. $\qquad\square$

9.2. Approximation categories for parabolic bundles.

Below, we will freely use the notation of section 1.4. In particular, we have a collection $c_i \in C(\mathsf{k})$, $i \in I$, of marked points and put $D = \sum_{i \in I} c_i$. Following the ideas of [Se],[Fa],[Bi], [KMM] we are going to use the relation between parabolic bundles and orbifold bundles on a ramified covering of the curve C.

In more detail, fix $r \geq 1$, the rank of the parabolic bundle. Then, it follows from the proof of Theorem 1-1-1 in [KMM] that one can choose an integer N, a simple effective divisor D', and a ramified Galois covering $p : \widetilde{C} \to C$ such that the following holds:

(1) We have $N > r$ and $(N, \mathrm{char}\,\mathsf{k}) = 1$.
(2) The geometric Galois group Γ, of the covering, is a cyclic group of order N.
(3) $\mathsf{D} := D + D'$ is the ramification divisor of the covering.
(4) The divisor $\mathsf{D} = D + D'$ is simple and it is linearly equivalent over k to $N \cdot H$, where H is a very ample divisor. The covering is totally ramified at every point of the support of D.

The covering is constructed as the scheme of N-th roots of the canonical section of $\mathcal{O}(\mathsf{D})$ in the total space of a line bundle L corresponding to the divisor H above, as in the proof of Theorem 1-1-1 in [KMM].

Let I a labeling set for the points in the support of D. Thus, the set I contains I as a subset, and we have $\mathsf{D} = \sum_{i \in \mathsf{I}} c_i$. Let z_i be the unique point of $\widetilde{C}$ over c_i. Thus, z_i is a Γ-fixed point and Γ acts on $\mathrm{T}_{z_i}\widetilde{C}$, the tangent space at z_i, via a multiplicative character ε_i. Since $\Gamma \cong \mathbb{Z}/(N)$, we have $(\varepsilon_i)^N = 1$. Furthermore, ε_i is the 'tautological' character $\Gamma \hookrightarrow \mathbb{G}$; in particular, the set $\{1, \varepsilon_i, (\varepsilon_i)^2, \ldots, (\varepsilon_i)^{N-1}\}$ is a complete set of multiplicative characters of the group Γ and we have $\varepsilon_i = \varepsilon_{i'}$ for all $i, i' \in \mathsf{I}$. Let $\mathcal{V}ect^\Gamma(\widetilde{C})$ be the k-linear category of Γ-equivariant vector bundles on $\widetilde{C}$. For any $\mathcal{E} \in \mathcal{V}ect^\Gamma(\widetilde{C})$, the group Γ acts naturally on the fiber of $\mathcal{E}$ at z_i and there is a weight decomposition $\mathcal{E}|_{z_i} = \oplus_{0 \le k \le N-1} \mathcal{E}_i^{(k)}$, where $\mathcal{E}_i^{(k)}$ is the weight space of weight $(\varepsilon_i)^k$.

Fix $\mathbf{m} = (m_i)_{i \in \mathsf{I}}$, $1 \le m_i \le r$, and let $\mathcal{V} \in \mathcal{V}ect(C, \mathsf{D}, \mathbf{m})$ be a parabolic bundle. Recall that we write V_i for the fiber of $\mathcal{V}$, resp. $V_i^\bullet$ for the partial flag in that fiber, at the point c_i, $i \in \mathsf{I}$. We view $\mathcal{V}$ as a locally free sheaf and, for each (i, j), let $\mathcal{V}_i^{(j)}$ be a torsion free subsheaf formed by the sections v of $\mathcal{V}$ such that $v|_{c_i} \in V_i^{(j)}$. Thus, there is a short exact sequence

$$0 \to \mathcal{V}_i^{(j)} \to \mathcal{V} \to (V/V_i^{(j)}) \otimes \mathsf{k}_{c_i} \to 0, \qquad (9.2.1) \quad \boxed{\texttt{coh-flag}}$$

where $\mathsf{k}_{c_i} = \mathcal{O}_C/\mathcal{O}_C(-c_i)$ is a skyscraper sheaf at c_i. We have $\mathcal{V}_i^{(j)}/\mathcal{V}_i^{(j+1)} = (V_i^{(j)}/V_i^{(j+1)}) \otimes \mathsf{k}_{c_i}$.

Let $\widetilde{\mathsf{D}} = \sum_{i \in \widetilde{\mathsf{I}}} z_i$ denote the simple divisor in $\widetilde{C}$ given by the collection of distinct ramification points $z_i = p^{-1}(c_i)$. We define a subsheaf $\mathsf{P}(\mathcal{V})$ of the sheaf $\mathcal{O}_{\widetilde{C}}(N \cdot \widetilde{\mathsf{D}}) \otimes p^*\mathcal{V}$, by the formula

$$\mathsf{P}(\mathcal{V}) := \sum_{i \in \widetilde{\mathsf{I}}} \sum_{j=0}^{m_i} \mathcal{O}_{\widetilde{C}}(j \cdot z_i) \otimes p^*\mathcal{V}_i^{(j)}.$$

The sheaf $\mathsf{P}(\mathcal{V})$ comes equipped with a natural Γ-equivariant structure and it is torsion free, hence, locally free. Thus, we have $\mathsf{P}(\mathcal{V}) \in \mathcal{V}ect^\Gamma(\widetilde{C})$.

Let $\mathcal{V}ect_{\mathbf{m}}^\Gamma(\widetilde{C})$ be a full subcategory of $\mathcal{V}ect^\Gamma(\widetilde{C})$ whose objects are the Γ-equivariant vector bundles $\mathcal{E}$, on $\widetilde{C}$, such that, for every $i \in \widetilde{\mathsf{I}}$ and $m_i \le k \le N-1$, one has $\mathcal{E}_i^{(k)} = 0$. Thus, we have $\mathcal{E}|_{z_i} = \oplus_{0 \le k \le m_i-1} \mathcal{E}_i^{(k)}$. Observe that for any $\mathcal{V} \in \mathcal{V}ect(C, \mathsf{D}, \mathbf{m})$ and $i \in \widetilde{\mathsf{I}}$ it follows from the definition of the sheaf $\mathsf{P}(\mathcal{V})$ that for any $i \in \widetilde{\mathsf{I}}$ we have canonical isomorphisms

$$\mathsf{P}(\mathcal{V})|_{z_i} \cong \bigoplus_j \mathcal{O}(j \cdot z_i)/\mathcal{O}((j+1) \cdot z_i) \bigotimes (\mathcal{V}_i^{(j)}/\mathcal{V}_i^{(j+1)}) \cong \bigoplus_j (\mathrm{T}_{z_i}\widetilde{C})^{\otimes j} \bigotimes (V_i^{(j)}/V_i^{(j+1)}) \otimes \mathsf{k}_{c_i}.$$

We deduce that $\mathsf{P}(\mathcal{V})$ is an object of the category $\mathcal{V}ect_{\mathbf{m}}^\Gamma(\widetilde{C})$.

The idea of the following proposition goes back to Seshadri [Se]; the statement below is a slight strengthening of [Fa, §5].

$\boxed{\texttt{biswas}}$ **Proposition 9.2.2.** *The functor* $\mathsf{P} : \mathcal{V}ect(C, \mathsf{D}, \mathbf{m}) \to \mathcal{V}ect_{\mathbf{m}}^\Gamma(\widetilde{C})$, $\mathcal{V} \mapsto \mathsf{P}(\mathcal{V})$, *is an equivalence.*

Proof. An inverse of P is constructed as follows. Observe that for any Γ-equivariant coherent sheaf $\mathcal{E}$, on $\widetilde{C}$, the sheaf $(p_*\mathcal{E})^\Gamma$, of Γ-invariants, is a direct summand of $p_*\mathcal{E}$, since the order of Γ is prime to $\mathrm{char}\,\mathsf{k}$. If $\mathcal{E}$ is locally free, then the sheaf $p_*\mathcal{E}$ is also locally free and hence so is $(p_*\mathcal{E})^\Gamma$. Further, $p_*[\mathcal{E}(-j \cdot z_i)]$ is a (not necessarily Γ-stable) subsheaf of $p_*\mathcal{E}$. The sequence $p_*[\mathcal{E}(-j \cdot z_i)] \cap (p_*\mathcal{E})^\Gamma$, $j = 0, 1, \ldots$, is a flag of locally free subsheaves of $(p_*\mathcal{E})^\Gamma$ which induces a flag of subspaces $(p_*\mathcal{E})^\Gamma|_{c_i} = V_i^{(0)} \supseteq V_i^{(1)} \supseteq \ldots$, in the fiber of $(p_*\mathcal{E})^\Gamma$ at c_i. Furthermore, the definition of category $\mathcal{V}ect_{\mathbf{m}}^\Gamma(\widetilde{C})$ insures that, for any $\mathcal{E} \in \mathcal{V}ect_{\mathbf{m}}^\Gamma(\widetilde{C})$ and $i \in \mathsf{I}$, one has $V_i^{(m_i)} = 0$. Moreover, it is not

difficult to show that the resulting functor $\mathcal{V}ect_{\mathbf{m}}^{\Gamma}(\widetilde{C}) \to \mathcal{V}ect(C, \mathrm{D}, \mathbf{m})$, $\mathcal{E} \mapsto (p_*\mathcal{E})^{\Gamma}$ is an inverse of the functor P, cf. eg. [Bi, section 2(c)]. $\qquad\square$

Recall that the destabilizing subsheaf $\mathrm{HN}_1(\mathcal{E})$ of a coherent sheaf $\mathcal{E}$ is defined as the sum of all subsheaves of $\mathcal{E}$ of the maximal slope. It follows that if the sheaf $\mathcal{E}$ is equipped with an equivariant structure then each term, $\mathrm{HN}_j(\mathcal{E})$, of the Harder-Narasimhan filtration of $\mathcal{E}$ is an equivariant subsheaf of $\mathcal{E}$. Therefore, for any real $\gamma > 0$ there is a well defined k-linear category $\mathcal{V}ect_{\mathbf{m},\gamma}^{\Gamma}(\widetilde{C}) = \mathcal{V}ect_{\mathbf{m}}^{\Gamma}(\widetilde{C}) \cap \mathcal{V}ect_{\gamma}(\widetilde{C})$, a full subcategory of $\mathcal{V}ect_{\mathbf{m}}^{\Gamma}(\widetilde{C})$ whose objects are Γ-equivariant vector bundles $\mathcal{E}$ on $\widetilde{C}$ such that for all k one has $-\gamma \le slope\,\mathrm{HN}_k\,\mathcal{E} \le \gamma$. Further, we use the functor P and let $\mathcal{V}ect_{\gamma}(C, \mathrm{D}, \mathbf{m})$ be a full subcategory of $\mathcal{V}ect(C, \mathrm{D}, \mathbf{m})$ whose objects are the parabolic bundles $\mathcal{V}$, on C, such that $\mathrm{P}(\mathcal{V}) \in \mathcal{V}ect_{\mathbf{m},\gamma}^{\Gamma}(\widetilde{C})$.

We claim that an analogue of Lemma 9.1.4 holds in the parabolic setting, specifically, we have

Lemma 9.2.3. *For any parabolic bundle $\mathcal{V}$ which is an indecomposable object of category $\mathcal{V}ect(C, \mathrm{D}, \mathbf{m})$, we have $\mathcal{V} \in \mathcal{V}ect_{\gamma}(C, \mathrm{D}, \mathbf{m})$, for any $\gamma > \gamma_0$ where γ_0 is the same constant as in Lemma 9.1.4.*

Proof. The functor P being an equivalence, it suffices to prove the corresponding result for the category $\mathcal{V}ect_{\mathbf{m}}^{\Gamma}(\widetilde{C})$. Further, since $\mathcal{V}ect_{\mathbf{m}}^{\Gamma}(\widetilde{C})$ is a full Karoubian subcategory of the category $\mathcal{V}ect^{\Gamma}(\widetilde{C})$, of all Γ-equivariant vector bundles, it is sufficient to establish an analogue of Lemma 9.1.4 in the case of category $\mathcal{V}ect^{\Gamma}(\widetilde{C})$.

It is clear from the proof that Lemma 9.1.4 is, essentially, a consequence of [Sh, Proposition 8]. That proposition is deduced in [Sh] from the vanishing of certain Ext^1-groups. Let $\mathrm{Coh}_{\Gamma}(\widetilde{C})$, resp. $\mathrm{Coh}(\widetilde{C})$, be the abelian category of Γ-equivariant, resp. non-equivariant, coherent sheaves on $\widetilde{C}$. Then, for any Γ-equivariant coherent sheaves $\mathcal{E}, \mathcal{E}'$ on $\widetilde{C}$, we have

$$\mathrm{Ext}^1_{\mathrm{Coh}_{\Gamma}(\widetilde{C})}(\mathcal{E}, \mathcal{E}') \;=\; \big(\mathrm{Ext}^1_{\mathrm{Coh}(\widetilde{C})}(\mathcal{E}, \mathcal{E}')\big)^{\Gamma}.$$

This isomorphism follows from the exactness of the functor $(-)^{\Gamma}$, of Γ-invariants, which holds since the order of the group Γ is prime to $\mathrm{char}\,\mathsf{k}$. We conclude that the vanishing of $\mathrm{Ext}^1_{\mathrm{Coh}(\widetilde{C})}(\mathcal{E}, \mathcal{E}')$ implies the vanishing of $\mathrm{Ext}^1_{\mathrm{Coh}_{\Gamma}(\widetilde{C})}(\mathcal{E}, \mathcal{E}')$. This insures that all the arguments in the proof of [Sh, Proposition 8] go through in our Γ-equivariant setting. $\qquad\square$

One has a decomposition $\mathfrak{X}(\mathcal{V}ect(C, \mathrm{D}, \mathbf{m})) = \sqcup_{\mathbf{r}} \mathcal{P}\mathcal{B}un_{\mathbf{r}}$, where is $\mathcal{P}\mathcal{B}un_{\mathbf{r}}$ is the stack parametrizing parabolic bundles of type $\mathbf{r}$. For any $\gamma > 0$, there is a similar decomposition $\mathfrak{X}(\mathcal{V}ect_{\gamma}(C, \mathrm{D}, \mathbf{m})) = \sqcup_{\mathbf{r}} \mathfrak{X}_{\mathbf{r}}(\mathcal{V}ect_{\gamma}(C, \mathrm{D}, \mathbf{m}))$.

From now on, we let $\mathbf{r}$ be a type such that $|\mathbf{r}| = r$, where the r is the integer involved in the choice of the covering $\widetilde{C} \to C$. Then, we have a parabolic analogue of Corollary 9.1.3:

Lemma 9.2.4. *For any $\mathbf{r}$ and γ, the stack $\mathfrak{X}_{\mathbf{r}}(\mathcal{V}ect_{\gamma}(C, \mathrm{D}, \mathbf{m}))$ is an open substack of $\mathcal{P}\mathcal{B}un_{\mathbf{r}}$ of finite type and we have $\mathcal{P}\mathcal{B}un_{\mathbf{r}} = \varinjlim_{\gamma} \mathfrak{X}_{\mathbf{r}}(\mathcal{V}ect_{\gamma}(C, \mathrm{D}, \mathbf{m}))$.*

Proof. By standard results, it is known that $\mathfrak{X}_{\mathbf{r},d}(\mathcal{V}ect_{\mathbf{m}}^{\Gamma}(\widetilde{C}))$, the stack parametrizing objects of category $\mathcal{V}ect_{\mathbf{m}}^{\Gamma}(\widetilde{C})$ of type $\mathbf{r}$ and degree d, is an Artin stack of finite type. We have a decomposition $\mathfrak{X}_{\mathbf{r}}(\mathcal{V}ect_{\mathbf{m}}^{\Gamma}(\widetilde{C})) = \sqcup_d \mathfrak{X}_{\mathbf{r},d}(\mathcal{V}ect_{\mathbf{m}}^{\Gamma}(\widetilde{C}))$. We observe that the proof of Lemma 9.1.2 as well as the argument leading to Corollary 9.1.3 extend to the Γ-equivariant setting word for word. We deduce that $\mathfrak{X}_{\mathbf{r}}(\mathcal{V}ect_{\mathbf{m},\gamma}^{\Gamma}(\widetilde{C}))$ is an open substack of $\mathfrak{X}_{\mathbf{r}}(\mathcal{V}ect_{\mathbf{m}}^{\Gamma}(\widetilde{C}))$ of finite type and $\mathfrak{X}_{\mathbf{r}}(\mathcal{V}ect_{\mathbf{m}}^{\Gamma}(\widetilde{C})) = \varinjlim_{\gamma} \mathfrak{X}_{\mathbf{r}}(\mathcal{V}ect_{\mathbf{m},\gamma}^{\Gamma}(\widetilde{C}))$. Transporting these observations via the equivalence P, we conclude that

$\mathfrak{X}_{\mathbf{r}}(\mathcal{V}ect_\gamma(C,\mathsf{D},\mathbf{m}))$ is an open substack of $\mathfrak{X}_{\mathbf{r}}(\mathcal{V}ect(C,\mathsf{D},\mathbf{m}))$ of finite type and $\mathfrak{X}_{\mathbf{r}}(\mathcal{V}ect(C,\mathsf{D},\mathbf{m})) = \varinjlim_\gamma \mathfrak{X}_{\mathbf{r}}(\mathcal{V}ect_\gamma(C,\mathsf{D},\mathbf{m}))$.

It remains to show that the stack $\mathfrak{X}_{\mathbf{r}}(\mathcal{V}ect(C,\mathsf{D},\mathbf{m}))$ is smooth. To see this, we use a natural morphism of stacks $\mathcal{PB}un_{\mathbf{r}} \to Bun_r$ that sends parabolic bundles to the underlying vector bundles and forgets the partial flag data. It is clear that this is a smooth morphism and its fiber is isomorphic to a product of partial flag varieties. The stack Bun_r being smooth, it follows that so is $\mathcal{PB}un_{\mathbf{r}}$. $\mathcal{PB}un_{\mathbf{r}}$. $\qquad\square$

Recall that $\mathsf{D} = \sum_{i\in\mathsf{I}} c_i$. Given an I-tuple $\mathbf{n} = (n_i)$, where $n_i \geq 1$, define an I-tuple $\mathbf{m} = (m_i)$ by letting $m_i = n_i$ for all $i \in I$, resp. $m_i = 1$ for all $i \in \mathsf{I} \smallsetminus I$. Let $\Xi(\mathbf{m}) = \{(i,j) \mid i \in \mathsf{I}, j \in [1,m_i]\}$. Note that parabolic bundles of any type in $\Xi(\mathbf{m})$ have a trivial flag data in the fibers over the point c_i for each $i \in \mathsf{I} \smallsetminus I$. Therefore, adding trivial flags at these points yields a fully faithful functor $\rho : \mathcal{V}ect(C,D,\mathbf{n}) \to \mathcal{V}ect(C,\mathsf{D},\mathbf{m})$. For any $\gamma > 0$, let $\mathsf{Vect}_\gamma(C,D,\mathbf{m})$ be a full subcategory of category $\mathcal{V}ect(C,D,\mathbf{m})$ whose objects are parabolic bundles $\mathcal{V}$ such that $\rho(\mathcal{V})$ is an object of the subcategory $\mathcal{V}ect_\gamma(C,\mathsf{D},\mathbf{m})$.

Proposition 9.2.5. *Let $\mathbf{r} \in I^{\mathbf{n}}$ be a type such that $|\mathbf{r}| = r$. Then, we have*

(i) For any $\gamma > 0$ the stack $\mathfrak{X}_{\mathbf{r}}(\mathsf{Vect}_\gamma(C,D,\mathbf{n}))$ is a smooth open substack of $\mathcal{PB}un_{\mathbf{r}}$ of finite type. Furthermore, we have $\mathcal{PB}un_{\mathbf{r}} = \varinjlim_\gamma \mathfrak{X}_{\mathbf{r}}(\mathsf{Vect}_\gamma(C,D,\mathbf{n}))$.

(ii) There exists $\gamma_0 > 0$ such that the natural map $\mathrm{AI}_{\mathbf{r}}(\mathsf{Vect}_\gamma(C,D,\mathbf{n})) \to \mathrm{AI}_{\mathbf{r}}(\mathcal{V}ect(C,D,\mathbf{n}))$ is a bijection for all $\gamma \geq \gamma_0$.

Proof. Part (i) is a consequence of Lemmas 9.2.3–9.2.4. Define a type $\widetilde{\mathbf{r}} \in \Xi(\mathbf{m})$ by $\widetilde{r}_i^{(j)} = r_i^{(j)}$ if $i \in I$, resp. $\widetilde{r}_i^{(j)} = 0$ if $i \in \mathsf{I} \smallsetminus I$, $j > 1$. It is immediate from definitions that the map $\mathfrak{X}_{\mathbf{r}}(\mathcal{V}ect(C,D,\mathbf{n})) \to \mathfrak{X}_{\widetilde{\mathbf{r}}}(\mathcal{V}ect(C,\mathsf{D},\mathbf{m}))$, induced by the functor ρ, is an isomorphism of stacks and, moreover, this maps restricts to an isomorphism $\mathfrak{X}_{\mathbf{r}}(\mathsf{Vect}_\gamma(C,D,\mathbf{n})) \xrightarrow{\sim} \mathfrak{X}_{\widetilde{\mathbf{r}}}(\mathcal{V}ect_\gamma(C,\mathsf{D},\mathbf{m}))$. The required statements now follow from the corresponding statements of Lemmas 9.2.3–9.2.4 via the functor ρ. $\qquad\square$

9.3. Reminder on the Hitchin base.

Let C be a smooth projective curve, and $D = \sum_{i\in I} c_i$, as above. Let $\mathcal{L} := \Omega^1_C(D)$, an invertible sheaf on C. Write L for the total space of the corresponding line bundle, $p_C : L \to C$ for the projection, and put $L_i := p_C^{-1}(c_i) = \mathcal{L}|_{c_i}$, a line in L. The sheaf $p_C^*\mathcal{L}$ has a canonical section λ whose value at any element $\ell \in L$ equals ℓ.

The residue map $Res_i : \Gamma(C,\Omega^1_C(D)) \to \mathsf{k}$ at the point c_i gives, for any $j = 1,\ldots,r$, a residue map $Res_i^{\otimes j} : \Gamma(C,\mathcal{L}^{\otimes j}) \to \mathsf{k}$; in particular, it induces a canonical isomorphism of the fiber $L_i \xrightarrow{\sim} \mathbb{A}$.

