Appendix: the case of a surface

A1. Below R is a commutative coefficient ring. Denote by $\mathcal{L}_R$ the Picard groupoid of $\mathbb{Z}$-graded super R-lines. So an object of $\mathcal{L}_R$ is a pair $(L, \deg_L)$, where L is an invertible R-module, $\deg_L \in \mathbb{Z}^{\mathrm{Spec} R}$:= the group of $\mathbb{Z}$-valued locally constant function on $\mathrm{Spec}\, R$; the operation on $\mathcal{L}_R$ is $(L, \deg_L) \otimes (L', \deg_{L'}) := (L \otimes L', \deg_L + \deg_{L'})$ with the commutativity corrected by the super sign $(-1)^{\deg_L \deg_{L'}}$.

Our X is a compact real-analytic surface with boundary Y whose connected components are denoted by Y_α, $\alpha \in A$. Let F be a complex of sheaves of R-modules on X whose fibers are perfect and the restriction of its cohomology to $X \smallsetminus Y$, and to Y minus finitely many points, is locally constant. Then $R\Gamma(X, F)$ is a perfect R-complex, so we have $\det R\Gamma(X, F) \in \mathcal{L}_R$. Let ν be any continuous nowhere vanishing 1-form on $X \smallsetminus Y$. Our aim is to assign to each component Y_α a graded super R-line $\mathcal{E}(F)_{\nu Y_\alpha}$ which has local origin, i.e., depends only on the restriction of our datum to a neighborhood of Y_α, and define the ε-factorization isomorphism

$$\otimes_\alpha \mathcal{E}(F)_{\nu Y_\alpha} \xrightarrow{\sim} \det R\Gamma(X, F). \qquad \text{(A1.1)}$$

The constructions are presented in A6; they are based on a theorem from A5.

A2. *A shot of abstract nonsense.* Let $\mathcal{A}$ be a Boolean algebra. Recall that it is realized canonically as the Boolean algebra of open compact subsets of a pro-finite set $\mathcal{P} = \mathrm{Spec}\, \mathcal{A}$.[1]

Let $\mathcal{L}$ be a Picard groupoid;[2] we write the operation in $\mathcal{L}$ as $\otimes$. The group of isomorphism classes of objects in $\mathcal{L}$ is denoted by $\pi_0\mathcal{L}$, the group of automorphisms of the unit object $1_\mathcal{L}$ is denoted by $\pi_1\mathcal{L}$; these groups are commutative (and written multiplicatively). For any $L \in \mathcal{L}$ one has a canonical identification $\pi_1 L \xrightarrow{\sim} \mathrm{Aut} L$, $\phi \mapsto id_L \otimes \phi$. There is a natural homomorphism $\epsilon : \pi_0\mathcal{L} \to \pi_1\mathcal{L}$ which sends an object $L \in \mathcal{L}$ to the symmetric constraint symmetry of $L \otimes L$; L is said to be *even* if $\epsilon(L) = 1$ and *odd* otherwise; let $\mathcal{L}^{ev} \subset \mathcal{L}$ be the Picard subgroupoid of even objects. $\mathcal{L}$ is said to be discrete if its $\pi_1\mathcal{L}$ is trivial; such an $\mathcal{L}$ amounts to an abelian group. For any $\mathcal{L}$ there is an evident morphism of Picard groupoids $\mathcal{L} \to \pi_0\mathcal{L}$.

(a) An $\mathcal{L}$-*measure* (λ, m) on $\mathcal{A}$ (or on $\mathcal{P}$) is a rule that assigns to every $Q \in \mathcal{A}$ an object $\lambda(Q) \in \mathcal{L}$, and to every finite set $\{Q_i\} \subset \mathcal{A}$ such that $Q_i \cap Q_{i'} = \emptyset$ for $i \neq i'$, an identification $m : \otimes\lambda(Q_i) \xrightarrow{\sim} \lambda(\cup Q_i)$. One demands m to be transitive in the obvious sense. We often drop m from the notation.

$\mathcal{L}$-valued measures form a Picard groupoid which we denote by $\mathcal{M}eas(\mathcal{A}, \mathcal{L})$ or $\mathcal{M}eas(\mathcal{P}, \mathcal{L})$. It is functorial with respect to morphisms of $\mathcal{P}$'s and $\mathcal{L}$'s. Notice that $\pi_1\mathcal{M}eas(\mathcal{P}, \mathcal{L}) = \mathcal{M}eas(\mathcal{P}, \pi_1\mathcal{L})$ (the usual group of $\pi_1\mathcal{L}$-valued measures on $\mathcal{P}$); the projection $\mathcal{L} \to \pi_0\mathcal{L}$ yields a homomorphism $\pi_0\mathcal{M}eas(\mathcal{P}, \mathcal{L}) \to \mathcal{M}eas(\mathcal{P}, \pi_0\mathcal{L})$ which is bijective if $\mathcal{A}$ is countable or if $\pi_1\mathcal{L}$ is finite.

