Centralizers of semisimple elements of a simply connected semisimple group G.

Let k be an algebraically closed field, and G a connected semisimple group over k. The centralizer of $g \in G$ will be denoted by $Z(g)$.

Reductivity of centralizers

Fact: if $g \in G$ is semisimple then $Z(g)$ is reductive.

Note that if $T \subset G$ is a maximal torus containing g then T is also a maximal torus of $Z(g)$. So the rank of $Z(g)$ equals that of G.

Connectedness of centralizers.

Fact: if G is simply connected and $g \in G$ is semisimple then $Z(g)$ is connected.

Remarks. (i) The simply-connectedness of G is essential: take $G = PGL(2)$, $g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(ii) The semisimplicity of g is also essential. In fact, it is known that if G is not isomorphic to a product $\prod_{i=1}^{m} SL(n_i)$ then G has a unipotent element whose centralizer is disconnected.

Centralizers and Levi subgroups.

Any Levi subgroup $L \subset G$ can be represented as $Z(g)$ for some semisimple $g \in G$ (take g to be a generic element of the center of L).

If $G = SL(n)$ then the centralizer of any semisimple $g \in G$ is a Levi subgroup. This is not true for a general simply connected semisimple group G:

(Assume that char $k \neq 2$). Let

Example 1. $G = Sp(2m+2n)$, $m, n > 0$, $g = \begin{pmatrix} 1_{2m} & 0 \\ 0 & -1_{2n} \end{pmatrix}$

Then $Z(g) = Sp(2m) \times Sp(2n) \subset Sp(2m+2n)$. This is not a Levi subgroup.

Example 2. Let char $k \neq 2$ and $G = Spin(m+n)$, $m, n > 0$. Assume that n is even, then $SO(n) \ni -1_n$. Let $\bar{g} := \begin{pmatrix} 1_m & 0 \\ 0 & -1_n \end{pmatrix} \in SO(m+n)$ and let $g \in Spin(m+n)$ be a preimage of $\bar{g}$. One has $Z(\bar{g}) = \{(A, B) \in O(m) \times O(n) \mid \det A = \det B\}$, so $Z(\bar{g})$ is disconnected. But $Z(g)$ has to be connected, so $Z(g)$ equals the preimage of $SO(m) \times SO(n)$ in $Spin(m+n)$. This is not a Levi subgroup unless $m = 2$ or $n = 2$.

Centralizers can be semisimple.

A Levi subgroup of G cannot be semisimple unless it equals G. On the other hand, in Example 1 the centralizer $Z(g)$ is semisimple. This is also true in Example 2 unless $m=2$ or $n=2$.

Remarks. (i) Note that if $Z(g)$ is semisimple then $Z(g)$ is not contained in any parabolic $P \subset G$ different from G.

(ii) Schur's lemma says that if $G = SL(n)$ and $\Gamma \subset G$ is a subgroup whose centralizer is bigger than the center of G then Γ is contained in a parabolic $P \subset G$, $P \neq G$. The above Remark (i) combined with Examples 1-2 show that this is not true for a general simply connected semisimple group G.

It may happen that $[Z(g), Z(g)]$ is not simply connected

If L is a Levi subgroup of a simply connected semisimple group G then $[L, L]$ is simply connected. On the other hand, in Example 2 with $m, n \neq 2$ the group $[Z(g), Z(g)] = Z(g)$ is not simply connected.

Semisimple elements $g \in G$ such that $Z(g)$ is semisimple.

From now on we assume that G is simply connected and almost simple. We also assume that char $k = 0$ (otherwise the statements would become slightly more complicated).

Let $A(G)$ be the set of conjugacy classes of semisimple elements $g \in G$ such that $Z(g)$ is semisimple. Let $B(G)$ be the set of vertices of the extended Dynkin diagram of G (so Card $B(G) = r+1$).