Fix an integer $r \geq 1$. The 'Hitchin base' $Hitch_r$ is a vector space defined as

$$Hitch_r = \bigoplus_{j=1}^r \Gamma(C, \Omega^{\otimes j}_C(j{\cdot}D)) = \bigoplus_{j=1}^r \Gamma(C, \mathcal{L}^{\otimes j}).$$

Associated with an element $\varkappa = (\varkappa^{(j)})_{j\in[1,r]} \in Hitch_r$ there is a spectral curve $\Sigma_\varkappa$, a closed subscheme of L defined as the zero locus of the section $\lambda^{\otimes r} + \sum_{j=1}^r p_C^*\varkappa^{(j)} \otimes \lambda^{\otimes(r-j)} \in \Gamma(L,\mathcal{L}^{\otimes r})$. The composition $\Sigma_\varkappa \hookrightarrow L \twoheadrightarrow C$ is a finite morphism of degree r. Thus, $\Sigma_\varkappa \cap L_i$, the fiber of this morphism over c_i, is a scheme of length r. Explicitly, it may be identified, via the canonical isomorphism $L_i \cong \mathbb{A}$, with a subscheme of $\mathbb{A}$ defined by the equation $z^r + \sum_{j=1}^r Res_i^{\otimes j}(\varkappa^{(j)}) \cdot z^{r-j} = 0$. Let $|\Sigma_\varkappa \cap L_i| \in \mathbb{A}^r /\!\!/ \mathfrak{S}_r$ be the unorderd r-tuple of roots of this polynomial counted with multiplicities, i.e., an effective divisor in $\mathbb{A}$ of degree r. We define a morphism

$$res_{Hitch} : Hitch_r \longrightarrow (\mathbb{A}^r /\!\!/ \mathfrak{S}_r)^I = \prod_{i\in I} \mathbb{A}^r /\!\!/ \mathfrak{S}_r, \quad \varkappa \mapsto \prod_{i\in I} |\Sigma_\varkappa \cap L_i|.$$

Let $(\mathbb{A}^r)_0^I$ be a codimension one hyperplane of $(\mathbb{A}^r)^I$ formed by the points with the vanishing total sum of all coordinates, i.e. by the tuples $(z_i^{(j)})_{i\in I, j\in[1,r]}$ such that $\sum_{i,j} z_i^{(j)} = 0$. Let

$Hitch_r^{\mathrm{red,irr}} \subset Hitch_r$ be the locus of points $\varkappa \in Hitch_r$ such that the spectral curve $\Sigma_\varkappa$ is reduced and irreducible. The following result is certainly known to the experts but we have been unable to find a convenient reference in the literature.

Lemma 9.3.1. (i) *For any $\varkappa \in Hitch_r$ one has* $res_{Hitch}(\varkappa) \in (\mathbb{A}^r)_0^I /\!\!/ \mathfrak{S}_r$.

  (ii) *Assume that $\varkappa \in Hitch_r$ is such that the following two conditions hold:*

- *For each $i \in I$, the divisor $|\Sigma_\varkappa \cap L_i|$ is a sum of r pairwise distinct points.*   (9.3.2)

- *For any $0 < r' < r$ and any collection of subsets $Z_i \subset \Sigma_\varkappa \cap L_i$, $i \in I$, such that $\#Z_i = r'$ for all i, one has $\sum_{i \in I} \sum_{z \in Z_i} z \neq 0$.*   (9.3.3)

Then, the spectral curve $\Sigma_\varkappa$ is reduced and irreducible, i.e., we have $\varkappa \in Hitch_r^{\mathrm{red,irr}}$.

Proof. For any $i \in I$, the sum of roots of the equation $z^r + \sum_{j=1}^r Res_i^{\otimes j}(\varkappa^{(j)}) \cdot z^{r-j} = 0$ equals $Res_i(\varkappa^{(1)})$. Hence, proving part (i) of the lemma amounts to showing that one has $\sum_{i \in I} Res_i(\varkappa^{(1)}) = 0$. The latter equation holds thanks to the Residue Theorem applied to $\varkappa^{(1)} \in \Gamma(C, \mathcal{L})$, a rational 1-form on the curve C.

To prove (ii), we use the canonical section λ of $p_C^* \mathcal{L}$. Specifically, let $C^\circ = C \smallsetminus (\cup_i c_i)$ and $L^\circ = p_C^{-1}(C^\circ)$. By definition, one has a canonical isomorphism $\mathcal{L}|_{C^\circ} = \Omega_C^1(D)|_{C^\circ} \cong \Omega_{C^\circ}^1$. Hence, $\lambda|_{L^\circ}$, the restriction of λ to the open set L°, may be identified with a 1-form on L°. This 1-form restricts further to give a 1-form λ_Σ° on $\Sigma_\varkappa^\circ := \Sigma_\varkappa \cap L^\circ$, a curve in L° (the form λ_Σ° may have poles at the points of the finite set $\Sigma_\varkappa \smallsetminus \Sigma_\varkappa^\circ = \bigsqcup_{i \in I} \Sigma_\varkappa \cap L_i$).

Assume now that conditions (9.3.2)-(9.3.3) hold. Then, one shows that λ_Σ° has a simple pole at any point $z \in \Sigma_\varkappa \cap L_i \subset \mathbb{A}$ and, moreover, for the corresponding residue we have $Residue_z(\lambda_\Sigma^\circ) = z$. Therefore, we deduce

$$\sum_{i \in I} \sum_{z \in \Sigma_\varkappa \cap L_i} z = \sum_{i \in I} \sum_{z \in \Sigma_\varkappa \cap L_i} Residue_z(\lambda_\Sigma^\circ) = \sum_{z \in \Sigma_\varkappa \smallsetminus \Sigma^\circ} Residue_z(\lambda_\Sigma^\circ) = 0, \qquad (9.3.4) \quad \boxed{\text{sum-eta}}$$

by the Residue Theorem. More generally, a similar calculation implies that for any irreducible component Σ' of the curve $\Sigma_\varkappa$ one must have $\sum_{i \in I} \sum_{z \in \Sigma' \cap L_i} z = 0$. Let r' be the degree of the morphism $\Sigma' \to C$. If $\Sigma' \neq \Sigma_\varkappa$ then for every $i \in I$ we have $\#(\Sigma' \cap L_i) = r' < r$, contradicting condition (9.3.3). It follows that the spectral curve can not be reducible.

To complete the proof observe that condition (9.3.2) clearly implies that the scheme $\Sigma_\varkappa$ is reduced at any point of $\Sigma_\varkappa \cap L_i$. Hence, being irreducible, $\Sigma_\varkappa$ is generically reduced. Further, the projection $\Sigma_\varkappa \to C$ is a finite, hence a flat, morphism. It follows that $\Sigma_\varkappa$ is reduced. $\qquad\square$

9.4. Parabolic Higgs bundles. Fix $\mathbf{m} = (m_i)$. Following Example 6.1.6, we view $\mathcal{V}ect(C, D, \mathbf{m})$ as a sheaf category equipped with the sheaf functor $F : \mathcal{V}ect(C, D, \mathbf{m}) \to \mathcal{V}ect^{\Xi(\mathbf{m})}$. Thus, given a type $\mathbf{r} \in \Xi(\mathbf{m})$, we have the framed stack $Y_\mathbf{r} = \mathfrak{X}_\mathbf{r}(\mathcal{V}ect(C, D, \mathbf{m}), F)$ of parabolic bundles of type $\mathbf{r}$. The stack $\mathrm{T}^* Y_\mathbf{r}$ classifies triples $(\mathcal{V}, b, u)$, where $(\mathcal{V}, u)$ is a parabolic Higgs bundle and b is a framing on $\mathcal{V}$. The assignment $(\mathcal{V}, b, u) \mapsto (\mathrm{gr}\, res_i(u))_{i \in I}$ gives a moment map $\mu_\mathbf{r} : \mathrm{T}^* Y_\mathbf{r} \to \mathfrak{g}_\mathbf{r}^*$, see §1.4. We remark that $\mathfrak{g}_\mathbf{r}^*$ is a codimension 1 hyperplane in $\mathfrak{gl}_\mathbf{r}^* \cong \mathfrak{gl}_\mathbf{r} \subset (\mathfrak{gl}_r)^I$ cut out by the equation $\sum_{i \in I} \mathrm{tr}\, g_i = 0$. The fact that the image of $\mu_\mathbf{r}$ is contained in this hyperplane may be seen as a consequence of the Residue theorem $\sum_{i \in I} \mathrm{gr}\, res_i(u) = 0$. The morphism $f_{\mathrm{bas}} : \mathrm{T}^* Y_\mathbf{r} \to Higgs_\mathbf{r}$, $(\mathcal{V}, b, u) \mapsto (\mathcal{V}, u)$ is a $G_\mathbf{r}$-torsor. Hence, we have $(\mathrm{T}^* Y_\mathbf{r})/G_\mathbf{r} = Higgs_\mathbf{r}$ and the map $\mu_\mathbf{r}$ descends to a well defined morphism $res_{Higgs} : Higgs_\mathbf{r} \to \mathfrak{g}_\mathbf{r}^*/G_\mathbf{r}$, of quotient stacks.

We briefly recall the construction of the *Hitchin map* in the case of Higgs bundles without parabolic structure. Let $Higgs_r$ be the stack that parametrizes pairs $(\mathcal{V}, u)$, where $\mathcal{V}$ is a rank r vector bundle on C and u is a morphism $\mathcal{V} \to \mathcal{V} \otimes \Omega_C^1(D)$. Given such a pair $(\mathcal{V}, u)$, one considers a vector bundle $p_C^* \mathcal{V}$ on L and a morphism $p_C^* u : p_C^* \mathcal{V} \to p_C^* \mathcal{V} \otimes \mathcal{L}$, where we have used the notation of §9.3.

The corresponding 'characteristic polynomial' $\det(\lambda - p_C^* u)$ is a morphism $\wedge^r p_C^* \mathcal{V} \to \wedge^r p_C^* \mathcal{V} \otimes \mathcal{L}^{\otimes r}$, that is, an element of $\mathrm{Hom}(\wedge^r p_C^* \mathcal{V}, \wedge^r p_C^* \mathcal{V} \otimes \mathcal{L}^{\otimes r}) = \Gamma(C, \mathcal{L}^{\otimes r})$. This element has an expansion of the form $\det(\lambda - p_C^* u) = \sum_{j=0}^{r} \kappa^{(j)}(u) \otimes \lambda^{\otimes(r-j)}$, where $\kappa^{(j)}(u)$ is a section of $\mathcal{L}^{\otimes j}$; in particular, we have $\kappa^{(0)}(u) = 1$. Then, the Hitchin map is a map $\kappa_r : Higgs_r \to Hitch_r$ defined by the assignment $(\mathcal{V}, u) \mapsto \oplus_{j=1}^{r} \kappa^{(j)}(u)$.

Each of the stacks $Higgs_{\mathbf{r}}$, $Y_{\mathbf{r}}$, and $\mathrm{T}^* Y_{\mathbf{r}}$, can be partitioned further by fixing a degree d of the underlying vector bundle, so one has $Higgs_r = \sqcup_{d \in \mathbb{Z}} Higgs_{r,d}$, etc. Below, we will use the following known result that amounts, essentially, to a statement about compactified Jacobians proved in [BNR].

Proposition 9.4.1. *For any degree d, the stack $Higgs_{r,d} \cap \kappa_r^{-1}(Hitch_r^{\mathrm{red,irr}}) = Higgs_{r,d} \times_{Hitch_r} Hitch_r^{\mathrm{red,irr}}$ has finite type.* $\qquad\square$

The standard Cartan subalgebra of the Lie algebra $\mathfrak{gl}_{\mathbf{r}}$ of block diagonal matrices is isomorphic to $\oplus_{(i,j) \in \Xi(\mathbf{m})} \mathbb{A}^{r_i^{(j)}} = (\mathbb{A}^r)^I$. Let $\mathfrak{t}_{\mathbf{r}}$ be the Cartan subalgebra of the Lie algebra $\mathfrak{g}_{\mathbf{r}}$. Thus, one has a natural identification $\mathfrak{t}_{\mathbf{r}}^* \cong (\mathbb{A}^r)_0^I$ and the Chevalley isomorphism $\mathfrak{g}_{\mathbf{r}}^* /\!\!/ G_{\mathbf{r}} \cong \mathfrak{t}_{\mathbf{r}}^* /\!\!/ \mathfrak{S}_{\mathbf{r}} \cong (\mathbb{A}^r)_0^I /\!\!/ \mathfrak{S}_{\mathbf{r}}$.

Proof of Theorem 1.4.5. Forgetting the flag data on parabolic bundles yields a morphism f_{Higgs} : $Higgs_{\mathbf{r}} \to Higgs_r$. Various maps considered above fit into the following diagram

$$
\begin{array}{ccccccc}
\mathrm{T}^* Y_{\mathbf{r},d} & \xrightarrow{\ f_{\mathrm{bas}}\ } & Higgs_{\mathbf{r},d} & \xrightarrow{\ f_{\mathrm{Higgs}}\ } & Higgs_{r,d} & \xrightarrow{\ \kappa\ } & Hitch_r \\
\downarrow{\scriptstyle \mu_{\mathbf{r}}} & \square & \downarrow{\scriptstyle res_{Higgs}} & & & & \downarrow{\scriptstyle res_{Hitch}} \\
\mathfrak{g}_{\mathbf{r}}^* & \longrightarrow & \mathfrak{g}_{\mathbf{r}}^*/G_{\mathbf{r}} & = & \mathfrak{t}_{\mathbf{r}}^* /\!\!/ \mathfrak{S}_{\mathbf{r}} & = & (\mathbb{A}^r /\!\!/ \mathfrak{S}_r)_0^I
\end{array}
\tag{9.4.2}
$$

It follows from definitions that this diagram commutes. Therefore, we have $res_{Higgs}^{-1}(\mathfrak{g}_{\mathbf{r}}^\diamond /\!\!/ G_{\mathbf{r}}) = (res_{Hitch} \circ \kappa_r \circ f_{\mathrm{Higgs}})^{-1}(\mathfrak{t}_{\mathbf{r}}^\diamond /\!\!/ \mathfrak{S}_{\mathbf{r}})$. The morphism f_{Higgs} is a proper schematic morphism. Hence, from Lemma 9.3.1 and Proposition 9.4.1 we deduce that the stack $\mathrm{T}^* Y_{\mathbf{r},d}/G_{\mathbf{r}} = res_{Higgs}^{-1}(\mathfrak{g}_{\mathbf{r}}^\diamond /\!\!/ G_{\mathbf{r}})$ has finite type. In particular, the stack $\mathcal{M}_O := \mu_{\mathbf{r}}^{-1}(O)/G_{\mathbf{r}}$ is a stack of finite type for any coadjoint orbit $O \subset \mathfrak{g}_{\mathbf{r}}^\diamond$.

Further, for any $\gamma > 0$ let $Y_{\mathbf{r},d,\gamma}$ be the preimage of the substack $\mathfrak{X}_{\mathbf{r},d}(\mathsf{Vect}_\gamma(C, D, \mathbf{m})) \subset \mathcal{PBun}_{\mathbf{r},d}$ in $Y_{\mathbf{r},d}$. By Proposition 9.2.5, $Y_{\mathbf{r},d,\gamma}$ is an open substack of $Y_{\mathbf{r},d}$ of finite type. The morphism $\mathrm{T}^* Y_{\mathbf{r},d} \to Y_{\mathbf{r},d}$ being schematic, we deduce that $(\mathrm{T}^* Y_{\mathbf{r},d,\gamma})/G_{\mathbf{r}}$ is an open substack of $\mathrm{T}^* Y_{\mathbf{r},d}/G_{\mathbf{r}}$ of finite type. Furthermore, Proposition 9.2.5 says that $\mathcal{PBun}_{\mathbf{r},d} = \varinjlim_\gamma \mathfrak{X}_{\mathbf{r},d}(\mathsf{Vect}_\gamma(C, D, \mathbf{m}))$. Put

$\mathcal{M}_{O,\gamma} = \mathcal{M}_O \times_{Higgs_{\mathbf{r},d}} (\mathrm{T}^* Y_{\mathbf{r},d,\gamma})/G_{\mathbf{r}}$, an open substack of $\mathcal{M}_O$. It follows that $(\mathrm{T}^* Y_{\mathbf{r},d})/G_{\mathbf{r}} = \varinjlim_\gamma (\mathrm{T}^* Y_{\mathbf{r},d,\gamma})/G_{\mathbf{r}}$ and hence $\mathcal{M}_O = \varinjlim_\gamma \mathcal{M}_{O,\gamma}$. Thus, we obtain a diagram

$$\varinjlim_\gamma \mathcal{M}_{O,\gamma} = \mathcal{M}_O \hookrightarrow (\mathrm{T}^* Y_{\mathbf{r},d})/G_{\mathbf{r}} = Higgs_{\mathbf{r},d} \to \mathcal{PBun}_{\mathbf{r},d} = \varinjlim_\gamma \mathfrak{X}_{\mathbf{r},d}(\mathsf{Vect}_\gamma(C, D, \mathbf{m})).$$

The stack $\mathcal{M}_O$ being of finite type, it follows that there exists γ such that $\mathcal{M}_{O,\gamma} = \mathcal{M}_O$ and, moreover, the image of $\mathcal{M}_O$ in $\mathcal{PBun}_{\mathbf{r},d}$ is contained in $\mathfrak{X}_{\mathbf{r},d}(\mathsf{Vect}_\gamma(C, D, \mathbf{m}))$. Increasing γ, if necessary, we may (and will) assume that $\gamma > \gamma_0$, where γ_0 is as in Lemma 9.2.3. Thus, using that Lemma, we get

$$\# \mathrm{AI}(\mathcal{PBun}_{\mathbf{r},d}, \mathbb{F}_q) = \# \mathrm{AI}_{\mathbf{r},d}(\mathsf{Vect}_\gamma(C, D, \mathbf{m})), \mathbb{F}_q)$$

$$= q^{-\frac{1}{2} \dim \mathcal{M}_{O,\gamma}} \cdot \mathsf{Tr}_{\mathsf{Fr}} | H_c^\bullet(\mathcal{M}_{O,\gamma}, \bar{\mathbb{Q}}_\ell)^{\langle \mathrm{sign} \rangle} = q^{-\frac{1}{2} \dim \mathcal{M}_O} \cdot \mathsf{Tr}_{\mathsf{Fr}} | H_c^\bullet(\mathcal{M}_O, \bar{\mathbb{Q}}_\ell)^{\langle \mathrm{sign} \rangle},$$

where the second equality follows from Theorem 1.1.5 applied to category $\mathsf{Vect}_\gamma(C, D, \mathbf{m})$. $\qquad\square$

quiv-sec

This section is devoted to the proof of Theorem 1.3.2. Although the theorem is stated under the assumption that $\mathscr{C}$ is either the category of quiver representations or the category of parabolic bundles, a large part of the theory may be developed in a much more general setting.

isible-sec

10.1. The 'indivisible' case. Let $(\mathscr{C}, F)$ be a data as in §1.1, and $\mathfrak{X}(\mathscr{C}, F) = \sqcup_{\mathbf{v}} \mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)$ the corresponding framed stack. Let $\mathbf{v}$ be a dimension vector. Below, we assume that there is a $G_{\mathbf{v}}$-variety $\mathcal{Z}$ equipped with a $G_{\mathbf{v}}$-equivariant morphism $\mu_{\mathcal{Z}} : \mathcal{Z} \to \mathfrak{g}_{\mathbf{v}}^*$, to such that the following holds:

Categorical quotient: There is a categorical quotient $\varpi : \mathcal{Z} \to \mathcal{M}$, by $G_{\mathbf{v}}$, such that $\mathcal{M}$ is quasi-projective and ϖ is a surjective affine morphism. In particular, the map $\mu_{\mathcal{Z}}$ induces a map $f_{\mathcal{M}}$ that fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{\ \mu_{\mathcal{Z}}\ } & \mathfrak{g}_{\mathbf{v}}^* \\ {\scriptstyle \varpi}\downarrow & & \downarrow \\ \mathcal{M} & \xrightarrow{\ f_{\mathcal{M}}\ } & \mathfrak{g}_{\mathbf{v}}^* /\!\!/ G_{\mathbf{v}} \end{array} \qquad (10.1.1)$$

cat-quot

Generic isomorphism: The open set $\mu_{\mathcal{Z}}^{-1}(\mathfrak{g}_{\mathbf{v}}^\diamond)$ is Zariski dense in $\mathcal{Z}$ and there is an isomorphism $\aleph_{\mathcal{M}}^\diamond$, resp. $G_{\mathbf{v}}$-equivariant isomorphism $\aleph_{\mathcal{Z}}^\diamond$, of stacks, such that the following diagram commutes

$$\begin{array}{ccccc} T^* X_{\mathbf{v}}(\mathscr{C}, F) & \longleftarrow\!\!\!\supset & \mu_{\mathbf{v}}^{-1}(\mathfrak{g}_{\mathbf{v}}^\diamond) & \xrightarrow[\cong]{\ \aleph_{\mathcal{Z}}^\diamond\ } & \mu_{\mathcal{Z}}^{-1}(\mathfrak{g}_{\mathbf{v}}^\diamond) \\ {\scriptstyle \mathrm{proj}}\downarrow & & {\scriptstyle \mathrm{proj}}\downarrow & & \downarrow{\scriptstyle \varpi} \\ T^* X_{\mathbf{v}}(\mathscr{C}, F)/G_{\mathbf{v}} & \longleftarrow\!\!\!\supset & \mu_{\mathbf{v}}^{-1}(\mathfrak{g}_{\mathbf{v}}^\diamond)/G_{\mathbf{v}} & \xrightarrow[\cong]{\ \aleph_{\mathcal{M}}^\diamond\ } & f_{\mathcal{M}}^{-1}(\mathfrak{g}_{\mathbf{v}}^\diamond /\!\!/ G_{\mathbf{v}}). \end{array}$$

The variety $\mathcal{Z}$ may be thought of as a coarse modili space associated with a substack of semistable objects of the stack $T^* \mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F)$ for some stability condition.

Let Nil be the nilpotent cone in $\mathfrak{g}_{\mathbf{v}}^* \cong \mathfrak{g}_{\mathbf{v}}$. We introduce the following notation

$$\mathcal{Z}^\diamond = \mu_{\mathcal{Z}}^{-1}(\mathfrak{g}_{\mathbf{v}}^\diamond), \quad \mathcal{Z}_{\mathrm{nil}} = \mu_{\mathcal{Z}}^{-1}(\mathrm{Nil}), \quad \mathcal{Z}_0 = \mu_{\mathcal{Z}}^{-1}(0), \quad \mathcal{M}^\diamond = f_{\mathcal{M}}^{-1}(\mathfrak{g}_{\mathbf{v}}^\diamond /\!\!/ G_{\mathbf{v}}), \quad \mathcal{M}_{\mathrm{nil}} = f_{\mathcal{M}}^{-1}(0).$$

The categorical quotient condition implies that $\mathcal{M}_0 := \varpi(\mathcal{Z}_0)$ is a closed subscheme of $\mathcal{M}_{\mathrm{nil}}$. The generic isomorphism condition implies that $\mathcal{Z}^\diamond \to \mathcal{M}^\diamond$, the restriction of the map ϖ to $\mathcal{Z}^\diamond$, is a $G_{\mathbf{v}}$-torsor.

