Tate - Nakayama duality

Let F be a local non-archimedian field. Let T be a torus over F and $X^*(T)$ its group of characters. $Gal(\bar{F}/F)$ acts on $X^*(T)$. The obvious pairing $T(\bar{F}) \times X^*(T) \to \bar{F}^\times$ induces a pairing $H^i(F,T) \times H^{2-i}(F, X^*(T)) \to H^2(F, \mathbb{G}_m) = Br(F) = \mathbb{Q}/\mathbb{Z}$.

<u>Theorem.</u> (i) This pairing is nondegenerate

(ii) $H^1(F,T)$ and $H^1(F, X^*(T))$ are finite.

What about H^0 and H^2? (It is known that F has cohomological dimension 2). Notation: $A^i := H^i(F, T)$, $B^{2-i} := H^{2-i}(F, X^*(T))$.

<u>Example</u>: $T = \mathbb{G}_m$

	$A^i := H^i(F, \mathbb{G}_m)$	$B^{2-i} := H^{2-i}(F, \mathbb{Z})$
$i=0$	$F^\times$	$Hom_{cont}(Gal(\bar{F}/F), \mathbb{Q}/\mathbb{Z}) = Hom_{cont}(F^\times, \mathbb{Q}/\mathbb{Z})$
$i=1$	0	0
$i=2$	$\mathbb{Q}/\mathbb{Z}$	$\mathbb{Z}$

<u>Claim</u>. The pairing $A^i \times B^{2-i} \to \mathbb{Q}/\mathbb{Z}$ induces isomorphisms $A^2 \xrightarrow{\sim} Hom(B^0, \mathbb{Q}/\mathbb{Z})$ and $B^2 \xrightarrow{\sim} Hom_{cont}(A^0, \mathbb{Q}/\mathbb{Z})$, where $A^0 = T(F)$ is equipped with the topology that comes from the topology on F. As said before, A^1 and B^1 are finite and dual to each other.

Proofs can be found in

J.P. Serre, Galois Cohomology, ch. II, §5.8

S. Shatz, Profinite groups, arithmetic, and geometry, ch. VI, §5.

At least, the statement about $H^1(F,T)$ is an exercise (once you believe that $Br(F) = \mathbb{Q}/\mathbb{Z}$).

Kottwitz' interpretation

(§§2,3 of his article in Duke Math. J., vol. 51 (1984), no. 3).

First, forget about the local field F. Let k be any algebraically closed field of characteristic 0. Let $\check{T}$ be a torus over k and $X_*(\check{T}) = \mathrm{Hom}(\mathbb{G}_m, \check{T})$ the group of cocharacters.

Kottwitz' lemma. $\mathrm{Hom}(H^1(\Gamma, X_*(\check{T})), \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\pi_0(\check{T}^\Gamma), k^\times)$.

Tate twist

Reformulation. $\pi_0(\check{T}^\Gamma) = H^1(\Gamma, X_*(\check{T}))(1)$ (most people take $k = \mathbb{C}$ and skip the Tate twist).

Combining the lemma with Tate-Nakayama duality Kottwitz immediately gets:

Corollary. Let F be a local nonarchimedian field and T a torus over F. Let $\check{T}$ be the dual torus over k, i.e., $X^*(\check{T}) = X_*(T)$. Let $\check{T}^\Gamma$ be the invariants of $\Gamma := \mathrm{Gal}(\bar{F}/F)$ acting on $\check{T}$. Then $H^1(F, T) = \mathrm{Hom}(\pi_0(\check{T}^\Gamma), k^\times)$.

(Note that k does not appear in the l.h.s. of this equality while in the r.h.s. it appears twice because $\check{T}$ depends on k).

Proof of Kottwitz' lemma for $k = \mathbb{C}$. We have to prove that $\pi_0(T^\Gamma) = H^1(\Gamma, X_*(T))$. One has $\pi_0(T^\Gamma) = \mathrm{Coker}(\mathrm{Lie}\, T^\Gamma \xrightarrow{\exp} T^\Gamma)$, $\mathrm{Lie}\, T^\Gamma = (\mathrm{Lie}\, T)^\Gamma = (X_*(T) \otimes \mathbb{C})^\Gamma$, $T^\Gamma = (X_*(T) \otimes \mathbb{C}/\mathbb{Z})^\Gamma$. Now use the exact sequence $(X_*(T) \otimes \mathbb{C})^\Gamma \to (X_*(T) \otimes \mathbb{C}/\mathbb{Z})^\Gamma \to$

$\to H^1(\Gamma, X_*(T)) \to H^1(\Gamma, X_*(T)\otimes \mathbb{C}) = 0$
(we have used the vanishing of H^i of a profinite group acting on a $\mathbb{Q}$-vector space, $i>0$).

<u>A variant of the proof which makes sense for any k.</u>

torsion in A_Γ

Notation: $A := X^*(T)$.

Then $X^*(T^\Gamma) = A_\Gamma$, $\mathrm{Hom}(\pi_0(T^\Gamma), k^*) = (A_\Gamma)_{tors}$

On the other hand, let $A^* := \mathrm{Hom}(A, \mathbb{Z}) = X_*(T)$, then $H^1(\Gamma, A^*) =$
$= \mathrm{Ker}\,(H^1(\Gamma, A^*) \to H^1(\Gamma, A^*\otimes \mathbb{Q})) =$
$= \mathrm{Coker}\,((A^*\otimes \mathbb{Q})^\Gamma \longrightarrow (A^*\otimes \mathbb{Q}/\mathbb{Z})^\Gamma)$.

But $(A^*\otimes \mathbb{Q})^\Gamma = \mathrm{Hom}(A_\Gamma, \mathbb{Q})$ and
$(A^*\otimes \mathbb{Q}/\mathbb{Z})^\Gamma = \mathrm{Hom}(A_\Gamma, \mathbb{Q}/\mathbb{Z})$, so
$H^1(\Gamma, A^*) = \mathrm{Ext}\,(A_\Gamma, \mathbb{Z}) = \mathrm{Hom}((A_\Gamma)_{tors}, \mathbb{Q}/\mathbb{Z})$.

<u>Remark</u>. The last part of the proof was devoted to showing that $(H^1(\Gamma, A^*))_{tors} = \mathrm{Hom}((A_\Gamma)_{tors}, \mathbb{Q}/\mathbb{Z})$. In fact, this is a particular case of the universal coefficients formula.