In 1969 Victor Kac constructed a bijection $B(G) \xrightarrow{\sim} A(G)$.

Notation: $\alpha_1, \dots, \alpha_r$ are the simple roots; $\check{\omega}_1, \dots, \check{\omega}_r$ are the fundamental coweights, i.e., $(\alpha_i, \check{\omega}_j) = \delta_{ij}$; $\alpha_{max} = \sum_{i=1}^{r} m_i \alpha_i$ is the maximal root. We choose an isomorphism $e: \mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} \{\text{roots of unity in } k^\times\}$.

<u>Construction of the map $B(G)\to A(G)$.</u> The vertex of the extended Dynkin diagram corresponding to α_{max} goes to $1\in G$. The vertex corresponding to α_i goes to

$$g_i := \check{\omega}_i\left(e\left(\frac{1}{m_i}\right)\right),$$

where $\check{\omega}_i$ is considered as a morphism $\mathbb{G}_m \to \{\text{maximal torus}\}$.

<u>Exercise.</u> $Z(g_i)$ is semisimple. The Dynkin diagram of $Z(g_i)$ is obtained from the <u>extended</u> Dynkin diagram of G by removing its i-th vertex.

<u>Theorem.</u> The above map $B(G)\to A(G)$ is bijective.

This theorem was proved by V. Kac as a part of his classification of elements of finite order in G up to conjugacy; see §3,6 of ch. 3 of the book:

V.V. Gorbatsevich, A.L. Onishchik and E.B. Vinberg, Structure of Lie groups and Lie algebras. Lie groups and Lie algebras, III. Encyclopaedia of Mathematical Sciences, 41. Springer-Verlag, Berlin, 1994.

<u>Consistency check.</u> Let $G=SL(n)$. Then $Z(g)$ is semisimple only if $g\in\{\text{center of } G\}$. So Card $A(G)=n$. On the other hand, the extended Dynkin diagram has n vertices.

Surprise no. 1. For each $n \in \mathbb{Z}$ one has the map $f_n : A(G) \to A(G)$ defined by $f_n(g) = g^n$. It is hard to imagine the corresponding map $B(G) \to B(G)$.

Surprise no. 2. Instead of considering $\check{\omega}_i(e(\frac{1}{m_i}))$ one could consider $g = \check{\omega}_i(e(\alpha))$, $\alpha \in \mathbb{Q}/\mathbb{Z}$. It is easy to show that if the denominator of α is $\leq m_i$ then $Z(g)$ is semisimple, so g defines an element of $A(G)$. It is hard to see why it comes from an element of $B(G)$.

Nevertheless, the theorem is true!

Sketch of the proof. Note that g belongs to the center of $Z(g)$, so if $Z(g)$ is semisimple then g has finite order. We have

$$\left\{\begin{array}{l}\text{Elements of finite}\\ \text{order in } G\end{array}\right\}/\text{conjugation} = \left\{\begin{array}{l}\text{Elements of}\\ \text{finite order in } T\end{array}\right\}/W =$$

$= (C \otimes \mathbb{Q}/\mathbb{Z})/W$, where C is the group generated by the coroots $\check{\alpha}_i$ (we have used the isomorphism $e: \mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} \{\text{roots of unity in } k^\times\}$). Now $(C \otimes \mathbb{Q}/\mathbb{Z})/W = (C \otimes \mathbb{Q})/W_{aff}$, where W_{aff} is the affine Weyl group. The theory of W_{aff} identifies $(C \otimes \mathbb{Q})/W_{aff}$ with the simplex

$$\Delta := \{x \in C \otimes \mathbb{Q} \mid (x, \alpha_{max}) \leq 1,\ (x, \alpha_i) \geq 0 \text{ for } 1 \leq i \leq r\}.$$

Finally, one checks that if $x \in \Delta$ and g_x is the corresponding element of G then $Z(g_x)$ is semisimple if and only if x is a vertex of Δ.