In the above setting, one can consider diagram (1.3.1), with $\mathcal{M}_{\mathbf{v}}^\theta$ being replaced by $\mathcal{M}$, and also the following commutative diagram

$$\begin{array}{ccccc} \mathcal{M}_{\mathrm{nil}} & \xrightarrow{\ \ \imath\ \ } & \mathcal{M}_{\mathrm{nil}} \times_{\mathfrak{g}_{\mathbf{v}}^*/\!\!/ G_{\mathbf{v}}} \mathfrak{t}_{\mathbf{v}}^* & \longleftarrow\!\!\!\supset^{\jmath} & \mathcal{M}^\diamond \times_{\mathfrak{g}_{\mathbf{v}}^*/\!\!/ G_{\mathbf{v}}} \mathfrak{t}_{\mathbf{v}}^\diamond \\ \downarrow & & \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle pr_2^\diamond} \\ \{0\} & \longhookrightarrow & \mathfrak{t}_{\mathbf{v}}^* & \longleftarrow\!\!\!\supset & \mathfrak{t}_{\mathbf{v}}^\diamond \end{array} \qquad (10.1.2)$$

la-diag

Thi diagram is essentially obtained from diagram (1.3.1) by base change via the projection $\mathfrak{t}_{\mathbf{v}}^* \to \mathfrak{t}_{\mathbf{v}}^*/W$, in view of a natural isomorphism $[\mathcal{M} \times_{\mathfrak{g}_{\mathbf{v}}^*/\!\!/ G_{\mathbf{v}}} \{0\}]_{\mathrm{red}} \cong \mathcal{M}_{\mathrm{nil}}$.

We use the Grothendieck-Springer resolution $\pi : \widetilde{\mathfrak{g}}_{\mathbf{v}} \to \mathfrak{g}$ and form a fiber product $\mathcal{Z} \times_{\mathfrak{g}_{\mathbf{v}}} \widetilde{\mathfrak{g}}_{\mathbf{v}}$. We now prove the following version of Theorem 1.3.2. The statement of part (iv) involves the notion of *semi-projective* algebraic variety introduced in of [HV2] and reminded in §11.2 below, cf. Definition 11.2.1.

Proposition 10.1.3. *Assume that:* (1) *the variety* $\mathcal{M}$ *is smooth,* (2) *the map* $\mu_{\mathcal{Z}}$ *is a smooth morphism,* (3) *the categorical quotient* $\mathcal{Z} \to \mathcal{M}$ *is a* geometric *quotient, and* (4) *there exists a categorical quotient* $\widetilde{\varpi} : \mathcal{Z} \times_{\mathfrak{g}_{\mathbf{v}}} \widetilde{\mathfrak{g}}_{\mathbf{v}} \to \widetilde{\mathcal{M}}$, *by* $G_{\mathbf{v}}$, *which is a surjective affine morphism. Then, the statements* (i) *and* (ii) *of Theorem 1.3.2 hold.*

Assume, in addition, that there is a $\mathbb{G}$-*action on* $\mathcal{M}$ *that makes it a semiprojective variety and such that the map* $f_{\mathcal{M}}$ *intertwines the* $\mathbb{G}$-*action on* $\mathcal{M}$ *and the* $\mathbb{G}$-*action on* $\mathfrak{t}_{\mathbf{v}}^*/\mathfrak{S}_{\mathbf{v}}$ *induced by the dilation action on* $\mathfrak{t}_{\mathbf{v}}^*$. *Then the conclusion of part* (iv) *of Theorem 1.3.2 holds.*

Example 10.1.4. (i) Let $\mathscr{C}$ be the category of representations of a quiver Q, let $\theta \in \mathbb{Q}^I$ be a stability condition. One takes $\mathcal{Z} = \mathrm{Rep}_{\mathbf{v}}^{\theta} \bar{Q}$, the open subset of $\mathrm{Rep}_{\mathbf{v}} Q$ formed by θ-semistable representations, resp. $\mathcal{M} = \mathrm{Rep}_{\mathbf{v}} \bar{Q} /\!/ \mathrm{PGL}_{\mathbf{v}}$. The $\mathbb{G}$-action on $\mathrm{Rep}_{\mathbf{v}} \bar{Q} /\!/ \mathrm{PGL}_{\mathbf{v}}$ is induced by the $\mathbb{G}$-action on $\mathrm{Rep}_{\mathbf{v}} \bar{Q}$ by scalar multiplication. The assumptions of the above proposition are known to hold for any indivisible dimension vector $\mathbf{v}$ whenever θ is sufficiently general.

(ii) If $\mathscr{C} = \mathcal{V}ect(C)$ is the category of vector bundles without parabolic structure, one takes $\mathcal{Z}$ to be the moduli space of semistable Higgs bundles of rank r and degree d equipped with a framing at a fixed (arbitrarily chosen) marked point, resp. $\mathcal{M}$ the corresponding moduli space of semistable Higgs bundles without a framing. The $\mathbb{G}$-action on $\mathcal{M}$ is induced by the dilation action along the fibers of the cotangent bundle, that is, given by rescaling the Higgs field. The assumptions of the above proposition are known to hold provided r and d are coprime.

The case of parabolic bundles of type $\mathbf{r}$ and degree d is similar; the assumptions of the proposition provided the vector $(\mathbf{r}, d)$ has coprime coordinates .

There is a functor Ψ_{pr_2}, resp. $\Psi_{f_{\mathcal{M}}}$, of nearby cycles over a *multi-dimensional smooth base* that sends sheaves on $\mathcal{M}^{\diamond} \times_{\mathfrak{g}_{\mathbf{v}}^*/G_{\mathbf{v}}} \mathfrak{t}_{\mathbf{v}}^{\diamond}$, resp. $\mathcal{M}^{\diamond}$, to sheaves constructible complexes on $\mathcal{M}_{\mathrm{nil}}$. This functor may be defined as the ordinary nearby cycles functor applied to the restrictiction of diagram (10.1.2), resp. (1.3.1), to a sufficiently general 1-dimensional subspace of $\mathfrak{t}_{\mathbf{v}}^*$, resp. the image of such a subspace in $\mathfrak{t}_{\mathbf{v}} /\!/ W$. In favorable situations, there is a monodromy action of the group $\pi_1^{et}(\mathfrak{t}_{\mathbf{v}}^{\diamond})$, resp. $\pi_1^{et}(\mathfrak{t}_{\mathbf{v}}^{\diamond} /\!/ W)$, on the sheaf of nearby cycles, cf. [Gri]. In the proof below, we will only use the following 'usual' properties of nearby cycles:

- Proper base change and smooth base change
- The nearby cycles functor associated with a smooth morphism is isomorphic to the restriction functor to the special fiber.
- For any sheaf $\mathcal{F}$ on $\mathcal{M}^{\diamond}$ one has a canonical isomorphism $\Psi_{pr_2}((pr_2^{\diamond})^* \mathcal{F}) \cong \Psi_{f_{\mathcal{Z}}}(\mathcal{F})$, of sheaves on $\mathcal{M}_{\mathrm{nil}}$, that respects the monodromy action of the group $\pi_1^{et}(\mathfrak{t}_{\mathbf{v}}^{\diamond})$.

We note that the monodromy action of the larger group $\pi_1^{et}(\mathfrak{t}_{\mathbf{v}}^{\diamond} /\!/ W) \supset \pi_1^{et}(\mathfrak{t}_{\mathbf{v}}^{\diamond})$ on $\Psi_{pr_2}((pr_2^{\diamond})^* \mathcal{F})$ can be recovered via the natural W-equivariant structure on $(pr_2^{\diamond})^* \mathcal{F}$, using that the subset $\mathcal{M}_{\mathrm{nil}}^{\theta} \subset \mathcal{M}^{\diamond} \times_{\mathfrak{g}_{\mathbf{v}}^*/G_{\mathbf{v}}} \mathfrak{t}_{\mathbf{v}}^W$ is pointwise fixed by W.

Proof of Proposition 10.1.3. We will use simplified notation $\mathfrak{g} = \mathfrak{g}_{\mathbf{v}}$, $\mathfrak{t} = \mathfrak{t}_{\mathbf{v}}$, etc. Since $\mu_{\mathcal{Z}}$ is a smooth morphism by assumption (2), the map $pr_{\widetilde{\mathfrak{g}}} : \mathcal{Z} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$, obtained by base change from $\mu_{\mathcal{Z}}$, is a smooth morphism as well. Hence $pr_{\widetilde{\mathfrak{g}}}$ is a smooth morphism and $\mathcal{Z} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}}$ is a smooth variety. Furthermore, the G-action on $\mathcal{Z} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}}$ is free since so is the action on $\mathcal{Z}$. It follows that $\widetilde{\varpi}$ is a universal geometric quotient. Also, assumptions (1) and (3) imply that $\mathcal{Z}$ is smooth.

By the universal property, the composite map $\mathcal{Z} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}} \to \mathcal{Z} \to \mathcal{M}$, factors through $\widetilde{\mathcal{M}}$. We obtain a commutative diagram

$$
\begin{array}{ccccc}
\mathcal{Z} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}} & \xrightarrow{\;u\;} & \mathcal{Z} \times_{\mathcal{M}} \widetilde{\mathcal{M}} & \longrightarrow & \mathcal{Z} \\
\Big\downarrow{\widetilde{\varpi}} & & \Big\downarrow{pr_2} \quad \square & & \Big\downarrow{\varpi} \\
\widetilde{\mathcal{M}} & =\!\!=\!\!=\!\!= & \widetilde{\mathcal{M}} & \longrightarrow & \mathcal{M}
\end{array}
$$

The map u in this diagram is a morphism of smooth G-varieties that induces an isomorphism on each geometric fiber. Hence, it is an isomorphism. Therefore, the perimeter of the diagram is cartesian and $\widetilde{\varpi}$ is a smooth morphism with fibers isomorphic to G. The composite map in the top row of the diagram is the first projection $\mathcal{Z} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}} \to \mathcal{Z}$. The latter is a projective morphism. It follows that the map $\widetilde{\mathcal{M}} \to \mathcal{M}$ is a projective morphism.

Next, we have the natural map $\nu : \widetilde{\mathfrak{g}} \to \mathfrak{t}^*$, similar to the map $\widetilde{\nu}$ in $(\overset{\text{GGTT}}{4.1.5})$. This map factors through a morphism $\widetilde{\mathfrak{g}} /\!/ G \to \mathfrak{t}^*$. The latter is an isomorphism since the morphism $\mathcal{O}_{\mathfrak{t}^*} \to (\nu_* \mathcal{O}_{\widetilde{\mathfrak{g}}})^G$ is known to be an isomorphism. Further, the composite map $\mathcal{Z} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}} /\!/ G$, factors through $\widetilde{\mathcal{M}}$, by the universal property of $\widetilde{\varpi}$. Thus, we obtain a diagram

$$
\begin{array}{ccccc}
\mathcal{Z} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}} & \xrightarrow{\widetilde{\varpi}} & \widetilde{\mathcal{M}} & \longrightarrow & \mathcal{M} \\
{\scriptstyle pr_{\widetilde{\mathfrak{g}}}}\downarrow & & {\scriptstyle \widetilde{f}}\downarrow & & {\scriptstyle f}\downarrow \\
\widetilde{\mathfrak{g}} & \xrightarrow{\widetilde{\mathfrak{w}}} & \widetilde{\mathfrak{g}} /\!/ G = \mathfrak{t}^* & \xrightarrow{\mathfrak{w}_{\mathfrak{t}}} & \mathfrak{g}^* /\!/ G = \mathfrak{t}^* /\!/ W.
\end{array}
\qquad (10.1.5) \quad \boxed{\texttt{TTg}}
$$

The left square in this diagram commutes by construction. Each of the maps $pr_{\widetilde{\mathfrak{g}}}$, $\widetilde{\mathfrak{w}}$, and $\widetilde{\varpi}$ is a smooth morphism. Hence, so is the map $\widetilde{f}$. The right square in $(\overset{\text{TTg}}{10.1.5})$ commutes thanks to the commutativity of $(\overset{\text{cat-quot}}{10.1.1})$. This gives a well-defined morphism $p : \widetilde{\mathcal{M}} \to \mathcal{M} \times_{\mathfrak{g}^* /\!/ G} \mathfrak{t}^*$. It is projective since so is the map $\widetilde{\mathcal{M}} \to \mathcal{M}$.

Let p_{nil}, resp. $p^\diamond$, be the restriction of p to $\widetilde{\mathcal{M}}_{\text{nil}}$, resp. $\widetilde{\mathcal{M}}^\diamond$. The map $p^\diamond$ is an isomorphism. Thus, we obtain a commutative diagram:

$$
\qquad (10.1.6) \quad \boxed{\texttt{grsp}}
$$

Here, we have used that the reduced scheme associated with $\mathcal{M} \times_{\mathfrak{g}^* /\!/ G} \{0\}$ is isomorphic to $\mathcal{M}_{\text{nil}}$.

The morphism $\widetilde{f}$ being smooth, the sheaf $\underline{\mathbf{C}}_{\widetilde{\mathcal{M}}}$ is locally acyclic with respect to $\widetilde{f}$. It follows from Lemma $\overset{\text{la-proper}}{11.1.4}$ that the sheaf $p_! \underline{\mathbf{C}}_{\widetilde{\mathcal{M}}}$ on $\mathcal{M} \times_{\mathfrak{g}^* /\!/ G} \mathfrak{t}^*$ is locally acyclic with respect to the map pr_2. Further, by the Decomposition theorem, the sheaf $p_! \underline{\mathbf{C}}_{\widetilde{\mathcal{M}}}$ is a direct sum of its perverse cohomology sheaves ${}^{\text{Perv}} R^k p_! \underline{\mathbf{C}}_{\widetilde{\mathcal{M}}}$ moreover, each of the sheaves ${}^{\text{Perv}} R^k p_! \underline{\mathbf{C}}_{\widetilde{\mathcal{M}}}$ is semisimple. It follows that each of the sheaves ${}^{\text{Perv}} R^k p_! \underline{\mathbf{C}}_{\widetilde{\mathcal{M}}}$ is also locally acyclic with respect to pr_2. Moreover, the map $p^\diamond$ being an isomorphism, the restriction of ${}^{\text{Perv}} R^k p_! \underline{\mathbf{C}}_{\widetilde{\mathcal{M}}}$ to $\mathcal{M} \times_{\mathfrak{g}^* /\!/ G} \mathfrak{t}^\diamond$ is a constant sheaf if $k = -\dim \mathcal{M}$ and is zero otherwise. Thus, applying Lemma $\overset{\text{mylem}}{11.1.5}$ we deduce that $p_! \underline{\mathbf{C}}_{\widetilde{\mathcal{M}}} = \text{IC}$, where we put $\text{IC} := \jmath_{!*} \underline{\mathbf{C}}_{\mathcal{M} \times_{\mathfrak{g}^* /\!/ G} \mathfrak{t}^\diamond} [-\dim \mathcal{M}]$, the minimal extension of the constant sheaf. It follows in particular that the sheaf IC is locally acyclic with respect to pr_2, proving part (i) of Theorem $\overset{\text{la-thm}}{1.3.2}$ in the indivisible case.

Similarly, since $\widetilde{f}$ is a smooth morphism, in the setting of nearby cycles we get $\Psi_{\widetilde{f}}(\mathbf{C}_{\widetilde{\mathcal{M}}}) = \underline{\mathbf{C}}_{\widetilde{\mathcal{M}}_{\text{nil}}}$. We obtain a chain of isomorphisms

$$
(p_{\text{nil}})_! \underline{\mathbf{C}}_{\widetilde{\mathcal{M}}_{\text{nil}}} = (p_{\text{nil}})_! \Psi_{\widetilde{f}} \underline{\mathbf{C}}_{\widetilde{\mathcal{M}}} = \Psi_{pr_2}(p_!^\diamond \underline{\mathbf{C}}_{\widetilde{\mathcal{M}}^\diamond}) = \Psi_{pr_2} \underline{\mathbf{C}}_{\mathcal{M} \times_{\mathfrak{g}^* /\!/ G} \mathfrak{t}^\diamond} = \Psi_{pr_2}(\mathfrak{w}_{\mathfrak{t}}^* \underline{\mathbf{C}}_{\mathfrak{t}^\diamond / W}) = \Psi_f \underline{\mathbf{C}}_{\mathfrak{t}^\diamond / W}.
$$
$$(10.1.7) \quad \boxed{\texttt{psi-base}}$$

Here the second equality follows by the proper base change for nearby cycles applied to the morphism p, the third equality holds since the map $p^\diamond$ is an isomorphism, and the last equality is a

property of the nearby cycles that we have assumed earlier. From the above isomorphisms, using (**??**) we deduce

$$H_c^\bullet(\mathcal{M}_{\mathrm{nil}}, \Psi_{f_\mathcal{M}}\mathbf{C}_{\mathfrak{g}^\diamond/\!\!/G}) \cong H_c^\bullet(\widetilde{\mathcal{M}}_{\mathrm{nil}}, \mathbf{C}). \tag{10.1.8}$$

Further, from diagram (10.1.6) we deduce

$$\Psi_{f_\mathcal{M}}\mathbf{C}_{\mathcal{M}^\diamond} = \Psi_{\widetilde{f}}\mathbf{C}_{\widetilde{\mathcal{M}}^\diamond} = (p_{\mathrm{nil}})_!\mathbf{C}_{\widetilde{\mathcal{M}}_{\mathrm{nil}}} = \imath^*(p_!\mathbf{C}_{\widetilde{\mathcal{M}}}) = \imath^*\,\mathrm{IC}\,.$$

The map $\widetilde{f}$ in diagram (10.1.6) is $\mathbb{G}$-equivariat. It follows, thanks to [HV2, Corollary 1.3.3] (cf. also [HLV2, Appendix B] and Theorem 11.2.2 of the present paper), that the restriction map $H^\bullet(\widetilde{\mathcal{M}}) \to H^\bullet(\widetilde{f}^{-1}(z))$ is an isomorphism for any $z \in \mathfrak{t}^*$ and that the sheaf $\widetilde{f}_!\mathbf{C}_{\widetilde{\mathcal{M}}}$ is geometrically constant, cf. Theorem 11.2.2. If $z = 0$ we have $\widetilde{f}^{-1}(0)) = \widetilde{\mathcal{M}}_{\mathrm{nil}}$. On the other hand, let $z \in \mathfrak{t}^\diamond$ and $\eta = \mathfrak{w}_\mathfrak{t}(z) \in \mathfrak{t}^\diamond/\!\!/W = \mathfrak{g}^*/\!\!/G$. The preimage of η under the quotient map $\mathfrak{w}_\mathfrak{g} : \mathfrak{g}^* \to \mathfrak{g}^*/\!\!/G$ is a regular semisimple coadjoint orbit O in $\mathfrak{g}^*$. The map $p^\diamond$ yields an isomorphism $\widetilde{f}^{-1}(z) \xrightarrow{\sim} f_\mathcal{M}^{-1}(\eta) = \mu_\mathcal{Z}^{-1}(O)/\!\!/G$. By our Generic isomorphism assumption we have $\mu_\mathcal{Z}^{-1}(O)/\!\!/G = \mu^{-1}(O)/G = \mathcal{M}_O$. Thus, using (10.1.8), we deduce

$$H_c^\bullet(\mathcal{M}_{\mathrm{nil}}, \Psi_{f_\mathcal{M}}\mathbf{C}_{\mathcal{M}^\diamond}) \cong H_c^\bullet(\widetilde{\mathcal{M}}_{\mathrm{nil}}, \mathbf{C}_{\mathcal{M}_{\mathrm{nil}}}) \cong H_c^\bullet(f_\mathcal{M}^{-1}(\eta), \mathbf{C}) \cong H_c^\bullet(\mu_\mathcal{Z}^{-1}(O)/\!\!/G) \cong H_c(\mathcal{M}_O). \tag{10.1.9}$$

It is clear from the construction that the composite isomorphism respects the natural monodromy actions. In particular, the monodromy action on $H_c^\bullet(\mu_\mathcal{Z}^{-1}(O)/\!\!/G, \mathbf{C})$ factors through a W-action. Alternatively, without reference to nearby cycles, this follows from the fact that the sheaf $\widetilde{f}_!\mathbf{C}_{\widetilde{\mathcal{M}}}$ is geometrically constant.

To complete the proof, we use the following commutative diagram

$$\tag{10.1.10}$$

Let $pr_\mathcal{Z} : \mathcal{Z} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}} \to \mathcal{Z}$, resp. $pr_\mathcal{M} : \mathcal{M} \times_{\mathfrak{g}^*/\!\!/G} \mathfrak{t}^* \to \mathcal{M}$ and $pr_{\mathfrak{g}^*} : \mathfrak{g}^* \times_{\mathfrak{g}^*/\!\!/G} \mathfrak{t}^* \to \mathfrak{g}^*$, be the first projection and $pr_\mathcal{Z}^\diamond$, resp. $pr_\mathcal{M}^\diamond$ and $pr_{\mathfrak{g}^*}^\diamond : \mathfrak{g}^* \times_{\mathfrak{g}^*/\!\!/G} \mathfrak{t}^\diamond \to \mathfrak{g}^\diamond$, its restriction to $\mathcal{Z}^\diamond \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}} \to \mathcal{Z}$, resp. $\mathcal{M} \times_{\mathfrak{g}^*/\!\!/G} \mathfrak{t}^\diamond$ and $\mathfrak{g}^* \times_{\mathfrak{g}^*/\!\!/G} \mathfrak{t}^\diamond$. We also use similar notation $pr_{\mathcal{Z},\mathrm{nil}}$, resp. $pr_{\mathcal{M},\mathrm{nil}}, pr_{\mathfrak{g}^*,\mathrm{nil}}$ for the corresponding maps over Nil. For $\rho \in \mathrm{Irr}(W)$, let $\mathrm{IC}_{\rho,\mathcal{Z}}$, resp. $\mathrm{IC}_{\rho,\mathcal{M}}$ and $\mathrm{IC}_{\rho,\mathfrak{g}^*}$, denote the IC-extension of the ρ-isotypic component of the local system $(pr_\mathcal{Z}^\diamond)_!\mathbf{C}_{\mathcal{Z}^\diamond \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}}}$, resp. $(pr_\mathcal{M}^\diamond)_!\mathbf{C}_{\mathcal{M} \times_{\mathfrak{g}^*/\!\!/G} \mathfrak{t}^\diamond}$ and $(pr_{\mathfrak{g}^*}^\diamond)_!\mathbf{C}_{\mathfrak{g}^* \times_{\mathfrak{g}^*/\!\!/G} \mathfrak{t}^\diamond}$. We have $\widetilde{f}_!\mathbf{C}_{\widetilde{\mathcal{M}}} = (pr_\mathcal{M})_!p_!\mathbf{C}_{\widetilde{\mathcal{M}}} = (pr_\mathcal{M})_!\,\mathrm{IC} = \oplus_\rho\,\mathrm{IC}_{\rho,\mathcal{M}}$. On the other hand, by smooth base change for the cartesian squares in the above diagram, we get

$$pr_{\mathcal{Z},\mathrm{nil}}^*\imath_{\mathcal{M}_{\mathrm{nil}}\to\mathcal{M}}^*\,\mathrm{IC}_{\rho,\mathcal{M}} = \imath_{\mathcal{Z}_{\mathrm{nil}}\to\mathcal{Z}}^*\,\mathrm{IC}_{\rho,\mathcal{Z}} = \mu_{\mathcal{Z},\mathrm{nil}}^*\imath_{\mathrm{Nil}\to\mathfrak{g}^*}^*\,\mathrm{IC}_{\rho,\mathfrak{g}^*}\,.$$

In the special case $\rho = \mathrm{sign}$, the sheaf $\imath_{\mathrm{Nil}\to\mathfrak{g}^*}^*\,\mathrm{IC}_{\rho,\mathfrak{g}^*}$ is known to be the sky-scrapper sheaf $\mathbf{C}_0$, at $\{0\} \subset \mathrm{Nil}$, by the theory of Springer representations. It follows that $\imath_{\mathcal{M}_{\mathrm{nil}}\to\mathcal{M}}^*\,\mathrm{IC}_{\mathrm{sign},\mathcal{M}} = \mathbf{C}_{\mathcal{M}_0}$ and, hence, $H_c^\bullet(\mathcal{M}_{\mathrm{nil}})^{(\mathrm{sign})} = H_c^\bullet(\mathcal{M}_0)$, as required in Theorem 1.3.2(ii).

Similarly, in the setting of nearby cycles, applying the smooth base change theorem for nearby cycles to the map pr, resp. $\mu_\mathcal{Z}$, we deduce

$$pr^*\Psi_{f_\mathcal{M}}\mathbf{C}_\mathcal{M} \cong \Psi_{f_\mathcal{M}\circ pr}\mathbf{C}_{\widetilde{\mathcal{M}}} \cong \Psi_{\mathfrak{w}_\mathfrak{g}\circ\mu_\mathcal{Z}}\mathbf{C} \cong \Psi_{\mu_\mathbf{v}}\mathbf{C} \cong (\mu_\mathbf{v})^*\Psi_{\mathfrak{w}_\mathfrak{g}}\mathbf{C}. \tag{10.1.11}$$

This completes the proof of Proposition 10.1.3. $\qquad\qquad\square$

Remark 10.1.12. In the case of quiver representations and parabolic Higgs bundles, the scheme $\mathcal{M}_{\mathrm{nil}}$ has a natural Poisson structure and the map $\widetilde{\mathcal{M}}_{\mathrm{nil}} \to \mathcal{M}_{\mathrm{nil}}$ is symplectic resolution.

new-sec

10.2. In the general, not necessarily indivisible case, the schemes $\mathcal{Z}$, $\widetilde{\mathcal{M}}$, and $\widetilde{\mathcal{M}}_{\mathrm{nil}}$ are not necessarily smooth, so the assumptions of Proposition 10.1.3 do not hold, in general. Below, we will adapt an idea of [HLV1] and construct an analogue of diagram (10.1.6) where $\mathcal{M} \times_{\mathfrak{g}_{\mathbf{v}}^*} \widetilde{\mathfrak{g}}_{\mathbf{v}}$ is replaced by a certain smooth scheme, that reduces to $\widetilde{\mathcal{M}}$ in the indivisible case. This will allow to adapt the arguments of the previous subsection and prove Theorem 1.3.2 in full generality.

We first consider quiver representations, as in Example 10.1.4. Let $\mathbf{v} \in \mathbb{Z}^I$ be a dimension vector. The commutator quotient $\mathrm{GL}_{\mathbf{v}}/[\mathrm{GL}_{\mathbf{v}}, \mathrm{GL}_{\mathbf{v}}]$ is isomorphic to $\mathbb{G}^I$ via the map $(g_i)_{i \in I} \mapsto (\det g_i)_{i \in I}$. We identify $\mathbb{K}$ the lattice $\mathrm{Hom}(\mathbb{G}, \mathrm{GL}_{\mathbf{v}}/[\mathrm{GL}_{\mathbf{v}}, \mathrm{GL}_{\mathbf{v}}])$, of 1-parameter subgroups of $\mathrm{GL}_{\mathbf{v}}/[\mathrm{GL}_{\mathbf{v}}, \mathrm{GL}_{\mathbf{v}}]$, with $\mathbb{Z}^I$. Let $\mathbb{G}^{\mathbf{v}}$ be the standard maximal torus of $\mathrm{GL}_{\mathbf{v}}$ so we have $\mathrm{Hom}(\mathbb{G}, \mathbb{G}^{\mathbf{v}}) = \mathbb{Z}^{\mathbf{v}}$ the lattice of 1-parameter subgroups of $\mathbb{G}^{\mathbf{v}}$. We have natural imbeddings $\mathbb{Z}_{\mathrm{diag}} \hookrightarrow \mathbb{Z}^I \hookrightarrow \mathbb{Z}^{\mathbf{v}}$ induced by the maps $\mathbb{G}_{\mathrm{diag}} \hookrightarrow \mathbb{G}^{\mathbf{v}} \to \mathrm{GL}_{\mathbf{v}}/[\mathrm{GL}_{\mathbf{v}}, \mathrm{GL}_{\mathbf{v}}])$. Let $T = \mathbb{G}^{\mathbf{v}}/\mathbb{G}_{\mathrm{diag}}$ be the maximal torus of $\mathrm{PGL}_{\mathbf{v}}$. We write $\mathbb{K} = \mathbb{Z}^I/\mathbb{Z}_{\mathrm{diag}} = \mathrm{Hom}(\mathbb{G}, \mathrm{PGL}_{\mathbf{v}}/[\mathrm{PGL}_{\mathbf{v}}, \mathrm{PGL}_{\mathbf{v}}])$, resp. $\widetilde{\mathbb{K}} = \mathbb{Z}^{\mathbf{v}}/\mathbb{Z}_{\mathrm{diag}} = \mathrm{Hom}(\mathbb{G}, T)$. Associated with an element of $\theta \in \mathrm{Hom}(\mathbb{K}, \mathbb{Z})$ viewed as an I-tuple $\theta = (\theta_i) \in \mathbb{Z}^I$ such that $\sum_{i \in I} \theta_i = 0$ there is a character $\chi_\theta : G \to \mathbb{G}$, $(g_i) \mapsto \prod_i \det(g_i)^{\theta_i}$. Let $\mathcal{B}$ be the flag variety of G. Let $\mathfrak{g} = \mathrm{Lie}\, G$ and $\widetilde{\mathfrak{g}} \to \mathfrak{g}^*$ the Grothendieck-Springer resolution, so $\widetilde{\mathfrak{g}}$ is a G-stable closed subvariety of $\mathfrak{g}^* \times \mathcal{B}$. Associated with an integral weight $\theta \in \mathrm{Hom}(\widetilde{\mathbb{K}}, \mathbb{Z})$ there is a G-equivariant line bundle $\mathcal{L}_{\mathcal{B}}^\theta$ on $\mathcal{B}$.

Let Q be a finite quiver with vertex set I and $\mathscr{C}$ the category of finite dimensional representations of Q eqipped with the forgetful functor F. Then, $\mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F) = \mathrm{Rep}_{\mathbf{v}} Q$ is the G-scheme of representations of Q of dimension $\mathbf{v}$ and $\mathrm{T}^* \mathfrak{X}_{\mathbf{v}}(\mathscr{C}, F) = \mathrm{Rep}_{\mathbf{v}} \bar{Q}$ is the G-scheme of representations of $\bar{Q}$, the double of Q. For any $\theta \in \mathrm{Hom}(\mathbb{K}, \mathbb{Z})$ let $\mathcal{L}_{\mathrm{Rep}_{\mathbf{v}} \bar{Q}}^\theta$ be the trivial line bundle $\mathrm{Rep}_{\mathbf{v}} \bar{Q} \times \mathbb{A} \to \mathrm{Rep}_{\mathbf{v}} \bar{Q}$ equipped with a G-equivariant structure defined by the G-action $g : (\rho, c) \mapsto (g\rho, \chi_\theta(g) \cdot c)$, for all $g \in G$, $\rho \in \mathrm{Rep}_{\mathbf{v}} \bar{Q}$, $c \in \mathbb{A}$. Let $\mu : \mathrm{Rep}_{\mathbf{v}} \bar{Q} \to \mathfrak{g}^*$ be the moment map. We consider a fiber product $\widetilde{\mathrm{Rep}_{\mathbf{v}}} \bar{Q} = \mathrm{Rep}_{\mathbf{v}} \bar{Q} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}}$ and let $p : \widetilde{\mathrm{Rep}_{\mathbf{v}}} \bar{Q} \to \mathrm{Rep}_{\mathbf{v}} \bar{Q}$ be the first projection. We also have the folowing composite map $\widetilde{\mu} : \widetilde{\mathrm{Rep}_{\mathbf{v}}} \bar{Q} \to \widetilde{\mathfrak{g}} \to \mathfrak{t}^*$, resp. $pr_{\widetilde{\mathrm{Rep}_{\mathbf{v}}} \bar{Q} \to \mathcal{B}} : \widetilde{\mathrm{Rep}_{\mathbf{v}}} \bar{Q} \to \widetilde{\mathfrak{g}} \to \mathcal{B}$. Using the identification $\mathfrak{g}^* = \mathfrak{g}$ one can view a closed point of $\widetilde{\mathrm{Rep}_{\mathbf{v}}} \bar{Q}$ as a pair $(\rho, \mathcal{F}l)$, where ρ is a representation of $\bar{Q}$ and $\mathcal{F}l \in \mathcal{B}$ is a $\mu(\xi)$-stable flag. We have $p(\rho, \mathcal{F}l) = \rho$, resp. $pr_{\widetilde{\mathrm{Rep}_{\mathbf{v}}} \bar{Q} \to \mathcal{B}}(\rho, \mathcal{F}l) = \mathcal{F}l$. Given $\theta \in \mathrm{Hom}(\widetilde{\mathbb{K}}, \mathbb{Z})$, one has a G-equivariant line bundle $pr_{\widetilde{\mathrm{Rep}_{\mathbf{v}}} \bar{Q} \to \mathcal{B}}^* \mathcal{L}_{\mathcal{B}}^\theta$ on $\widetilde{\mathrm{Rep}_{\mathbf{v}}} \bar{Q}$. Note that for θ in $\mathrm{Hom}(\mathbb{K}, \mathbb{Z})$, a sublattice in $\mathrm{Hom}(\widetilde{\mathbb{K}}, \mathbb{Z})$, the line bundle $\mathcal{L}_{\mathcal{B}}^\theta$ is trivial as a non-equivariant line bundle on $\mathcal{B}$, furthermore, one has a canonical isomorphism $pr_{\widetilde{\mathrm{Rep}_{\mathbf{v}}} \bar{Q} \to \mathcal{B}}^* \mathcal{L}_{\mathcal{B}}^\theta \cong p^* \mathcal{L}_{\mathrm{Rep}_{\mathbf{v}} \bar{Q}}^\theta$, of G-equivariant line bundles on $\widetilde{\mathrm{Rep}_{\mathbf{v}}} \bar{Q}$.

Let $\theta \in \mathrm{Hom}(\mathbb{K}, \mathbb{Z})$, resp. $\theta \in \mathrm{Hom}(\widetilde{\mathbb{K}}, \mathbb{Z})$. An element $\rho \in \mathrm{Rep}_{\mathbf{v}} \bar{Q}$, resp. $\xi \in \mathrm{Rep}_{\mathbf{v}} \bar{Q}$, is called θ-semistable if it is semistable (in the sense of GIT) with respect to the G-linearization $\mathcal{L}_{\mathrm{Rep}_{\mathbf{v}} \bar{Q}}^\theta$, resp. $pr_{\widetilde{\mathrm{Rep}_{\mathbf{v}}} \bar{Q} \to \mathcal{B}}^* \mathcal{L}_{\mathcal{B}}^\theta$. Stable objects are defined similarly. Note that θ-(semi)stability is equivalent to $N\theta$-(semi)stability, for any integer $N > 0$. This allows to define θ-(semi)stable objects of $\mathrm{Rep}_{\mathbf{v}} \bar{Q}$, resp. $\widetilde{\mathrm{Rep}_{\mathbf{v}}} \bar{Q}$, for an arbitrary element $\theta \in \mathbb{K}^* := \mathrm{Hom}(\mathbb{K}, \mathbb{Q})$, resp. $\theta \in \widetilde{\mathbb{K}}^* := \mathrm{Hom}(\widetilde{\mathbb{K}}, \mathbb{Q})$. Let $\mathcal{Z}^\theta \subset \mathrm{Rep}_{\mathbf{v}} \bar{Q}$, resp. $\widetilde{\mathcal{Z}}^\theta \subset \widetilde{\mathrm{Rep}_{\mathbf{v}}} \bar{Q}$, be the θ-semistable locus.

Next, we consider the setting of parabolic bundles. Recall the notation $\Xi(\mathbf{m})$ and fix a type $\mathbf{r} \in \mathbb{Z}_{\geq 0}^{\Xi(\mathbf{m})}$. Following Mehta and Seshadri [MeSe], a stability condition for parabolic Higgs bundles of type $\mathbf{r}$ is given by an I-tuple $\theta_i^{(1)} \leq \theta_i^{(2)} \leq \ldots \leq \theta_i^{(m_i)}$, of rational numbers, such that $\theta_i^{(m_i)} - \theta_i^{(1)} < 1$.

The slope of the parabolic bundle of type $\mathbf{r}$ and degree d is defined as $\frac{1}{r}(d + \sum_{i,j} r_i^{(j)} \cdot \theta_i^{(j)})$, where $r = |\mathbf{r}|$ is the rank of the corresponding vector bundle.

The It will be convenient for us to allow nonstrict inequalities as well. and to use the normalization $\sum_{i,j} \theta_i^{(j)} = 0$. The center $Z(\mathrm{GL}_\mathbf{r})$ of the group $\mathrm{GL}_\mathbf{r} = \prod_{(i,j)\in\Xi(\mathbf{m})} \mathrm{GL}_{r_i^{(j)}}$ is naturally identified with $\mathbb{G}^\mathbf{r}$ and the stability condition $(\theta_i^{(j)}) \in \mathbb{Q}^\mathbf{r}$ may be viewed as an element in the fundamental alcove in $\mathbb{Q} \otimes_\mathbb{Z} \mathrm{Hom}(\mathbb{G}, Z(\mathrm{GL}_\mathbf{r}))$. The element $\theta = (\theta_i^{(j)})$ and let $Z(\mathrm{GL}_\mathbf{r})$ be the center of the Lie algebra Let $\mathbb{Z}^\mathbf{r} = \mathrm{Hom}(\mathrm{GL}_\mathbf{r}, \mathbb{G})$ be the character lattice

let $G = G_\mathbf{r} = \prod_{(i,j)\in\Xi(\mathbf{m})} \mathrm{GL}_{r_i^{(j)}}$ and $\mathfrak{g} = \mathrm{Lie}\, G$. Let $\mathfrak{t} \cong (\mathbb{A}^r)^I/\mathbb{A}$ be the standard Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{z} = \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. Then, similarly to the above, one defines $\mathfrak{t}^*_\mathbb{Z}$, $\mathfrak{t}^+_\mathbb{Q}$, $\mathfrak{z}^*_\mathbb{Z}$, etc. Following Mehta and Seshadri [MeSe], associated with any element θ (in the fundamental alcove) of $\mathfrak{t}^+_\mathbb{Q}$, resp. $\mathfrak{z}^*_\mathbb{Q}$, there is a stability condition for parabolic Higgs bundles of type $\mathbf{r}$, Given an integer $d \in \mathbb{Z}$ write $\mathbf{v} = (d, \mathbf{r})$, let $\mathrm{T}^*\mathfrak{X}_\mathbf{v}(\mathscr{C}, F) = \mathrm{T}^*\mathfrak{X}_{\mathbf{r},d}(\mathcal{V}ect(C, D, \mathbf{m}), F)$ be the stack of framed parabolic Higgs bundles of type $\mathbf{r}$ and degree d. Let $\mathrm{T}^*\mathfrak{X}_\mathbf{v}(\mathscr{C}, F)^\theta \subset \mathrm{T}^*\mathfrak{X}_\mathbf{v}(\mathscr{C}, F)$, resp. be the substacks of θ-semistable objects, $\mathcal{Z}^\theta$ the corresponding coarse moduli space.

Below, we will consider the setting of quiver representations and of parabolic Higgs bundles at the same time. Let $\mathcal{Z}^{\theta\text{-stab}}$, resp. $\mathrm{T}^*\mathfrak{X}_\mathbf{v}(\mathscr{C}, F)^{\theta\text{-stab}}$, the θ-stable locus of $\mathcal{Z}^\theta$, resp. $\mathrm{T}^*\mathfrak{X}_\mathbf{v}(\mathscr{C}, F)^\theta$. The following result uses a special feature that there is a well behaved notion of a subobject of a given object of the category $\bar{Q}$-mod, resp. the category of parabolic Higgs bundles, in particular, the category in question is an exact category.

Lemma 10.2.1. *Let ξ be a closed point of $\mathrm{T}^*\mathfrak{X}_\mathbf{v}(\mathscr{C}, F)/G_\mathbf{v}$ such that $\underline{\mu}_\mathbf{v}(\xi) \in \mathfrak{g}^\diamond_\mathbf{v}/G_\mathbf{v}$. Then, ξ has no nonzero proper subobjects. In particular, ξ is θ-stable, for any stability condition θ.*

Proof. Let ξ' be a subobject of ξ and let $\xi'' = \xi/\xi'$. Let x, resp. x', x'', be the image of ξ, resp. ξ', ξ'', in $\mathfrak{X}(\mathscr{C})$ and let $\mathbf{v}' = \dim x'$, $\mathbf{v}'' = \dim x''$. Clearly, we have $\mathbf{v} = \mathbf{v}' + \mathbf{v}''$, moreover, one can choose compatible framings of x and x', x'', i.e. choose a basis of the vector space $F(x)$ such that a subset of that basis is a basis of the vector space $F(x') \subset F(x)$ and the images of the remaining base vectors form a basis of $F(x'') = F(x)/F(x')$. These choices provide a lift of ξ, resp. ξ', ξ'', to an object $\widetilde{\xi}$, resp. $\widetilde{\xi}', \widetilde{\xi}''$, of the stack $\mathrm{T}^*\mathfrak{X}_\mathbf{v}(\mathscr{C}, F)$. Let $g = \mu(\widetilde{\xi}) \in \mathfrak{g}^*_\mathbf{v} \cong \mathfrak{sl}_\mathbf{v}$, resp. $g' = \mu(\widetilde{\xi}') \in \mathfrak{g}^*_{\mathbf{v}'} \cong \mathfrak{sl}_{\mathbf{v}'}$, $g'' = \mu(\widetilde{\xi}'') \in \mathfrak{g}^*_{\mathbf{v}''} \cong \mathfrak{sl}_{\mathbf{v}''}$. By construction, the matrix g has a block form $g = \begin{pmatrix} g' & * \\ 0 & g'' \end{pmatrix}$. The assumption of the lemma that $g \in \mathfrak{g}^\diamond_\mathbf{v}$ forces one of the two diagonal blocks to have zero size. We conclude that either $\mathbf{v}' = \mathbf{v}$ and $\xi' = \xi$, or $\mathbf{v}' = 0$ and $\xi' = 0$. Finally, applying the Schur lemma, one deduces that $\mathrm{Aut}\,\xi = \mathbb{G}$. $\square$

The following result is well known.

Lemma 10.2.2. *The canonical morphism $\mathrm{T}^*\mathfrak{X}_\mathbf{v}(\mathscr{C}, F)^{\theta\text{-stab}} \to \mathcal{Z}^{\theta\text{-stab}}$ is an isomorphism of G-stacks. Furthermore, $\mathcal{Z}^{\theta\text{-stab}}$ is a smooth symplectic variety, the G-action on $\mathcal{Z}^{\theta\text{-stab}}$ is free, and the moment map $\mathcal{Z}^{\theta\text{-stab}} \to \mathfrak{g}^*$ is a smooth morphism.*

Sketch of Proof. The case of quiver representations is clear, so we only consider the case of parabolic Higgs bundles. Let ξ be a stable closed point $\mathrm{T}^*\mathfrak{X}_\mathbf{v}(\mathscr{C}, F)$. Then, we have $\mathrm{Hom}(\xi, \xi) = \mathbf{k}$. In particular, we have $\mathrm{Aut}(x) = \mathbb{G}$, hence $\mathrm{T}^*\mathfrak{X}_\mathbf{v}(\mathscr{C}, F)^{\theta\text{-stab}}$ is an algebraic space. An explicit construction of the coarse moduli space $\mathcal{Z}^\theta$ shows that the scheme $\mathcal{Z}^{\theta\text{-stab}}$ represents the functor associated with the stack $\mathrm{T}^*\mathfrak{X}_\mathbf{v}(\mathscr{C}, F)^{\theta\text{-stab}}$, so this algebraic space is in fact a scheme.

To prove that $\mathrm{T}^*\mathfrak{X}_\mathbf{v}(\mathscr{C}, F)^{\theta\text{-stab}}$ is smooth, it suffices to show that the function $\xi \mapsto \dim \mathrm{Ext}^1(\xi, \xi)$ is constant on $\mathrm{T}^*\mathfrak{X}_\mathbf{v}(\mathscr{C}, F)^{\theta\text{-stab}}$. To this end, one observes that the stack of parabolic bundles being smooth, the corresponding cotangent stack has cohomological dimension ≤ 2. It follows that

for all objects ξ of $\mathcal{Z}^\theta$ the group $\mathrm{Ext}^j(\xi,\xi) = \mathrm{R}^j\,\mathrm{Hom}(\xi,\xi)$ vanishes for all $j \neq 0,1,2$. Hence, the Euler characteristic $\dim\mathrm{Ext}^0(\xi,\xi) - \dim\mathrm{Ext}^1(\xi,\xi) + \dim\mathrm{Ext}^2(\xi,\xi)$ is independent of ξ. Furthermore, thanks to the (derived) symplectic structure on a cotangent stack, one has $\dim\mathrm{Ext}^2(\xi,\xi) = \dim\mathrm{Ext}^0(\xi,\xi) = 1$. We deduce that the function $\xi \mapsto \dim\mathrm{Ext}^1(\xi,\xi)$ is constant on the stable locus, as required. The remaining statements follow from standard results of symplectic geometry. $\qquad\square$

Let $\varpi : \mathcal{Z}^\theta \to \mathcal{M}^\theta = \mathcal{Z}^\theta/\!\!/G$, resp. $\widetilde{\varpi} : \widetilde{\mathcal{Z}}^\theta \to \widetilde{\mathcal{M}}^\theta = \widetilde{\mathcal{Z}}^\theta/\!\!/G$ be the corresponding categorical quotient. Further, let $\varpi : \mathcal{Z}^\theta \to \mathcal{M}^\theta = \mathcal{Z}^\theta/\!\!/G$, resp. $\widetilde{\varpi} : \widetilde{\mathcal{Z}}^\theta \to \widetilde{\mathcal{M}}^\theta = \widetilde{\mathcal{Z}}^\theta/\!\!/G$, be the corresponding categorical quotient. Thus, $\mathcal{M}^\theta = Higgs^\theta_{\mathbf{r},d}$ is the coarse moduli space of θ-semistable representations of $\bar{Q}$ of dimension $\mathbf{v}$, resp. parabolic Higgs bundles (without framing) of degree d and type $\mathbf{r}$. Let μ^θ be the restriction of μ to $\mathcal{Z}^\theta$ and $f : \mathcal{M}^\theta \to \mathfrak{g}^*/\!\!/G$ the map induced by μ^θ.

Corollary 10.2.3. *For any $\theta \in \mathfrak{z}^*_\mathbb{Q}$, resp. $\theta \in \mathfrak{t}^*_\mathbb{Q}$. one has an isomorphism $\mu^{-1}(\mathfrak{g}^\diamond)/G \cong f^{-1}(\mathfrak{g}^\diamond/\!\!/G)$, of stacks.* $\qquad\square$

Fix $d \in \mathbb{Z}$ and let $\mathfrak{Z}$, resp. $\widetilde{\mathfrak{Z}}$, be the stack of framed parabolic Higgs bundles of degree d and type $\mathbf{r}$, resp. $\widetilde{\mathbf{r}}$. There is a natural forgetful map $p : \widetilde{\mathfrak{Z}} \to \mathfrak{Z}$. We keep the same notation p for the corresponding map of unframed parabolic Higgs bundles.

An analogue of the map $\widetilde{\mathrm{Rep}_\mathbf{v}}\bar{Q} \to \mathrm{Rep}_\mathbf{v}\bar{Q}$ in the Higgs bundle setting is provided by the natural forgetful map $\mathrm{T}^*\mathfrak{X}_{\widetilde{\mathbf{r}}}(\mathcal{V}ect(C,D,\mathbf{m})) \to \mathrm{T}^*\mathfrak{X}_\mathbf{r}(\mathcal{V}ect(C,D,\mathbf{m}))$, where $\widetilde{\mathbf{r}}$ is the parabolic type that corresponds to taking complete flags at every marked points and the forgetful map sends a complete flag to a partial flag of type $\mathbf{r}$. The group $G_{\widetilde{\mathbf{r}}}$ is a maximal torus in $G_\mathbf{r}$. Given $\theta \in \mathfrak{t}^+_\mathbb{Q}$ let $\widetilde{\mathcal{Z}}^\theta$ one has the coarse moduli space of framed θ-semistable parabolic Higgs bundles of type $\widetilde{\mathbf{r}}$ and degree d and $\widetilde{\mathcal{M}}^\theta = \widetilde{\mathcal{Z}}^\theta/G_{\widetilde{\mathbf{r}}} = Higgs^\theta_{\widetilde{\mathbf{r}},d}$, the corresponding categorical quotient.

Given $\theta \in \mathfrak{t}^+_\mathbb{Q}$, resp. $\theta \in \mathfrak{t}^+_\mathbb{Q} \cap \mathfrak{z}^*_\mathbb{Q}$, let $\widetilde{\mathfrak{Z}}^\theta \subset \widetilde{\mathfrak{Z}}$, resp. $\mathfrak{Z}^\theta \subset \mathfrak{Z}$, be the θ-semistable locus. Let $\widetilde{\mathcal{Z}}^\theta$, resp. $\mathcal{Z}^\theta$, be the corrsponding coarse moduli space of *framed* parabolic Higgs bundles, and $\mathcal{M}^\theta := Higgs^\theta_{\mathbf{r},d}$, resp. $\widetilde{\mathcal{M}}^\theta := Higgs^\theta_{\widetilde{\mathbf{r}},d}$, the coarse moduli space of θ-semistable parabolic Higgs bundles (without framing), of degree d and type $\mathbf{r}$, resp. $\widetilde{\mathbf{r}}$. Thus, one has $\mathcal{M}^\theta = \mathcal{Z}^\theta/\!\!/G$, resp. $\widetilde{\mathcal{M}}^\theta = \widetilde{\mathcal{Z}}^\theta/\!\!/G$. Let $\varpi : \mathcal{Z}^\theta \to \mathcal{M}^\theta$, resp. $\widetilde{\varpi} : \widetilde{\mathcal{Z}}^\theta \to \widetilde{\mathcal{M}}^\theta$ be the categorical quotient map. We have a moment map $\mu^\theta : \mathcal{Z}^\theta \to \mathfrak{g}^*$, resp. $\widetilde{\mu}^\theta : \widetilde{\mathcal{Z}}^\theta \to \mathfrak{t}^*$ and also a map $f : \mathcal{M}^\theta \to \mathfrak{g}^*/\!\!/G$ induced by μ^θ.

Fix a sufficiently general element θ_{small} in the interior of $\mathfrak{t}^+_\mathbb{Q}$.

Proposition 10.2.4. *Let $\theta \in \mathfrak{z}^+_\mathbb{Q}$ and $\xi \in \widetilde{\mathcal{Z}} = \mathrm{Rep}_\mathbf{v}\bar{Q} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}}$, resp. $\xi \in \widetilde{\mathfrak{Z}}$. Then, we have*
(i) *ξ is θ-semistable iff $p(\xi)$ is θ-semistable.*
(ii) *For all $\varepsilon \in \mathbb{Q}$ such that $0 < \varepsilon \ll 0$ we have*

- *If ξ is $(\theta + \varepsilon\theta_{\mathrm{small}})$-semistable then $p(\xi)$ is θ-semistable;*
- *If $p(\xi)$ is θ-stable then ξ is $(\theta + \varepsilon\theta_{\mathrm{small}})$-stable;*
- *Any $(\theta + \varepsilon\theta_{\mathrm{small}})$-semistable element is $(\theta + \varepsilon\theta_{\mathrm{small}})$-stable.*

Sketch of Proof. In the quiver case, the required statement follows from results of Seshadri [Se2], Propositions 5.1 and 5.2. Seshadri only considers the case where all schemes are projective. However, one checks that for his arguments to go through in our setting, it suffices that the map $(\xi, F) \to \xi$ gives a projective morphism $\widetilde{\mathcal{Z}} \to \mathcal{Z}$. In the case of Higgs bundles, the statement follows from a result of Reineke [Rei, Theorem 5.1] which is general enough to be applicable in the setting of Higgs bundle. This res $\qquad\square$

We choose and fix ε such that part (ii) of the proposition holds and put $\vartheta = \theta + \varepsilon\theta_{\mathrm{small}}$. Thanks to the proposition, there is a well defined map $\widetilde{\mathcal{Z}}^\vartheta \to \widetilde{\mathcal{Z}}^\theta$. We obtain a commutative diagram

$$
\begin{array}{ccccc}
\widetilde{\mathcal{Z}}^\vartheta & \longrightarrow & \mathcal{Z}^\theta \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}} & \longrightarrow & \widetilde{\mathfrak{g}} \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{\mathcal{Z}}^\theta & \longrightarrow & \mathcal{Z}^\theta \times_{\mathfrak{g}^* /\!\!/ G} \mathfrak{t}^* & \longrightarrow & \mathfrak{t}^*
\end{array}
\qquad (10.2.5) \quad \boxed{\texttt{vth-diag}}
$$

$\boxed{\texttt{GIT-quot}}$ **Lemma 10.2.6.** (i) *The first map in the bottom row of diagram (10.2.5)* $\overset{\texttt{vth-diag}}{}$ *is an isomorphism for any* $\theta \in \mathfrak{z}^+_{\mathbb{Q}}$.

(ii) *If the vector* $\mathbf{v}$, *resp.* $(\mathbf{r}, d)$, *is indivisible and* $\theta \in \mathfrak{z}^+_{\mathbb{Q}}$ *is sufficiently general, then the first map in the top row of diagram (10.2.5)* $\overset{\texttt{vth-diag}}{}$ *is an isomorphism.*

(iii) *The composite map in the top row is a smooth morphism.*

Taking categorical quotients by G, from the above proposition we deduce

$\boxed{\texttt{indiv-cor}}$ **Corollary 10.2.7.** (i) *One has an isomorphism* $\widetilde{\mathcal{M}}^\theta \overset{\sim}{\to} [\mathcal{M}^\theta \times_{\mathfrak{g}^* /\!\!/ G} \mathfrak{t}^*]_{\mathrm{red}}$.

(ii) *Assume that the vector* $\mathbf{v}$, *resp.* $(\mathbf{r}, d)$, *is indivisible and* $\theta \in \mathfrak{z}^+_{\mathbb{Q}}$ *is sufficiently general. Then, the composite map*

$$
\mathcal{Z}^\vartheta \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}} \overset{\sim}{\to} \widetilde{\mathcal{Z}}^\vartheta \to \widetilde{\mathcal{Z}}^{\widetilde{\theta}} /\!\!/ G = \widetilde{\mathcal{M}}^{\widetilde{\theta}}
$$

is a geomtric quotient of $\mathcal{Z}^\theta_{\mathbf{v}} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}}$ *by* G.

(iii) *The map* $\widetilde{f} : \widetilde{\mathcal{M}}^\vartheta \to \widetilde{\mathfrak{g}} /\!\!/ G \cong \mathfrak{t}^*$, *induced by the composite map in the top row of diagram (10.2.5)* $\overset{\texttt{vth-diag}}{}$, *is a smooth morphism. In particular,* $\widetilde{\mathcal{M}}^\vartheta$ *is a smooth algebraic variety.*

Sketch of Proof. The proof of (i) is based on the isomorphism $\mathsf{k}[\widetilde{\mathfrak{g}}]^G = \mathsf{k}[\mathfrak{t}^*]$. Using Lemma $\overset{\texttt{GIT-quot}}{10.2.6(\mathrm{i})}$, we compute

$$
\widetilde{\mathcal{M}}^\theta = \widetilde{\mathcal{Z}}^\theta /\!\!/ G \cong (\mathcal{Z}^\theta \times_{\mathfrak{g}^* /\!\!/ G} \mathfrak{t}^*) /\!\!/ G \cong [(\mathcal{Z}^\theta /\!\!/ G) \times_{\mathfrak{g}^* /\!\!/ G} \mathfrak{t}^*]_{\mathrm{red}} = [\mathcal{M}^\theta \times_{\mathfrak{g}^* /\!\!/ \mathrm{GL}_{\mathbf{v}}} \mathfrak{t}^*]_{\mathrm{red}}.
$$

Here, the operation $[\ldots]_{\mathrm{red}}$, of taking the reduced scheme structure, is only necessary in the case where the ground field has positive characteristic, so the functor of taking G-invariants is not necessarily exact.

Part (ii) is immediate from Lemma $\overset{\texttt{GIT-quot}}{10.2.6(\mathrm{ii})}$. To prove part (iii), we observe that the vector $(\widetilde{\mathbf{r}}, d)$ is indivisible. It follows that all points in $\widetilde{\mathcal{Z}}^\vartheta$ are stable. Therefore, the categorical quotient $\widetilde{\mathcal{Z}}^\vartheta \to \widetilde{\mathcal{M}}^\vartheta$ is a *geometric* quotient. $\qquad\square$

Let $\widetilde{\mathcal{M}}^{\vartheta,\diamond} := \widetilde{f}^{-1}(\mathfrak{t}^\diamond)$, resp. $\widetilde{\mathcal{M}}^\vartheta_0 = (\widetilde{\mu}^\vartheta)^{-1}(0) /\!\!/ G$. The map $\widetilde{\mathcal{Z}}^\vartheta \to \widetilde{\mathcal{Z}}^\theta$ induces a a proper morphism $p^\vartheta : \widetilde{\mathcal{M}}^\vartheta \to \widetilde{\mathcal{M}}^\theta$, of GIT quotients. Thus, from Lemma $\overset{\texttt{GIT-quot}}{10.2.6}$ and diagram (10.2.5) $\overset{\texttt{vth-diag}}{}$ we obtain the following analogue of diagram (10.1.6) $\overset{\texttt{grsp}}{}$

$$
\begin{array}{ccccc}
\widetilde{\mathcal{M}}_{\mathrm{nil}} \lhook\joinrel\longrightarrow & \widetilde{\mathcal{M}}^\vartheta & \longleftarrow\joinrel\rhook & \widetilde{\mathcal{M}}^{\vartheta,\diamond} & (10.2.8) \quad \boxed{\texttt{grsp2}} \\
\downarrow \qquad\qquad & \downarrow{\scriptstyle p^\vartheta} & & \downarrow{\scriptstyle \cong} & \\
\mathcal{M}^\theta_{\mathrm{nil}} \overset{\imath}{\lhook\joinrel\longrightarrow} & \widetilde{\mathcal{M}}^\theta = [\mathcal{M}^\theta \times_{\mathfrak{g}^* /\!\!/ G} \mathfrak{t}^*]_{\mathrm{red}} & \overset{\jmath}{\longleftarrow\joinrel\rhook} & \mathcal{M}^\theta \times_{\mathfrak{g}^* /\!\!/ G} \mathfrak{t}^\diamond & \\
\downarrow \qquad\qquad & \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle pr_2} & \\
\{0\} \lhook\joinrel\longrightarrow & \mathfrak{t}^* & \longleftarrow\joinrel\rhook & \mathfrak{t}^\diamond &
\end{array}
$$

By Lemma $\overset{\texttt{GIT-quot}}{10.2.6((\mathrm{iii})}$ and Corollary $\overset{\texttt{indiv-cor}}{10.2.7(\mathrm{iii})}$, the results of §$\overset{\texttt{indivisible-sec}}{10.1}$ apply to the variety $\widetilde{\mathcal{M}}^\vartheta$ and the map $\widetilde{f}$. Let $p^\vartheta_{\mathrm{nil}} := p^\vartheta|_{(\widetilde{f})^{-1}(0)}$. Aplying the proper base change theorem for nearby cycles to

78

diagram (10.2.8) one deduces the following analogue of (10.1.8):

$$\Psi_f\mathbb{C}_{\mathcal{M}^\theta} \cong (p^\vartheta_{\mathrm{nil}})_!\Psi_{\widetilde{f}}\underline{\mathbf{C}}_{\widetilde{\mathcal{M}}^\vartheta}, \quad \text{and} \quad H^\bullet_c(\mathcal{M}^\theta_{\mathrm{nil}},\, \Psi_f\mathbb{C}_{\mathcal{M}^\theta}) \cong H^\bullet_c((\widetilde{f})^{-1}(0)) = H^\bullet_c(\widetilde{\mathcal{M}}_{\mathrm{nil}}). \quad (10.2.9) \quad \boxed{\texttt{ind2}}$$

Further, using diagram (10.2.8), we repeat the arguments of §10.1 and obtain the following result

Proposition 10.2.10. *We have canonical isomorphisms*

$$p^\vartheta_!\underline{\mathbf{C}}_{\widetilde{\mathcal{M}}^\vartheta} = \jmath_{!*}\underline{\mathbf{C}}_{\mathcal{M}^\theta\times_{\mathfrak{g}^*/\!/G}\mathfrak{t}^\diamond}, \quad (p^\vartheta_{\mathrm{nil}})_!\underline{\mathbf{C}}_{\widetilde{\mathcal{M}}^\vartheta_{\mathrm{nil}}} = i^*\jmath_{!*}\underline{\mathbf{C}}_{\mathcal{M}^\theta\times_{\mathfrak{g}^*/\!/G}\mathfrak{t}^\diamond}.$$

Using the definition of an IC-sheaf, from the first isomorphism of the proposition we also deduce the following corollary that resembles a result of Reineke [Rei]:

Corollary 10.2.11. *The map* $p^\vartheta : \mathcal{M}^\vartheta_{\widetilde{\mathbf{v}}} \to \widetilde{\mathcal{M}}^\theta$ *is a small resolution.*

10.3. Completing the proof of Theorem ??.
In the special case where $\mathscr{C}$ is the category of representations of a finite quiver Q with vertex set I, our construction reduces to the original construction of [HLV1]. Specifically, given a dimension vector $\mathbf{v}$, in *loc cit* the authors introduce a new quiver $\widetilde{Q}$ obtained from Q by attaching legs. The moduli stack $X_{\widetilde{\mathbf{v}}}(\widetilde{\mathscr{C}}, \widetilde{F})$ is then easily identified with $\mathrm{Rep}_{\widetilde{\mathbf{v}}}\widetilde{Q}$, the representation scheme of $\widetilde{\mathbf{v}}$-dimensional representations of $\widetilde{Q}$.

Next, let $\mathscr{C}$ be the category of parabolic bundles on a curve C with parabolic structure at a collection of marked points c_i, $i \in I$. In that case, an object of $X_{\widetilde{\mathbf{v}}}(\mathscr{C}, \widetilde{F})$ is a vector bundle equipped with a *complete flag* and a basis in the fiber over each of the marked points. The projection $X_{\widetilde{\mathbf{v}}}(\mathscr{C}) \to X_{\mathbf{v}}(\mathscr{C})$ is a map that forgets some of the components of the complete flags. Thus, this map sends a vector bundle with complete flags in the fibers to the same vector bundle with partial flags of the type determined by the type of the parabolic structure of $\mathscr{C}$.

We conclude that in the cases where $\mathscr{C}$ is either the category of quiver representations or the category of parabolic bundles, the variety $\mathcal{M}^{\widetilde{\theta}}_{\widetilde{\mathbf{v}}}$ is semi-projective. Thus, one can mimic the argument used in §10.1. Specifically, we get an isomorphism $H^\bullet_c((f^{\widetilde{\theta}}_{\widetilde{\mathbf{v}}})^{-1}(0), \mathbb{C}) \cong H^\bullet_c((f^{\widetilde{\theta}}_{\widetilde{\mathbf{v}}})^{-1}(z), \mathbb{C})$ for any $z \in \mathfrak{t}^\diamond_{\mathbf{v}}$. Hence, similarly to (10.1.9), we deduce

$$H^\bullet_c(\mathcal{M}^\theta_{\mathrm{nil}},\, \Psi_f\underline{\mathbf{C}}) \cong H^\bullet_c((f^{\widetilde{\theta}}_{\widetilde{\mathbf{v}}})^{-1}(0),\, \underline{\mathbf{C}}) \cong H^\bullet_c((f^{\widetilde{\theta}}_{\widetilde{\mathbf{v}}})^{-1}(z),\, \underline{\mathbf{C}}) \cong H^\bullet_c((\mu^\theta_{\mathbf{v}})^{-1}(O)/\!/G_{\mathbf{v}},\, \underline{\mathbf{C}}). \quad (10.3.1) \quad \boxed{\texttt{tf-iso}}$$

Parts (i) and (iii) of Theorem ?? follow from the above. It remains to prove that the support of the sheaf $\Psi(\underline{\mathbf{C}})^{\mathrm{sign}}$ is contained in $(\mu^\theta_{\mathbf{v}})^{-1}(0)/\!/G_{\mathbf{v}}$ in the case where the vector $\mathbf{v}$ is divisible. We reduce this to the case indivisible dimension vectors that has been settled in §10.1. To this end, we perform constructions of previous sections in a slightly different setting where we 'frame' by legs all elements of I except one.

Specifically, assume that $\#I > 1$, fix $i \in I$ and let $\mathscr{C}_\flat$ be a category whose objects are pairs (x, ρ), where $x \in \mathscr{C}$ and ρ is a diagram like (??) where $F(x) = \oplus_{i\in I} F_i(x)$, an I-graded vector space, is replaced by $\oplus_{i'\in I\smallsetminus\{i\}} F_{i'}(x)$, $I\smallsetminus\{i\}$-graded vector space, and where each F^j is an $I\smallsetminus\{i\}$-graded vector space as well. The assignment $F_\flat : (x, \rho) \mapsto F(x) \oplus (\otimes_j F^j)$ gives a functor from $\mathscr{C}_\flat$ to the category of finite dimensional vector spaces graded by the set $I \sqcup (I\smallsetminus\{i\}) \times \mathbb{Z}$. Thus, we have moduli stacks $X_{\mathbf{v},\mathbf{v}_{\mathrm{small}}}(\mathscr{C}_\flat, F_\flat)$, of framed objects.

Fix a dimension vector $\mathbf{v} \in \mathbb{Z}^I$. We mimic §?? in the setting where J_{small} is replaced by the set $J_\flat = \sqcup_{i'\in I\smallsetminus\{i\}} [1, v_{i'}-1]$. Thus, we let $\mathbf{v}_\flat = (v^j_{i'}) \in \mathbb{Z}^{J_\flat}$ be defined, for each $i' \in I \smallsetminus \{i\}$, by the formula $v^j_{i'} = v_{i'} - j$, $j = 0, 1, \ldots, v_{i'}-1$. Let $\mathbf{v}' := (\mathbf{v}, \mathbf{v}_\flat)$. We put $G_\flat = \prod_{i'\in I\smallsetminus\{i\}} GL_{v_{i'}}$, resp. $W_\flat := \prod_{i'\in I\smallsetminus\{i\}} \mathfrak{S}_{v_{i'}}$, and $G_i = \mathrm{PGL}_{v_i}$, resp. $W_i = \mathfrak{S}_{v_i}$. Let $\mathfrak{g}_\flat$ and $\mathfrak{t}_\flat$, resp. $\mathfrak{g}_i$ and $\mathfrak{t}_i$, be the Lie algebra and the Cartan subalgebra of $G_\flat$, resp. G_i. The natural imbedding $G_\flat \hookrightarrow \prod_{i'\in i\in I} GL_{v_{i'}}$ induces a short exact sequence $1 \to G_\flat \to G_{\mathbf{v}} \to G_{v_i} \to 1$. The identifications $\mathfrak{g} = \mathfrak{sl}_{\mathbf{v}}$, resp. $\mathfrak{g}_i = \mathfrak{sl}_{v_i}$, provide

79

a natural splitting $\mathfrak{g} = \mathfrak{g}_\flat \oplus \mathfrak{g}_i$ of the corresponding Lie algebra extension $\mathfrak{g}_\flat \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_i$. To simplify the exposition, we will use this splitting to identify $\mathfrak{t}_{\mathbf{v}} = \mathfrak{t}_\flat \oplus \mathfrak{t}_i$, $\mathfrak{g}^* /\!/ G_{\mathbf{v}} = \mathfrak{g}_\flat^* /\!/ G_\flat \times \mathfrak{g}_i^* /\!/ G_i$, etc.

Further, let $\widetilde{\mathbb{K}} = \mathbb{K} \oplus \mathbb{Z}^{J_{\mathrm{small}}}$. We have an obvious decomposition $J_{\mathrm{small}} = \sqcup_{i' \in I} [1, \ldots, v_{i'} - 1] = J_\flat \sqcup [1, \ldots, v_i - 1]$ and an induced direct sum decomposition $\widetilde{\mathbb{K}} = \mathbb{K}_\flat \oplus \mathbb{Z}^{v_i - 1}$, resp. $\widetilde{\mathbb{K}}^* = \mathbb{K}_\flat^* \oplus \mathbb{Q}^{v_i - 1}$. Let $0_\flat \in \oplus_{i' \in I \smallsetminus \{i\}} \mathbb{Q}^{v_{i'} - 1}$, resp. $0_i \in \mathbb{Q}^{v_i - 1}$, denote the origin.

The vector $\mathbf{v}' := (\mathbf{v}, \mathbf{v}_\flat)$ is indivisible and, moreover , $v_{i'} \geq 2$, $\forall i' \in I \smallsetminus \{i\}$, since the vector $\mathbf{v}$ is assumed to be divisible. Hence, given $\theta \in \mathbb{K}^*$, we may apply Lemma **??** in the case of the lattice $\mathbb{K}_\flat$ and the stability condition $(\theta, 0_\flat) \in \mathbb{K}_\flat^* = \mathbb{K}^* \times (\oplus_{i' \in I \smallsetminus \{i\}} \mathbb{Q}^{v_{i'} - 1})$. We deduce that there exists a generic deformation, $\theta_i' \in \mathbb{K}_\flat^*$, of $(\theta, 0_\flat)$. Applying Lemma **??** one more time, we deduce further that there exists a generic deformation, $\widetilde{\theta} \in \widetilde{\mathbb{K}}^*$, of the stability condition $(\theta_i', 0_i) \in \widetilde{\mathbb{K}}^* = \mathbb{K}_\flat^* \oplus \mathbb{Q}^{v_i - 1}$. Clearly, $\widetilde{\theta}$ is also a generic deformation of $(\theta, 0)$ where $0 = (0_\flat, 0_i)$, so the notation $\widetilde{\theta}$ agrees with the one used earlier.

The construction of section **??** produces a $G_{\mathbf{v}}$-scheme $\mathcal{Z}_{\mathbf{v}'}^{\theta_i'}$, resp. $\mathcal{Z}_{\widetilde{\mathbf{v}}}^{\theta, 0_\flat}$ and $\mathcal{Z}_{\widetilde{\mathbf{v}}}^{\theta_i', 0_i}$, equipped with a map to $\mathfrak{g}^*$ induced by a moment map. There are canonical $G_{\mathbf{v}}$-equivariant proper morphisms $\mathcal{Z}_{\mathbf{v}'}^{\theta_i'} \to \mathcal{Z}_{\widetilde{\mathbf{v}}}^{\theta, 0_\flat}$ and $\mathcal{Z}_{\widetilde{\mathbf{v}}}^{\widetilde{\theta}} \to \mathcal{Z}_{\widetilde{\mathbf{v}}}^{\theta_i', 0_i} \to \mathcal{Z}^{(\theta, 0_\flat), 0_i} = \mathcal{Z}^{\theta, 0}$. Each of these morphisms restricts to an isomorphism over $\mathfrak{g}^\diamond$, in particular, it is a birational isomorphism. A slight modification of Lemma 10.2.6(i) yields natural isomorphisms

$$\mathcal{Z}_{\mathbf{v}'}^{\theta, 0_\flat} \cong [\mathcal{Z}_{\mathbf{v}}^{\theta} \times_{\mathfrak{g}_{\mathbf{v}}^*} ((\mathfrak{g}_\flat^* \times_{\mathfrak{g}_\flat^* /\!/ G_\flat} \mathfrak{t}_\flat^*) \times \mathfrak{g}_i^*)]_{\mathrm{red}} \cong [\mathcal{Z}_{\mathbf{v}}^{\theta} \times_{\mathfrak{g}_\flat^* /\!/ G_\flat} \mathfrak{t}_\flat^*]_{\mathrm{red}},$$

$$\mathcal{Z}_{\widetilde{\mathbf{v}}}^{\theta_i', 0_i} \cong [\mathcal{Z}_{\mathbf{v}'}^{\theta_i'} \times_{\mathfrak{t}_\flat^* \times \mathfrak{g}_i^* /\!/ G_i} \mathfrak{t}_i^*]_{\mathrm{red}} = [\mathcal{Z}_{\mathbf{v}'}^{\theta_i'} \times_{\mathfrak{t}_{\mathbf{v}}^* /\!/ W_i} \mathfrak{t}_{\mathbf{v}}^*]_{\mathrm{red}},$$

where $(-) \times_{\mathfrak{g}_\flat^* /\!/ G_\flat} \mathfrak{t}_\flat^*$ stands for $(-) \times_{\mathfrak{g}_\flat^* /\!/ G_\flat \times \mathfrak{g}_i^* /\!/ G_i} (\mathfrak{t}_\flat^* \times \mathfrak{g}_i^* /\!/ G_i)$, etc. One also has a GIT quotient $\mathcal{M}_{\mathbf{v}'}^{\theta_i'} = \mathcal{Z}_{\mathbf{v}'}^{\theta_i'} /\!/ G_{\mathbf{v}}$, resp. $\mathcal{M}_{\widetilde{\mathbf{v}}}^{\theta, 0_\flat}$ and $\mathcal{M}_{\widetilde{\mathbf{v}}}^{\theta_i', 0_i} = \mathcal{Z}_{\widetilde{\mathbf{v}}}^{\theta_i', 0_i} /\!/ G_{\mathbf{v}}$. Similarly to the above, a variant of Corollary 10.2.7(i) yields isomorphisms $\mathcal{M}_{\mathbf{v}'}^{\theta, 0_\flat} \cong [\mathcal{M}_{\mathbf{v}}^{\theta} \times_{\mathfrak{g}_\flat^* /\!/ G_\flat} \mathfrak{t}_\flat^*]_{\mathrm{red}}$, resp. $\mathcal{M}_{\widetilde{\mathbf{v}}}^{\theta_i', 0_i} \cong [\mathcal{M}_{\mathbf{v}'}^{\theta_i'} \times_{\mathfrak{t}^* /\!/ W_i} \mathfrak{t}_{\mathbf{v}}^*]_{\mathrm{red}}$.

The dimension vector $\mathbf{v}'$ being indivisible, the moment map $\mathcal{Z}_{\mathbf{v}'}^{\theta_i'} \to \mathfrak{g}^* = \mathfrak{g}_\flat^* \oplus \mathfrak{g}_i^*$ is smooth. Furthermore, an analogue of Lemma 10.2.6(ii) yields an isomorphism $\mathcal{Z}_{\widetilde{\mathbf{v}}}^{\widetilde{\theta}} = \mathcal{Z}_{\mathbf{v}'}^{\theta_i'} \times_{\mathfrak{g}_i^*} \widetilde{\mathfrak{g}}_i$ where $\pi_i : \widetilde{\mathfrak{g}}_i \to \mathfrak{g}_i^*$ is the Grothendieck -Springer resolution for the Lie algebra $\mathfrak{g}_i$ and $(-) \times_{\mathfrak{g}_i^*} \widetilde{\mathfrak{g}}_i$ stands for $(-) \times_{\mathfrak{g}_\flat^* \times \mathfrak{g}_i^*} (\mathfrak{g}_\flat^* \times \widetilde{\mathfrak{g}}_i)$. Thus, all squares in the following diagram, which is an analogue of diagram (10.1.10), are cartesian

$$
\begin{array}{ccccccc}
\mathfrak{g}_\flat^* \times \widetilde{\mathfrak{g}}_i & \xleftarrow{\ \mu_1\ } & \mathcal{Z}_{\widetilde{\mathbf{v}}}^{\widetilde{\theta}} & \longrightarrow & \mathcal{M}_{\widetilde{\mathbf{v}}}^{\widetilde{\theta}} & \longrightarrow & \mathfrak{t}_{\mathbf{v}}^* \\
\downarrow{\scriptstyle \mathrm{Id} \times \pi_i} & & \downarrow & & \downarrow & & \| \\
\mathfrak{g}_\flat^* \times (\mathfrak{g}_i^* \times_{\mathfrak{g}_i^* /\!/ G_i} \mathfrak{t}_i^*) & \xleftarrow{\ \mu_i\ } & X_{\widetilde{\mathbf{v}}}^{\theta_i', 0_i} & \xrightarrow{\ p_i\ } & \mathcal{M}_{\widetilde{\mathbf{v}}}^{\theta_i', 0_i} & \xrightarrow{\ f_i\ } & \mathfrak{t}_{\mathbf{v}}^*
\end{array}
\tag{10.3.2}
$$

Write $\pi_{\mathrm{Nil}_i} : \widetilde{\mathrm{Nil}}_i \to \mathrm{Nil}_i$ for the Springer resolution of the nilpotent cone of $\mathfrak{g}_i^*$ and pr_i for the second projection $\mathcal{Z}_{\mathrm{nil}}^{\theta_i'} \times_{\mathrm{nil}} \mathrm{Nil}_i \to \mathrm{Nil}_i$. Further, let $\mathcal{Z}_{\mathrm{nil}}^{\theta_i', 0_i} := \mathcal{Z}_{\widetilde{\mathbf{v}}}^{\theta_i', 0_i} \times_{\mathfrak{g}_{\mathbf{v}}^*} \mathrm{Nil}$, resp. $\mathcal{Z}_{\mathrm{nil}}^{\theta_i'} := \mathcal{Z}_{\mathbf{v}'}^{\theta_i'} \times_{\mathfrak{g}_{\mathbf{v}}^*} \mathrm{Nil}$. From earlier isomorphisms we deduce $\mathcal{Z}_{\mathrm{nil}}^{\theta_i', 0_i} = \mathcal{Z}_{\mathrm{nil}}^{\theta_i'}$, resp. $\mathcal{Z}_{\mathrm{nil}}^{\widetilde{\theta}} \cong \mathcal{Z}_{\mathrm{nil}}^{\theta_i'} \times_{\mathrm{nil}} (\mathrm{Nil}_\flat \times \widetilde{\mathrm{Nil}}_i)$.

All the spaces fit into the following commutative diagram

$$
\begin{array}{ccccccccc}
X^{\widetilde{\theta}}_{\mathrm{nil}} & \xrightarrow{\ a\ } & X^{\theta'_i,0_i}_{\mathrm{nil}} \\[2pt]
\downarrow{\scriptstyle G_{\mathbf{v}}} & & \downarrow{\scriptstyle G_{\mathbf{v}}}\ {\scriptstyle p_{\mathrm{Nil},i}} \\[2pt]
\mathcal{M}^{\widetilde{\theta}}_{\mathrm{nil}} & \xrightarrow{b^1_i} & [\mathcal{M}^{\theta'_i,0_i}_{\mathrm{nil}}]=\mathcal{M}^{\theta'_i}_{\mathrm{nil}} & \xrightarrow{b^2_i} & \mathcal{M}^{(\theta,0_\flat),0_i}_{\mathrm{nil}} & \xrightarrow{b^3_i} & [\mathcal{M}^{\theta,0_\flat}_{\mathrm{nil}}] & \xrightarrow{b^4_i} & \mathcal{M}^{\theta}_{\mathrm{nil}} \\[2pt]
\downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle q_3} & & \downarrow{\scriptstyle q_4} & & \downarrow{\scriptstyle q_5} \\[2pt]
\mathcal{M}^{\widetilde{\theta}} & \xrightarrow{c^1_i} & \mathcal{M}^{\theta'_i,0_i}=[\mathcal{M}^{\theta'_i}\times_{\mathfrak{t}^*/W_i}\mathfrak{t}^*] & \xrightarrow{c^2_i} & [\mathcal{M}^{\theta,0_\flat}\times_{\mathfrak{t}^*/W_i}\mathfrak{t}^*] & = & \mathcal{M}^{\theta,0}=[\mathcal{M}^{\theta}\times_{\mathfrak{t}^*/W}\mathfrak{t}^*] & \xrightarrow{c^5_i} & \mathcal{M}^{\theta} \\[2pt]
\downarrow & & \downarrow{\scriptstyle f_i} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f^{\theta}_{\mathbf{v}}} \\[2pt]
\mathfrak{t}^* & = & \mathfrak{t}^* & = & \mathfrak{t}^* & = & \mathfrak{t}^* & \longrightarrow & \mathfrak{t}^*/W
\end{array}
$$

Then, similarly to (10.1.11), from diagram (10.3.2) we obtain W_i-equivariant isomorphisms of sheaves on $\mathcal{Z}^{\theta'_i,0_i}_{\mathrm{nil}}$

$$
p^*_{\mathrm{Nil},i}\Psi_{f_i}\underline{\mathbf{C}} = \Psi_{f_i\circ p_i}\underline{\mathbf{C}} = pr^*_i(\pi_{\mathrm{Nil}_i})_!\underline{\mathbf{C}}.
$$

Now let sign_i be the sign representation of the group W_i. According to the theory of Springer representations, the sign_i-isotypic component of the sheaf $(\pi_{\mathrm{Nil}_i})_!\underline{\mathbf{C}}$ is supported at $0\in\mathrm{Nil}_i$. Hence, from the above isomorphisms we deduce $\mathrm{supp}(p^*_{\mathrm{Nil},i}(\Psi_{f_2}\underline{\mathbf{C}})^{\mathrm{sign}_i}) = \mathrm{supp}(pr^*_2((\pi_{\mathrm{Nil}_i})_!\underline{\mathbf{C}}))^{\mathrm{sign}_i}) = pr^{-1}_i(0)$. Hence, we get $\mathrm{supp}((\Psi_{f_i}\underline{\mathbf{C}})^{\mathrm{sign}_i}) \subset p_{\mathrm{Nil},i}(pr^{-1}_i(0))$. Further, using the notation of the last diagram, proper base change yields $(b^3_i)^*(b^4_i)^*\Psi_{f^{\theta}_{\mathbf{v}}}\underline{\mathbf{C}} = (b^2_i)_!\Psi_{f_i}\underline{\mathbf{C}}$. Therefore, we deduce $\mathrm{supp}((\Psi_{f^{\theta}_{\mathbf{v}}}\underline{\mathbf{C}})^{\mathrm{sign}_i}) \subset (b^4_i\circ b^3_i\circ b^2_i\circ p_{\mathrm{Nil},i})(pr^{-1}_i(0))$.

We can use the above argument for all elements $i\in I$ at the same time provided we choose (as we may) $\widetilde{\theta}$ in such a way that it is a generic deformation of θ'_i for every $i\in I$. This way, we obtain that $\mathrm{supp}((\Psi_{f^{\theta}_{\mathbf{v}}}\underline{\mathbf{C}})^{\mathrm{sign}_i}) \subset \cap_{i\in I}\ (b^4_i\circ b^3_i\circ b^2_i\circ p_{\mathrm{Nil},i})(pr^{-1}_i(0))$. The intersection on the right is clearly equal to $(\mu^{\theta}_{\mathbf{v}})^{-1}(0)/\!\!/G_{\mathbf{v}}$. This completes the proof of Theorem ??(ii), as well as the corresponding statement of Theorem 1.3.2, provided the set I contains at least 2 elements.

It remains to consider the case where $\#I = 1$. In the setting of parabolic bundles, this is the case where there is only one marked point and the type at that point is trivial. Then, one can add a second marked point with trivial parabolic type. This procedure clearly doesn't change the category $\mathscr{C}$ in question and creates a new functor F' on $\mathscr{C}$ with values in vector spaces graded by a new set I' that consists of *two* elements. The required statement is then deduced from the one for the pair $(\mathscr{C}, F')$, that has been already established above.

There is a similar trick in the general case of an arbitrary pair $(\mathscr{C}, F)$. The idea is to consider the functor $F^2 : x \mapsto F(x)\oplus F(x)$ where we view $F(x)\oplus F(x)$ as a vector space graded by the set $\{1,2\}$ with dimension vector $(v,v)\in\mathbb{Z}^2$ where $v := \dim F(x)\in\mathbb{Z}$. One has the corresponding $G_v\times G_v$-stack $\mathfrak{X}_{v,v}(\mathscr{C}, F^2)$ and a moment map $\mathrm{T}^*\mathfrak{X}_{v,v}(\mathscr{C}, F^2)\to\mathfrak{g}^*_v\times\mathfrak{g}^*_v$. The required statement for the pair $(\mathscr{C}, F)$ can then be deduced from the one for the pair $(\mathscr{C}, F^2)$.

Namely, for the pair $(\mathscr{C}, F^2)$ consider stability conditions $\theta,\theta'_i,\widetilde{\theta}$ as before for $i=1$. Then the second-to-last row of the last diagram above gives maps $\mathcal{M}^{\widetilde{\theta}}_{F^2} \xrightarrow{c'} \mathcal{M}^{\theta'_1,0_1}_{F^2} \xrightarrow{c} \mathcal{M}^{\theta,0}_{F^2} \to \mathfrak{t}_{F^2}$. The family $\mathcal{M}^{\theta,0}_{F^2} \to \mathfrak{t}_{F^2}$ descends to $\mathfrak{t}_{F^2,(v,v)}/\!\!/W_{F^2,(v,v)} = (\mathfrak{t}_{F,v}/\!\!/W_{F,v})\times(\mathfrak{t}_{F,v}/\!\!/W_{F,v})$ and the family $\mathcal{M}^{\theta'_1,0_1}_{F^2} \to \mathfrak{t}_{F^2}$ descends to $\mathfrak{t}_{F^2,(v,v)}/\!\!/(W_{F,v}\times\{e\}) = (\mathfrak{t}_{F,v}/\!\!/W_{F,v})\times\mathfrak{t}_{F,v}$. Then, if $\mathrm{IC},\mathrm{IC}',\mathrm{IC}''$ denote generically constant IC sheaves on $\mathcal{M}^{\theta,0}_{F^2}, \mathcal{M}^{\widetilde{\theta}}_{F^2}, \mathcal{M}^{\theta'_1,0_1}_{F^2}$, we have $\mathrm{IC} = c_*\,\mathrm{IC}'$, $\mathrm{IC}' = c'_*\,\mathrm{IC}''$ as explained before. Now, applying the above argument to $(\mathscr{C}, F^2)$, we get that $\mathrm{IC}^{W_{F^2,(v,v)},\mathrm{sign}}$ (anti-invariants) is supported on [preimage of zero under the moment map]. On the other hand,

$\mathcal{M}_{F^2}^{\theta_1',0_1}$ can be identified with $\mathcal{M}_{\mathscr{C}',F'}^{\theta_1''}$, where $\mathscr{C}'$ is the full subcategory of $\mathscr{C}_1$ consisting of objects for which the maps ρ_i are all surjective, F' is the composition of the forgetful functor $\mathscr{C}' \to \mathscr{C}$ with F, and θ_1'' corresponds to θ_1'. We see from this that the anti-invariants to do is a constant sheaf on
    Details will be explained in an expanded version of the paper.

11. APPENDIX A: PURITY

11.1. **Local acyclicity.** Throughout this section we fix a smooth geometrically irreducible scheme S and a morphism $p : X \to S$.
    We will use the following version of the standard definition of local acyclicity.

Definition 11.1.1. A sheaf $\mathcal{F}$ on X is called *locally acyclic* with respect to p if for any cartesian diagram

$$\begin{array}{ccc} C \times_S X & \xrightarrow{\;\widetilde{g}\;} & X \\ \downarrow{\scriptstyle\widetilde{p}} & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{\;g\;} & S \end{array} \qquad\qquad (11.1.2) \quad \boxed{\texttt{lac}}$$

where C is a smooth curve, the sheaf $\widetilde{g}^*\mathcal{F}$ has zero vanishing cycles with respect to the map $\widetilde{g}$ at any closed point of the curve C.

Local acyclicity behaves well under proper push-forwards in the following sense. Let π be a proper morphism, so we have a diagram

$$\widetilde{X} \xrightarrow{\;\pi\;} X \xrightarrow{\;p\;} S. \qquad\qquad (11.1.3) \quad \boxed{\texttt{pip}}$$

$\boxed{\texttt{la-proper}}$ **Lemma 11.1.4.** *Let $\mathcal{F}$ be a sheaf on $\widetilde{X}$ which is locally acyclic with respect to the composite map $f = p \circ \pi$. Then, the sheaf $\pi_!\mathcal{F}'$ is locally acyclic with respect to p.*

Proof. This follows from a similar property of the vanishing cycles functor. $\qquad\square$

Next, for each $n \in \mathbb{Z}$, let $\mathscr{H}^n$ denote the n-th perverse cohomology sheaf of the complex $\pi_!\underline{\mathbf{C}}_{\widetilde{X}}$ and, given a Zariski open subset $U \subset S$ let $\jmath_U : p^{-1}(U) \hookrightarrow X$ denote the corresponding open imbedding. For any closed point $s \in S$ write $\widetilde{X}_s := f^{-1}(s)$, resp. $X_s := p^{-1}(s)$.
    The main result of this subsection, which is equivalent to Proposition 1.3.5, is as follows

$\boxed{\texttt{mylem}}$ **Proposition 11.1.5.** *Assume that π is a projective morphism, the composite $f = p \circ \pi$, in (11.1.3), is a smooth morphism, and $U \subset S$ is a Zariski open and dense subset such that $\pi : f^{-1}(U) \to p^{-1}(U)$ is a smooth morphism. Then, we have*
    (i) $\mathscr{H}^n = \mathrm{IC}(\mathscr{H}^n|_{p^{-1}(U)})$ for all n, equivalently, the map π is virtually small *in the sense of [Rei].*
    (ii) Assume in addition that π is generically finite. Then π is small. Furthermore, for any $s \in S$ the map $\widetilde{X}_s \to X_s$, induced by π, is semi-small.

Proof. By smooth base change the morphism $\widetilde{f} : C \times_S \widetilde{X} \to C$ is smooth for any smooth curve $g : C \to S$. Furthermore, $\widetilde{X}$ and $C \times_S \widetilde{X}$ are smooth schemes. It follows that the sheaf $\underline{\mathbf{C}}_{\widetilde{X}}$ is locally acyclic with respect to f. Hence, the sheaf $\pi_!\underline{\mathbf{C}}_{\widetilde{X}}$ is locally acyclic with respect to p, by Lemma 11.1.4. Decomposition theorems implies that $\pi_!\underline{\mathbf{C}}_{\widetilde{X}} = \oplus_n \mathscr{H}^n$ and, moreover, for each n one has $\mathscr{H}^n = \oplus_\alpha \mathrm{IC}(X_{n,\alpha}, \mathcal{L}_{n,\alpha})$ for some smooth connected locally closed and pairwise distinct subvarieties $X_\alpha \subset X$ and some local systems $\mathcal{L}_{n,\alpha}$ on $X_{n,\alpha}$. We deduce that each of the sheaves $\mathrm{IC}(X_{n,\alpha}, \mathcal{L}_{n,\alpha})$ must be locally acyclic with respect to p.
    To prove (i) we must show that the direct sum $\oplus_\alpha \mathrm{IC}(X_{n,\alpha}, \mathcal{L}_{n,\alpha})$ consists of a single summand $\mathrm{IC}(X_{n,\alpha_0}, \mathcal{L}_{n,\alpha_0})$ where X_{n,α_0} is a Zariski open and dense subset of X. Assume that $X_{n,\alpha}$ is not dense in X. The sheaf $\mathscr{H}^n|_{p^{-1}(U)}$ being a local system, it follows that $\bar{X}_{n,\alpha}$, the closure of $X_{n,\alpha}$, is

82

contained in $p^{-1}(S \smallsetminus U)$. We choose $x \in X_{n,\alpha}$ such that $s = p(x)$ is a sufficiently general smooth point of $p(X_{n,\alpha})$, a constructible subset of S. Further, we choose (as we may) a smooth connected curve C in S such that $C \cap (S \smallsetminus U) = \{s\}$. Let $\phi_{\widetilde{p},s}$ denote the functor of vanishing cycles for the map $\widetilde{p} : C \times_S X \to C$, in diagram 11.1.2, at the point $s \in C \subset S$. In our case, the map $\widetilde{g} : C \times_S X \to X$, in diagram 11.1.2, is a locally closed imbedding and $\bar{X}_{n,\alpha} \subset \{s\} \times_S X = p^{-1}(s)$. It follows that $\phi_{\widetilde{p},s}(\widetilde{g}^* \operatorname{IC}(X_{n,\alpha}, \mathcal{L}_{n,\alpha})) = \widetilde{g}^* \operatorname{IC}(X_{n,\alpha}, \mathcal{L}_{n,\alpha})$. The restriction of this sheaf to the point $\widetilde{g}^{-1}(x)$ is isomorphic to $\mathcal{L}_{n,\alpha}|_x$, hence nonzero. This contradicts the local acyclicity of $\operatorname{IC}(X_{n,\alpha}, \mathcal{L}_{n,\alpha})$. Thus, $\operatorname{IC}(X_{n,\alpha}, \mathcal{L}_{n,\alpha}) = 0$ and part (i) is proved.

Assume now that the map π is generically finite. It follow that the morphism $\pi : (p \circ \pi)^{-1}(U) \to p^{-1}(U)$ is finite. Hence, $\dim \widetilde{X} = \dim X$ and $\mathcal{H}^n|_{p^{-1}(U)} = 0$ unless $n = -\dim X$. We conclude that $\pi_! \mathbf{C}_{\widetilde{X}} = \operatorname{IC}(\mathcal{H}^{-\dim X}|_{p^{-1}(U)})$, that is, the map π is small.

The last statement in (ii) is clear for all $s \in U$. Thus, let $s \in S \smallsetminus U$ and let $\pi_s : \widetilde{X}_s \to X_s$ be the restriction of π. Proving that π_s is semismall amounts to showing that $(\pi_s)_! \mathbf{C}_{\widetilde{X}_s}[\dim \widetilde{X}_s]$ is a perverse sheaf. To this end, choose a smooth curve C in S such that $s \in C$ and $C \smallsetminus \{s\} \subset U$. The map $\widetilde{g} : C \times_S X \to X$ is a locally closed imbedding such that $(C \smallsetminus \{s\}) \times_S X \subset p^{-1}(U)$, cf. diagram 11.1.2. It follows that the restriction of the sheaf $\widetilde{g}^* \pi_! \mathbf{C}_{\widetilde{X}}$ to $(C \smallsetminus \{s\}) \times_S X = \widetilde{p}^{-1}(C \smallsetminus \{s\})$ is a local system, say $\mathcal{L}_{C \smallsetminus \{s\}}$. Let $\psi_{F,s}$ denote the functor of nearby cycles for a map $F : Y \to C$ at the point $s \in C$. This functor takes perverse sheaves to perverse sheaves. We deduce that $\psi_{\widetilde{p},s} \mathcal{L}_{C \smallsetminus \{s\}}[\dim X_s]$ is a perverse sheaf on X_s. On the other hand, let $\widetilde{\pi} : C \times_S \widetilde{X} \to C \times_S X$ be the map induced by π. The map $\widetilde{p} \circ \widetilde{\pi} : C \times_S \widetilde{X} \to C$ being smooth, we have $\psi_{\widetilde{p} \circ \widetilde{\pi}, s} \mathbf{C}_{C \times_S \widetilde{X}} = \mathbf{C}_{\widetilde{X}_s}$. Hence, the proper base change theorem for nearby cycles yields

$$\psi_{\widetilde{p},s} \mathcal{L}_{C \smallsetminus \{s\}} = (\pi_s)_! \psi_{\widetilde{p} \circ \widetilde{\pi}, s} \mathbf{C}_{C \times_S \widetilde{X}} = (\pi_s)_! \mathbf{C}_{\widetilde{X}_s}.$$

Thus, $(\pi_s)_! \mathbf{C}_{\widetilde{X}_s}[\dim \widetilde{X}_s]$ is a perverse sheaf, as required. $\qquad\square$

The proof of part (ii) also yields the following result. Let $i_s : X_s \hookrightarrow X$ denote the closed imbedding

Corollary 11.1.6. *In the setting of Proposition 11.1.5(ii), the sheaf $i_s^* \operatorname{IC}(\mathcal{H}^{-\dim X}|_{p^{-1}(U)})$ is a perverse sheaf which is isomorphic to $(\pi_s)_! \mathbf{C}_{\widetilde{X}_s}[\dim X_s]$.*

11.2. The following definition was introduced by Hausel and Rodriuez-Villegas [HV2].

Definition 11.2.1. A smooth quasi-projective variety X is called *semi-projective* if X is equipped with a $\mathbb{G}$-action such that $X^{\mathbb{G}}$, the fixed point set, is a projective vatiety and, moreover, the $\mathbb{G}$-action contracts X to $X^{\mathbb{G}}$, i.e., for any $x \in X$ the map $\mathbb{G} \to X$, $x \mapsto zx$ extends to a regular map $\mathbb{A} \to X$.

In the above setting, $X^{\mathbb{G}}$ is necessarily smooth and we have $\lim_{z \to 0} z \cdot x \in X^{\mathbb{G}}$.

The goal of this section is to give an algebraic proof of [HV2, Corollary 1.3.3], which is stated below. The original proof, essentially contained in [HV], works over the complex numbers and it involves a C^∞-argument and a compatification of X constructed earlier by C. Simpson. The statement for étale cohomology is then deduced from the corresponding result for the ordinary cohomology by certain comparison arguments. In the special case of quiver varieties, the theorem was proved earlier in [CBvdB] where the proof also used transcendental methods (the hyper-Kähler trick) .

Theorem 11.2.2. *Let $\mathfrak{h}$ be a finite dimensional vector space equpped with a linear $\mathbb{G}$-action with positive weights, so the $\mathbb{G}$-action is a contraction to $0 \in \mathfrak{h}$, the origin. Let $\mathcal{Z}$ be a semi-projective variety and $f : \mathcal{Z} \to \mathfrak{h}$ a smooth $\mathbb{G}$-equivariant morphism. Then, for any $h \in \mathfrak{h}$ and $k \geq 0$, we have*

(i) *The restriction map $H^\bullet(\mathcal{Z}) \to H^\bullet(X_h)$, where $\mathcal{Z}_h := \phi^{-1}(h)$, is an isomorphism and the cohomology group $H^k(\mathcal{Z}_h)$ is pure of weight $k/2$.*

(ii) *The sheaf $R^k \phi_* \mathbb{C}_\mathcal{Z}$ is a constant sheaf on $\mathfrak{h}$.*

Proof. Following [HV], for a $\mathbb{G}$-variety Y one defines $\mathrm{Core}(Y)$, the *core* of Y, to be the set of all points $y \in Y$ such that $\lim_{z \to \infty} z \cdot y$ exists. It is clear that $\mathcal{Z}_0^{\mathbb{G}} = \mathcal{Z}^{\mathbb{G}}$ and $\mathrm{Core}(\mathcal{Z}_0) = \mathrm{Core}(\mathcal{Z})$. Hence, X_0 is a semi-projective variety and one has restriction maps $H^\bullet(\mathcal{Z}) \to H^\bullet(\mathcal{Z}_0) \to H^\bullet(\mathrm{Core}(\mathcal{Z}_0))$. It is not difficult to show, see e.g. [HV, Theorem 1.3.1], that the second map, as well as the composite of the two maps, is an isomorphism. It follows that the first map is an isomorphism. Hence, the cohomology groups of $\mathcal{Z}$, resp. $\mathcal{Z}_0$, are isomorphic to those of $\mathcal{Z}^{\mathbb{G}}$, in particular, they are pure, [HV, Corollary 1.3.2].

Next, fix $h \neq 0$ and put $\mathcal{Z}' := \mathbb{A} \times_\mathfrak{h} \mathcal{Z}$, where the fiber product is taken with respect to the action map $\mathbb{A} \to \mathfrak{h}$, $z \mapsto z(h)$. We let $\mathbb{G}$ act on $\mathcal{Z}'$ via dilations on the first factor. Thus, we have a commutative diagram of $\mathbb{G}$-equivariant maps

$$
\begin{array}{ccccc}
pr_1^{-1}(0) & \stackrel{i}{\hookrightarrow} & \mathcal{Z}' & \stackrel{pr_2}{\longrightarrow} & \mathcal{Z} \\
\downarrow & & \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle f} \\
\{0\} & \hookrightarrow & \mathbb{A} & \stackrel{z \mapsto z(h)}{\longrightarrow} & \mathfrak{h}
\end{array}
$$

Note that $pr_1^{-1}(0) \cong \mathcal{Z}_0$ and the $\mathbb{G}$-action on $\mathcal{Z}'$ is a contraction. We deduce similarly to the above that each of the pull-back morphisms $H^\bullet(\mathcal{Z}) \xrightarrow{pr_2^*} H^\bullet(\mathcal{Z}') \xrightarrow{i^*} H^\bullet(\mathcal{Z}_0)$ is an isomorphism. In particular, the cohomology of $\mathcal{Z}'$ is pure. Note further that $pr_1^{-1}(1) \cong \mathcal{Z}_h$ and the $\mathbb{G}$-action provides a $\mathbb{G}$-equivariant isomorphism $\mathbb{G} \times \mathcal{Z}'_h \xrightarrow{\sim} pr_1^{-1}(\mathbb{G})$. Thus, we have a diagram

$$\mathcal{Z}_0 \stackrel{i}{\hookrightarrow} \mathcal{Z}' \stackrel{j}{\hookleftarrow} \mathcal{Z}' \smallsetminus \mathcal{Z}_0 = \!\!= \mathbb{G} \times \mathcal{Z}_h, \qquad (11.2.3) \quad \boxed{\text{kun}}$$

where i and j are a closed and an open imbedding, respectively.

The cohomology groups $H^0(\mathbb{G}) = \underline{\mathbf{C}}(0)$ and $H^1(\mathbb{G}) = \underline{\mathbf{C}}(2)$ are 1-dimensional vector spaces which have weights 0 and 2, respectively. Hence, one has the following Künneth decomposition

$$H^\bullet(\mathcal{Z}' \smallsetminus \mathcal{Z}_0) \cong [H^0(\mathbb{G}) \otimes H^\bullet(\mathcal{Z}_h)] \oplus [H^1(\mathbb{G}) \otimes H^{\bullet - 1}(\mathcal{Z}_h)] = H^\bullet(\mathcal{Z}_h) \oplus H^{\bullet - 1}(\mathcal{Z}_h)(2).$$

Using the Künneth decomposition, the long exact sequence associated with diagram (11.2.3), takes the following form,

$$\ldots \to H^{k-1}(\mathcal{Z}_0)(1) \xrightarrow{i_!} H^k(\mathcal{Z}') \xrightarrow{j^*} H^k(\mathcal{Z}_h) \oplus H^{k-1}(\mathcal{Z}_h)(2) \xrightarrow{[1]} H^{k+1}(\mathcal{Z}_0) \to \ldots \quad (11.2.4) \quad \boxed{\text{pure\_k}}$$

Let $\mathrm{gr}_w H^\bullet(-)$ denote an associated graded term of weight $w \in \mathbb{Z}$ in the weight filtration on the cohomology. Applying the functor $\mathrm{gr}_w(-)$, which is an exact functor, to (11.2.4), one obtains an exact sequence of spaces of weight w. Thanks to the purity result proved earlier, we know that $\mathrm{gr}_w H^n(\mathcal{Z}_0) = \mathrm{gr}_w H^n(\mathcal{Z}') = 0$, whenever $w \neq n$. Hence, for any $w \geq k + 2$, the fragment of the resulting exact sequence of spaces of weight w that corresponds to (11.2.4) reads

$$\ldots \to 0 \xrightarrow{i_!} 0 \xrightarrow{j^*} \mathrm{gr}_w H^k(\mathcal{Z}_h) \oplus [\mathrm{gr}_{w-2} H^{k-1}(\mathcal{Z}_h)](2) \xrightarrow{[1]} 0 \to \ldots \quad (11.2.5) \quad \boxed{\text{pure2}}$$

From (11.2.5), we see that $\mathrm{gr}_{w-2} H^{k-1}(\mathcal{Z}_h) = 0$. It follows that the group $\mathrm{gr}_w H^k(\mathcal{Z}_h)$ vanishes for any pair of integers w, k, such that $w > k$. On the other hand, since $\mathcal{Z}_h$ is smooth, we also have $\mathrm{gr}_w H^k(\mathcal{Z}_h) = 0$ for all $w < k$. Thus, we conclude that, for each k, the group $H^k(\mathcal{Z}_h)$ is pure of weight k.

The purity implies that the long exact sequence (11.2.4) breaks up into a direct sum $\oplus_{w\in\mathbb{Z}} E^{(w)}$, of long exact sequences $E^{(w)}$, $w \in \mathbb{Z}$, such that, for any w, all terms in $E^{(w)}$ are pure of weight w. Furthermore, one finds that each of these long exact sequences actually splits further into length two exact sequences. Specifically, for $k \in \mathbb{Z}$, the long exact sequence $E^{(k)}$ reduces, effectively, to the following pair of isomorphisms:

$$\jmath^* : H^k(\mathcal{Z}') \stackrel{\sim}{\to} H^k(\mathcal{Z}_h) \quad \text{and} \quad i_! : H^{k-1}(\mathcal{Z}_0)(2) \stackrel{\sim}{\to} H^{k+1}(\mathcal{Z}'). \tag{11.2.6}$$

Here, the isomorphism $\jmath^*$ is induced by the imbedding $\jmath: \mathcal{Z}_h = \{1\} \times \mathcal{Z}_h \hookrightarrow \mathcal{Z}'$.

To complete the proof of part (i) of the theorem, we factor the restriction map $H^\bullet(\mathcal{Z}) \to H^\bullet(\mathcal{Z}_h)$ as a composition $H^\bullet(\mathcal{Z}) \xrightarrow{pr_2^*} H^\bullet(\mathcal{Z}') \xrightarrow{\jmath^*} H^\bullet(\mathcal{Z}_h)$. We have shown that each of these maps is an isomorphism, proving (i).

To prove (ii), write $i_h : \{h\} \hookrightarrow \mathfrak{h}$, resp. $\imath_h : \mathcal{Z}_h \hookrightarrow \mathcal{Z}$, for the corresponding closed imbeddings and $p : \mathcal{Z} \times \mathfrak{h} \twoheadrightarrow \mathfrak{h}$ for the second projection. Also, define a map $\varepsilon : \mathcal{Z} \hookrightarrow \mathcal{Z} \times \mathfrak{h}$ by the assignment $x \mapsto (x, f(x))$. Thus, ε is a closed embedding via the graph of f, so one has a factorization $f = p \circ \varepsilon$.

Now, we begin the proof by noting that each cohomology sheaf $R^k p_* \mathbf{C}_{\mathcal{Z}\times\mathfrak{h}}$ is a constant sheaf, by the Künneth formula. Next, we observe that there is a canonical morphism

$$u : p_*\mathbf{C}_{\mathcal{Z}\times\mathfrak{h}} \longrightarrow p_*(\varepsilon_*\varepsilon^*\mathbf{C}_{\mathcal{Z}\times\mathfrak{h}}) = f_*\mathbf{C}_{\mathcal{Z}}.$$

Thus, we would be done provided we can prove that the morphism u is, in fact, an isomorphism. We will prove this by showing that, for every $h \in \mathfrak{h}$, the morphism $i_h^!(u) : i_h^!(p_*\mathbf{C}_{\mathcal{Z}\times\mathfrak{h}}) \to i_h^!(f_*\mathbf{C}_{\mathcal{Z}})$, induced by u, is an isomorphism. This is known to be sufficient to conclude that u is an isomorphism,

The argument below involves the following diagram, where $f_h := f|_{\mathcal{Z}_h}$ and p_h stands for a constant map,

$$\begin{array}{ccccc}
\mathcal{Z}_h & \xrightarrow{\quad\varepsilon|_{\mathcal{Z}_h}\quad} & \mathcal{Z} \times \{h\} & \xrightarrow{\quad p_h \quad} & \{h\} \\
\cap\downarrow{\scriptstyle\imath_h} & & \cap\downarrow{\scriptstyle \mathrm{Id}\times i_h} & & \downarrow{\scriptstyle i_h} \\
\mathcal{Z} & \xrightarrow{\quad\varepsilon\quad} & \mathcal{Z} \times \mathfrak{h} & \xrightarrow{\quad p \quad} & \mathfrak{h}.
\end{array} \tag{11.2.7}$$

It is clear that all commutative squares in the diagram are cartesian. Also, f is a smooth morphism, so $\mathcal{Z}_h$ is a smooth subvariety of codimension $n := \dim \mathfrak{h}$ in $\mathcal{Z}$. Therefore, applying proper base change to the above diagram one gets the following canonical isomorphisms:

$$i_h^!(p_*\mathbf{C}_{\mathcal{Z}\times\mathfrak{h}}) = (p_h)_*(\mathrm{Id}\times i_h)^!\mathbf{C}_{\mathcal{Z}\times\mathfrak{h}} = (p_h)_*(\mathbf{C}_{\mathcal{Z}}[2n]) = H^{\bullet+2n}(\mathcal{Z});$$
$$i_h^!(f_*\mathbf{C}_{\mathcal{Z}}) = (f_h)_*(\imath_h^!\mathbf{C}_{\mathcal{Z}}) = (f_h)_*(\mathbf{C}_{\mathcal{Z}_h}[2n]) = H^{\bullet+2n}(\mathcal{Z}_h).$$

Thus, the morphism $i_h^!(u)$ goes, via base change, to a morphism $H^{\bullet+n}(\mathcal{Z}) \to H^{\bullet+n}(\mathcal{Z}_h)$. One can check that the latter morphism is the restriction morphism $\imath_h^!$ induced by the imbedding $\imath_h : \mathcal{Z}_h \hookrightarrow \mathcal{Z}$. Furthermore, the first isomorphism in (11.2.6) implies that $\imath_h^!$ is an isomorphism, for any $h \in \mathfrak{h}$. It follows that the morphism u is an isomorphism, completing the proof of the theorem. $\square$

12. APPENDIX B: AN APPLICATION TO THE CALOGERO-MOSER VARIETY OF TYPE A

The goal of this Appendix is to illustrate the relation between indecomposable representations and the geometry of the moment map in a special case of the Calogero-Moser quiver.

12.1. Fix a quiver Q with vertex set I and a dimension vector $\mathbf{v} \in \mathbb{Z}^I$. Let $\bar{Q}$ be the double of the quiver Q and $\mathcal{M}_O$ be as in the formulation of Theorem 1.1.5. Further, let $(\eta_i)_{i \in I} \in \mathsf{k}^I$ and write $\eta = (\eta_i \cdot \mathrm{tr}_{v_i})_{i \in I} \in \mathfrak{g}_{\mathbf{v}}^*$. One has a diagram

$$
\begin{array}{ccc}
\mu_{\mathbf{v}}^{-1}(\eta) \hookrightarrow \mathrm{Rep}_{\mathbf{v}}\,\bar{Q} \longrightarrow \mathrm{Rep}_{\mathbf{v}}\,Q & & (12.1.1) \quad \boxed{\texttt{pqr}} \\
\downarrow q \qquad\qquad\qquad\qquad\qquad \downarrow \mathfrak{w} & & \\
\mu_{\mathbf{v}}^{-1}(\eta)/\!\!/G_{\mathbf{v}} \xrightarrow{\quad \pi \quad} \mathrm{Rep}_{\mathbf{v}}/\!\!/G_{\mathbf{v}} & &
\end{array}
$$

Here the first map in the top row is the natural inclusion and the second map is a restriction of represenations of $\bar{Q}$ to Q, viewed as a subquiver of $\bar{Q}$. Let p denote the composite of these two maps. Further, for any $x \in \mathrm{Rep}_{\mathbf{v}}/\!\!/G_{\mathbf{v}}$ let $\mathrm{Ind}(\mathrm{x})$ be the set of $G_{\mathbf{v}}$-orbits in $\mathfrak{w}^{-1}(x)$ formed by the indecomposable representations.

It was shown by Crawley-Boevey that in the case where $\mathbf{v}$ is indivisible and the I-tuple (η_i) is sufficiently general the image of p is equal to the set of indecomposable representations. Moreover, η is a regular value of the moment map $\mu : \mathrm{Rep}_{\mathbf{v}}\,\bar{Q} \to \mathfrak{g}_{\mathbf{v}}^*$. In particular, $\mu_{\mathbf{v}}^{-1}(\lambda)$ is a smooth scheme, the $G_{\mathbf{v}}$-action on this scheme is free, and the map $\mu_{\mathbf{v}}^{-1}(\lambda) \to \mu_{\mathbf{v}}^{-1}(\lambda)/\!\!/G_{\mathbf{v}}$ is a geometric quotient.

The following result is essentially [CBvdB, Proposition 2.2.1].

$\boxed{\texttt{nochar}}$ **Proposition 12.1.2.** *Let $\mathbf{v}$ be an indivisible dimension vector and assume η is sufficiently general. Then we have*

(i) *For any $G_{\mathbf{v}}$-orbit $y \subset \mathrm{Rep}_{\mathbf{v}}Q$, of indecomposable representations, we have $q(p^{-1}(y)) \cong \mathbb{A}^{\frac{1}{2}\dim \mathcal{M}_O}$, where $\mathcal{M}_O$ is as in Theorem 1.1.5.*

(ii) *For any $\mathbb{F}_q$-rational point $x \in \mathrm{Rep}_{\mathbf{v}}Q/\!\!/G_{\mathbf{v}}$, the number, $\#\,\mathrm{Ind}(\mathrm{x})(\mathbb{F}_q)$, of isomorphism classes of absolutely idecomposable representations defined over $\mathbb{F}_q$, is given by the formula*

$$
\#\,\mathrm{Ind}(\mathrm{x})(\mathbb{F}_q) = q^{\frac{1}{2}\dim \mathcal{M}_O} \cdot \mathsf{Tr}_{\mathsf{Fr}}|\,H_c^\bullet(\pi^{-1}(x)).
$$

(iii) *If the the set $\mathrm{Ind}(\mathrm{x})$ is finite then we have a partition $\pi^{-1}(x) = \sqcup_{y \in \mathrm{Ind}(\mathrm{x})}\, q(p^{-1}(y))$ into a disjoint union of finitely many locally closed subvarieties, each isomorphic to $\mathbb{A}^{\frac{1}{2}\dim \mathcal{M}_O}$.*

Proof. (i) According to Lemma 3.1 in [CB1], for $y \in \mathrm{Rep}_{\mathbf{v}}Q$, which is in the image of $p \circ i$, the fiber $(p \circ i)^{-1}(y)$ is a quotient of a vector space V isomorphic to $\mathrm{Ext}^1_{\mathsf{k}Q}(y,y)^*$ by an affine-linear free action on V of a unipotent group U, where $U = G_y$. The latter group is unipotent since y is absolutely indecomposable. representation y. Any such quotient V/U is known to be isomorphic an affine space of dimension $\dim V - \dim U$. Thus, we have $(p \circ i)^{-1}(y) \cong \mathbb{A}^m$ where $m = \dim \mathrm{Ext}^1_{\mathsf{k}Q}(y,y) - \dim Aut(y) = \dim \mathrm{Rep}_{\mathbf{v}}\,Q - \dim G_{\mathbf{v}} = \frac{1}{2}\dim \mathcal{M}_O$. This proves (i). In particular, for any $\mathbb{F}_q$-rational point $x \in \mathrm{Rep}_{\mathbf{v}}Q/\!\!/G_{\mathbf{v}}$, we have

$$
\mathsf{Tr}_{\mathsf{Fr}}|\,H_c^\bullet(\pi^{-1}(x)) = \#\pi^{-1}(x)(\mathbb{F}_q) = q^{\frac{1}{2}\dim \mathcal{M}_O} \cdot \#r^{-1}(x)(\mathbb{F}_q).
$$

Part (ii) follows. In the case where the number of somorphism classes of absolutely indecomposable representations is finite we deduce from (i) that the scheme $\pi^{-1}(x)$ is a disjoint union of finitely many pieces each of which is isomorphic to $\mathbb{A}^{\frac{1}{2}\dim \mathcal{M}_O}$. $\qquad\square$

$\boxed{\texttt{CM}}$

12.2. Let Q be the Calogero-Moser quiver Q, i.e. a quiver with two vertices, a loop at one of the vertices, and an edge joining the vertices, which we fix to point toward the vertex with the loop. We fix the the dimension vector $(n,1)$, where n labels the vertex with the loop. A representation with this dimension vector is a pair (u,v), where $u \in \mathfrak{gl}(V), v \in V$, and V is an n-dimensional vector space. It is known that the map that assigns to $(u,v) \in \mathfrak{gl}_n \times V$ the unordered n-tuple of

eigenvalues of the operator u yields an isomorphism

$$\mathrm{Rep}_{n,1}(Q)/\!\!/\mathrm{GL}_n = (\mathfrak{gl}_n \times V)/\!\!/\mathrm{GL}(V) \overset{\sim}{\to} \mathfrak{gl}(V)/\!\!/\mathrm{GL}(V) \cong \mathbb{A}^n/\mathfrak{S}_n.$$

Let $\mathrm{Nil} = \mathrm{Nil}(\mathfrak{gl}(V))$. Conjugacy classes in Nil are parametrized by partitions of n according to Jordan normal form. Given a partition $\nu = (\nu_1 \geq \nu_2 \geq \ldots)$, write $|\nu| = \sum \nu_i$, resp. $\ell(\nu)$ for the number of parts, i.e. $\ell(\nu) = \#\{i \mid \nu_i \neq 0\}$. Let Σ be the set of pairs, (λ, μ), of partitions such that

  (1) $|\lambda| + |\mu| = n$;
  (2) The parts of each of the partitions λ and μ strictly decrease;
  (3) One has that either $\ell(\lambda) = \ell(\mu)$ or $\ell(\lambda) = \ell(\mu) + 1$.

The definition of the set Σ and the lemma below were also obtained in a recent paper by Bellamy and Boos, see [BeBo, Lemma 5.3], where the authors use the notation $P_F(n)$ for our Σ.

Lemma 12.2.1. *For any field* k, *there is a natural bijection* $(\lambda, \mu) \mapsto O_{\lambda,\mu}$ *between the set* Σ *and the set of* k-*rational* $\mathrm{GL}(V)$-*orbits in* $\mathrm{Nil} \times V$ *which consist of absolutely indecomposable representations such that for* $(u, v) \in O_{\lambda,\mu}$ *the Jordan normal form of* u *is given by the partition with parts* $\lambda_i + \mu_i = \nu_i$, *i.e. the size of the* i-*th block of the nilpotent* u *equals* $\lambda_i + \mu_i = \nu_i$.

Proof. First we recall the description of the $GL(V)$-orbits in the space of pairs $(u, v) \in \mathcal{N} \times V$, where V is a vector space and $\mathcal{N} \subset \mathfrak{gl}(V)$ is the nilpotent cone. This classification is the result of Theorem 1 in [Tr] or Proposition 2.3 in [AH], and is described as follows (we follow [Tr]). The orbits in $\mathcal{N} \times V$ are in bijection with pairs of partitions λ, μ such that $|\lambda| + |\mu| = \sum \lambda_i + \sum \mu_i = n$. This bijection has the property that the type of the nilpotent u is equal to $\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots)$, and is constructed in the following way. Given a pair of partitions (λ, μ) as above, let $\nu = \lambda + \mu$, and let u be a nilpotent of type ν. Let D_ν be the set of boxes of the Young diagram ν, so that $D_\nu = \{(i, j) : 1 \leq j \leq \nu_i\}$. We choose a basis of V with basis vectors $e_{i,j}$, where $(i, j) \in D_\nu$ such that $ue_{i,j} = e_{i,j-1}$ for $2 \leq j \leq \nu_i$ and $ue_{i,1} = 0$. Let $v = \sum e_{i,\lambda_i}$, where we put $e_{i,0} = 0$.

To single out the orbits consisting of indecomposable representations, we start by recalling another bijection in Proposition-Construction 1 in [Tr] between the orbits in $\mathcal{N} \times V$ and pairs of partitions ν, θ, where ν is the type of the nilpotent u and θ is the type of the nilpotent u acting on the quotient vector space $V/\langle v, uv, u^2v, \ldots\rangle$. Note that ν and θ have the property that their columns with the same number differ in length at most by one.

We observe that a representation is reducible if and only if ν and θ have a row of the same length. Indeed, a representation is reducible if and only if u has a Jordan block in its Jordan normal form such that the component of v is zero in a basis of generalized eigenvectors for u, which translates into the condition that ν and θ have a row of the same length. Because of the equalities $\lambda_i - \lambda_{i+1} = \nu_i - \theta_i$, $\mu_i - \mu_{i+1} = \nu_i - \theta_{i+1}$ we obtain that λ and μ have to have distinct parts.

Finally we remark that because of the explicit description of indecomposables, which by the above are indexed by the same data over any field, we have that absolutely indecomposable representations coincide with indecomposable ones. $\square$

An element of $\mathrm{Rep}_{n,1}\bar{Q}$ is a quadruple $(B_1, B_2, v, w) \in \mathfrak{gl}(V) \times \mathfrak{gl}(V) \times V \times V^*$. The moment map $\mu : \mathrm{Rep}_{n,1}\bar{Q} \to \mathfrak{gl}_n$ is given by the formula $\mu(B_1, B_2, v, w) = [B_1, B_2] + vw$. It is known that for any $c \neq 0$ the element $c \cdot \mathrm{Id} \in \mathfrak{gl}(V)$ is sufficiently general. $\mathrm{Id} \in \mathfrak{gl}(V)$ is a regular value of the moment map. Thus, $\mathrm{Id} \in \mathfrak{gl}(V)$ is a regular value of the moment map and $\mathcal{M} = \mu^{-1}(\mathrm{Id})/\!\!/GL_n$ is a smooth symplectic variety of dimension $2n$, called the *Calogero-Moser space*. We use the notation of diagram (12.1.1) and put

$$\mathcal{M}_0 := \mu^{-1}(\mathrm{Id}) \cap \pi^{-1}(0) = \{(B_1, B_2, v, w) \in \mathrm{Nil} \times \mathfrak{gl}(V) \times V \times V^* \mid [B_1, B_2] = I + vw = \mathrm{Id}\}/\!\!/\mathrm{GL}(V).$$

The above variety has been studied by G. Wilson [Wi]. He constructed a parametrization of connected components of the variety $\mathcal{M}_0$ by partitions of n and showed that each connected component of $\mathcal{M}_0$ is isomorphic to $\mathbb{A}^n$. On the other hand, the number of $\mathrm{GL}(V)$-orbits in $\mathfrak{w}^{-1}(0) = \mathrm{Nil} \times V$ being finite, Proposition 12.1.2(iii) is applicable and it yields an alternative proof of the latter isomorphism. Furthermore, Proposition 12.1.2(iii) provides a natural parametrization of connected components of $\mathcal{M}_0$ by the $\mathrm{GL}(V)$-orbits of indecomposables in $\mathrm{Nil} \times V$. Applying Lemma 12.2.1 we obtain a parametrization of connected components of $\mathcal{M}_0$ by the elements of the set Σ.

The latter parametrization is related to the one given by Wilson via a bijection between the set Σ and the set of partitions of n, viewed as Young diagrams with n boxes. The bijection amounts, essentially, to describing Young diagrams by its 'Frobenius form' , Specifically, let $(\lambda, \mu) \in \Sigma$. First, shift the rows of the Young diagram μ so that the i-th row is shifted by i units to the right (where the first row is the longest, the second row is the second longest, and so on). Next, shift the columns of the Young diagram λ so that the i-th column is hifted by $i - 1$ units up (where the first column is the longest, the second column is the second longest, and so on). Now, the bijection in question is defined by assigning to (λ, μ) the Young diagram with n boxes obtained by gluing the resulting shifts of λ and μ along the line that becomes the main diagonal of the new diagram. Lemma 12.2.1 ensures that the diagram thus defined has the shape of a Young diagram.

Remark 12.2.2. Frobenius form was also mentioned by Wilson, see [Wi, p.28]. Our Lemma 12.2.1 gives a geometric interpretation, in terms of indecomposable quiver representations, of somewhat mysterious inequalities in Wilson's work [Wi, Lemma 6.9].

Remark 12.2.3. One can generalize[1] the case of the Calogero-Moser quiver to a cyclic quiver with one additional vertex connected by an edge to one of the vertices of the cycle, where the component of the dimension vector is equal to n at each of the vertices of the cycle and 1 at the additional vertex. Below we describe the bijection which is the content of Proposition 12.1.2 for the cyclic quiver (i.e. the analog for the cyclic quiver of the bijection of Proposition 12.1.2(iii) for the Calogero-Moser quiver).

The torus-fixed points in a quiver variety for the cyclic quiver are described, for example, in [Ne], and the description is as follows. Let $(v_1, ..., v_n)$ be the dimension vector at the vertices of the cycle so that $\dim V_k = v_k$ for the standard vector spaces V_k associated to the quiver variety (note that we will take k modulo n below), and let the framing be given by 1 at the first vertex and 0 at all the other vertices of the cycle. Then the torus-fixed points in the corresponding quiver variety are labelled by Young diagrams in which each box is colored in one of the n colors corresponding to the content of the box (i.e. the difference of the x- and the y-coordinates of the center of the box, when the Young diagram is placed in the first quadrant in the standard way) in such a way that in total there are exactly v_i boxes of the i-th color in the Young diagram.

Now we describe the other side of the bijection in Proposition 12.1.2(iii). The absolutely indecomposable representations of the undoubled quiver in this case coincide with the indecomposable ones, as in the above case of the Calogero-Moser quiver. The indecomposable (nilpotent when one goes around the circle enough times) representations of our undoubled quiver (with an oriented cycle and the remaining arrow pointing inside) can be described as follows.

We call a "snake" a nilpotent representation of the cyclic quiver which consists of a string of vectors $u_1, ..., u_N$ where u_k lies in $V_{k \bmod n}$ and u_k is mapped to u_{k+1} by the arrow $V_{k \bmod n} \to V_{(k+1) \bmod n}$) with a marked point in each snake at a place with residue 1 modulo n (i.e. the place $\bmod\ n$ corresponding to where the nonzero component of the framing vector lies). Then an indecomposable representation of our undoubled quiver is labelled by a collection of "snakes," so that when all snakes are unrolled and placed on the coordinate line so that the marked place of each

---

[1]This has also been done in [BeBo].

snake is at the origin, any two snakes have the following property: one of them lies strictly within the other inside of the coordinate line (i.e. the longer is the head of a snake, the longer is its tail).

The bijection in Proposition 12.1.2(iii) of the above data describing an indecomposable representation of the undoubled quiver with the data describing a torus-fixed point in the quiver variety for the cyclic quiver is obtained by taking each snake, bending it at the marked point to form the right angle, and placing snakes which have been bent in this way on top of each other to form the Young diagram.

References

[AH]   P. N. Achar, A. Henderson, Orbit closures in the enhanced nilpotent cone, Adv. Math. 219 (2008), 27-62.

[Al]   J. Alper, *Good moduli spaces for Artin stacks.* arXiv:0804.2242.

[AHL]   _____, J. Hall, D. Rydh, *A Luna etale slice theorem for algebraic stacks,* http://arxiv.org/abs/1504.06467.

[An]   Y. André, *Slope filtrations*, Confluentes Math. 1 (2009), 1-85.

[BNR]   A. Beauville, M. Narasimhan, and S. Ramanan, *Spectral curves and generalized theta divisor*, J. Reine Angew. Math. 398 (1989), 169179.

[B1]   K. Behrend, *Derived ℓ-adic categories for algebraic stacks*, Memoirs of the A.M.S. 163 (2003).

[B2]   _____, *The Lefschetz trace formula for the moduli stack of principal bundles*, Ph.D. University of California, Berkeley 1991.

[BBD]   A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers*, Asterisque, **100** (1982).

[BD]   _____, V. Drinfeld, *Quantization of Hitchin's hamiltonians.*

[BeBo]   G. Bellamy, M. Boos, *The (cyclic) enhanced nilpotent cone via quiver representations.* arXiv:1609.04525.

[BB]   R. Bezrukavnikov, A. Braverman, *Geometric Langlands correspondence for D-modules in prime characteristic: the $GL(n)$ case*, Pure Appl. Math. Q. 3 (2007), 153-179.

[Bi]   I. Biswas, *Parabolic bundles as orbifold bundles.* Duke Math. J. 88 (1997), 305-325.

[BMW]   _____, S. Majumder, M. Wong, *Parabolic Higgs bundles and Γ-Higgs bundles.* J. Aust. Math. Soc. 95 (2013), 315-328.

[BvdB]   A. Bondal, M. van der Bergh, *Generators and representability of functors in commutative and noncommutative algebraic geometry*, Mosc. Math. J. 3 (2003) 1-36.

[BM]   W. Borho, R. MacPherson, *Représentations des groupes de Weyl et homologie d'intersection pour les variétés nilpotentes.* C. R. Acad. Sci. Paris Sér. I Math. 292 (1981), 707-710.

[BG]   A. Braverman, D. Gaitsgory, *Geometric Eisenstein series.* Invent. Math. 150 (2002), 287-384.

[Br]   J.-L. Brylinski, *Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques.* Géométrie et analyse microlocales. Astérisque No. 140-141 (1986), 3-134, 251.

[CHM]   M. de Cataldo, T. Hausel, L. Migliorini, *Exchange between perverse and weight filtration for the Hilbert schemes of points of two surfaces.* arXiv:1012.2583.

[CI]   D. Chan, C. Ingalls, *Noncommutative coordinate rings and stacks.* Proc. London Math. Soc. (3) **88** (2004), 63-88.

[CL]   P.-H. Chaudouard, G. Laumon, *Un théorème du support pour la fibration de Hitchin*, Preprint, arXiv:1407.3562.

[CYZ]   X.W. Chen, Y. Ye, P. Zhang, *Algebras of derived dimension zero.* Comm. Algebra 36 (2008), no. 1, 1–10.

[CB1]   W. Crawley-Boevey, *Geometry of the moment map for representations of quivers*, Compositio Mathematica **126** (2001), 257-293.

[CB2]   _____, *Normality of Marsden-Weinstein reductions for representations of quivers.* Math. Ann. 325 (2003), 55-79.

[CBEG]   _____, P. Etingof, and V. Ginzburg, *Noncommutative geometry and quiver algebras*, Adv. Math. **209** (2007), 274-336, arXiv:math.AG/0502301.

[CBvdB]   _____, M. van den Bergh, *Absolutely indecomposable representations and Kac-Moody Lie algebras* (with an appendix by Hiraku Nakajima), Invent. Math. 155 (2004), 537-559.

[CQ]   J. Cuntz and D. Quillen, *Algebra extensions and nonsingularity*, J. Amer. Math. Soc. **8** (1995), 251-289.

[Da1]   B. Davison, *The critical CoHA of a self dual quiver with potential,* arXiv:1311.7172.

[Da2]   _____, *Purity of critical cohomology and Kac's conjecture,* arXiv:1311.6989.

[DM]   _____, S. Meinhardt, *Cohomological Donaldson-Thomas theory of a quiver with potential and quantum enveloping algebras.* arXiv:1601.02479.

[De]   P. Deligne, *Comptage de faisceaux ℓ-adiques.* IAS Preprint, May 2013.

[DL]   _____, G. Lusztig, *Representations of reductive groups over finite fields.* Ann. of Math. (2) 103 (1976), 103-161.

[Dr]   V. Drinfeld, *The number of two-dimensional irreducible representations of the fundamental group of a curve over a finite field.* (Russian) Funktsional. Anal. i Prilozhen. 15 (1981), no. 4, 75-76.

[Fa]   G. Faltings, *Stable G-bundles and projective connections.* J. Algebraic Geom. 2 (1993), 507-568.

[FG]   J. Francis, D. Gaitsgory, *Chiral Koszul duality.* Selecta Math. (N.S.) 18 (2012), 27-87.

[Ga]   D. Gaitsgory, *The notion of category over an algebraic stack.* arXiv:math/0507192.

[Gi]   V. Ginzburg, *Lectures on Nakajima's quiver varieties.* Geometric methods in representation theory. I, 145-219, Sémin. Congr., 24-I, Soc. Math. France, Paris, 2012.

[Gri]   M. Grinberg, *On the specialization to the asymptotic cone.* J. Algebraic Geom. 10 (2001), 1, 1-17.

[J]   D. Joyce, *Configurations in abelian categories. I. Basic properties and moduli stacks.* Adv. Math. 203 (2006), 194-255.

[HL]   D. Halpern-Leistner, *On the structure of instability in moduli theory,* https://arxiv.org/abs/1411.0627

[Ha]   T. Hausel, *Betti numbers of holomorphic symplectic quotients via arithmetic Fourier transform.* Proc. Natl. Acad. Sci. USA 103 (2006), no. 16, 6120–6124.

[HV]   ——— , F. Rodriguez-Villegas, *Mixed Hodge polynomials of character varieties.* With an appendix by Nicholas M. Katz. Invent. Math. 174 (2008), 555-624.

[HV2]   ——— , ——— *Cohomology of large semiprojective hyperkähler varieties.* Astérisque No. 370 (2015), 113-156.

[HLV1] T. Hausel, E. Letellier, and F. Rodriguez-Villegas, *Positivity of Kac polynomials and DT-invariants of quivers*, Ann. of Math. (2) 177 (2013), 1147-1168.

[HLV2] ——— , ——— ,——— , *Arithmetic harmonic analysis on character and quiver varieties.* Duke Math. J. 160 (2011), 323-400.

[HLV3] ——— , ——— ,——— , *Arithmetic harmonic analysis on character and quiver varieties II.* Adv. Math. 234 (2013), 85-128.

[H]   J. Hua, *Counting representations of quivers over finite fields.* J. Algebra 226, (2000) 1011-1033.

[Ka]   D. Kaledin, *Derived equivalences by quantization.* Geom. Funct. Anal. 17 (2008), no. 6, 1968-2004.

[KL]   N. Katz and G. Laumon, *Transformation de Fourier et majoration de sommes exponentielles.* Inst. Hautes Etudes Sci. Publ. Math. 62 (1985), 361418.

[KMM]   Y. Kawamata, K. Matsuda, and K. Matsuki, *Introduction to the minimal model problem.* in Algebraic Geometry Sendai 1985, Advanced Studies in Pure Math. 10(1987), Kinokuniya and North-Holland, 283-360.

[KS1]   M. Kontsevich, Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations.* arXiv:0811.2435.

[KS2]   ——— , ——— , *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants.* Commun. Number Theory Phys. 5 (2011), 231-352.

[LO]   Y. Laszlo and M. Olsson, *The six operations for sheaves on Artin stacks* I, II, Publ. Math. IHES 107 (2008), 109–210.

[La]   G. Laumon, *Faisceaux characteres (d'apres Lusztig)*, Seminaire Bourbaki, Vol. 1988/89, Asterisque, 177-178 (1989), 231-260.

[La2]   ——— , *Correspondance de Langlands geometrique pour les corps de fonctions*, Duke Math. J. 54 (1987), 309–359.

[LM]   ——— , L. Moret-Bailly, *Champs algébriques.*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. 39. Springer-Verlag, Berlin, 2000.

[LP]   L, Le Bruyn, C. Procesi, *Semisimple representations of quivers.* Trans. Amer. Math. Soc. 317 (1990), 585-598.

[Le]   E. Letellier, *DT-invariants of quivers and the Steinberg character of GL_n.* Int. Math. Res. Not. IMRN 2015, no. 22, 11887-11908.

[LvdB] W. Lowen, M. Van den Bergh, *Deformation theory of abelian categories.* Trans. Amer. Math. Soc. 358 (2006), 5441-5483.

[Lur]   J. Lurie, *Higher algebra.*

[L1]   G. Lusztig, *Green polynomials and singularities of unipotent classes.* Adv. in Math. 42 (1981), 169-178.

[L2]   ——— , *Character sheaves I-V*.

[Me]   S. Meinhardt, *Donaldson-Thomas invariants vs. intersection cohomology for categories of homological dimension one.* arXiv:1512.03343.

[MR]   ——— , M. Reineke, *Donaldson-Thomas invariants versus intersection cohomology of quiver moduli.* arXiv:1411.4062.

[MeSe] V. B. Mehta, C. S. Seshadri, *Moduli of vector bundles on curves with parabolic structures.* Math. Ann. 248 (1980), 205-239.

[MS]   S. Mozgovoy, O. Schiffmann, *Counting Higgs bundles.* arXiv:1411.2101.

[MR]   S. Mozgovoy, M. Reineke, *Moduli spaces of stable pairs and non-abelian zeta functions of curves via wall-crossing.* arXiv:1310.4991.

[MY]   M. Maruyama, K. Yokogawa. *Moduli of parabolic stable sheaves.* Math. Ann., 293 (1992), 77-99.

[Ne]   A. Negut, *Quantum algebras and cyclic quiver varieties.* arXiv:1504.06525.

[Ols]   M. Olsson, *Algebraic Spaces and Stacks.* A.M.S. Colloquium Publications. 2016.

[Rei]   M. Reineke, *Quiver moduli and small desingularizations of some GIT quotients.* arXiv:1511.08316.

[Ro]   M. Romagny, *Group actions on stacks and applications.* Michigan Math. J. 53 (2005), 209-236.

[Sch]   O. Schiffmann, *Indecomposable vector bundles and stable Higgs bundles over smooth projective curves.* arXiv:1406.3839.

[Sc]   J. P. Schneiders, *Quasi-abelian categories and sheaves.* Mém. Soc. Math. France 76 (1999).

[Se]   C. S. Seshadri, *Moduli of π-vector bundles over an algebraic curve.* 1970 Questions on Algebraic Varieties (C.I.M.E., III Ciclo, Varenna, 1969) pp. 139-260 Edizioni Cremonese, Rome.

[Se2]   ———— , *Quotient spaces modulo reductive algebraic groups.* Ann. of Math. (2) 95 (1972), 511-556.

[Sh]   S. Shatz, *The decomposition and specialization of algebraic families of vector bundles.* Compositio Math. 35 (1977), 163-187.

[So]   P. Sosna, *Scalar Extensions of Triangulated Categories,* Applied Categorical Structures 22 (2014), 211-227. arXiv:1109.6515

[To]   B. Toën, *Moduli of objects in dg-categories.* Ann. Sci. cole Norm. Sup. (4) 40 (2007), 387-444.

[TV]   B. Toen, M. Vaquie, Moduli of objects in dg-categoies, https://arxiv.org/pdf/math/0503269v5.pdf.

[Tr]   R. Travkin, *Mirabolic Robinson-Schensted-Knuth correspondence.* Selecta Math. (N.S.) 14 (2009), 727-758.

[Wi]   G. Wilson, *Collisions of Calogero-Moser particles and an adelic Grassmannian (with an appendix by I.G. Macdonald).* Invent. Math. 133 (1998), 1-41.

[Y]   K, Yokogawa, *Compactification of moduli of parabolic sheaves and moduli of parabolic Higgs sheaves.* J. Math. Kyoto Univ. 33 (1993), 451-504.