Exercise. Let $\mathbb{Z}^\mathcal{P}$ be the group of $\mathbb{Z}$-valued locally constant functions on $\mathcal{P}$; for $Q \in \mathcal{A}$ let $1_Q \in \mathbb{Z}^\mathcal{P}$ be the characteristic function of $Q \subset \mathcal{P}$. Consider the Picard

[1] Here $\mathcal{A}$ is considered as a commutative $\mathbb{Z}/2$-algebra with the operations $QQ' := Q \cap Q'$, $Q + Q' := (Q \cup Q') \smallsetminus (Q \cap Q')$.

[2] I.e., a symmetric monoidal category all of whose objects and morphisms are invertible.

groupoid $\mathrm{Hom}(\mathbb{Z}^{\mathcal{P}},\mathcal{L})$ of Picard groupoid morphisms $\phi:\mathbb{Z}^{\mathcal{P}}\to\mathcal{L}$. There is a fully faithful embedding of Picard groupoids $\mathrm{Hom}(\mathbb{Z}^{\mathcal{P}},\mathcal{L})\hookrightarrow\mathcal{M}eas(\mathcal{P},\mathcal{L})$, $\phi\mapsto\lambda_\phi$, where $\lambda_\phi(Q)=\phi(1_Q)$ and m comes since $1_{\sqcup Q_i}=\Sigma 1_{Q_i}$. Show that its essential image equals $\mathcal{M}eas(\mathcal{P},\mathcal{L})^{ev}=\mathcal{M}eas(\mathcal{P},\mathcal{L}^{ev})$, i.e., in the $\mathcal{L}$-setting an outright integration of functions makes sense only for even λ's.

Remark. Suppose that we write $\mathcal{P}$ as the projective limit of a directed family of finite sets $\mathcal{P}_\alpha$ and surjections $\pi_{\alpha\alpha'}:\mathcal{P}_{\alpha'}\twoheadrightarrow\mathcal{P}_\alpha$, $\alpha'\geq\alpha$; so for every $b\in\mathcal{P}_\alpha$ we have an open $Q_{\alpha b}\subset\mathcal{P}$. An $\mathcal{L}$-measure λ is the same as a datum of objects $\lambda_\alpha(b)=\lambda(Q_{\alpha b})\in\mathcal{L}$, $a\in\mathcal{P}_\alpha$, together with identifications $m_{\alpha\alpha' b}:\otimes\lambda_{\alpha'}(b')\overset{\sim}{\to}\lambda_\alpha(b)$, where $b\in\mathcal{P}_\alpha$ and b' run the set of all elements of $\mathcal{P}_{\alpha'}$ such that $\pi_{\alpha\alpha'}(b')=b$; the $m_{\alpha\alpha'}$'s should satisfy the transitivity property.

(b) *Inclusion-exclusion formula:* Take $Q\in\mathcal{A}$ and a finite collection $\{Q_\beta\}$, $\beta\in B$, such that $Q=\cup Q_\beta$. For a non-empty subset $S\subset B$ set $Q_S:=\underset{\beta\in S}{\cap}Q_\beta$.

Lemma. *If all $\lambda(Q_S)$ are even, then one has a canonical isomorphism*

$$\mu_Q^{\{Q_\beta\}}:\underset{\emptyset\neq S\subset B}{\otimes}\lambda(Q_S)^{\otimes(-1)^{|S|+1}}\overset{\sim}{\to}\lambda(Q).\tag{A2.1}$$

Proof. For a non-empty $S\subset B$ let $Q_{(S)}$ be the complement in Q_S to the union of $Q_{S'}$ where $S'\supset S$, $S'\neq S$. Then Q is the disjoint union of all $Q_{(S)}$'s, $S\subset B$. Intersecting our datum with $Q_{(S)}$, we are reduced to the situation where all Q_β's are equal. Here (A2.1) is immediate. □

Remark. Let $Q=\cup Q_\gamma$, $\gamma\in\Gamma$, be another presentation of Q such that $\lambda(Q_T)\in\mathcal{L}^{ev}$ for every non-empty $T\subset\Gamma$. Let us compare $\mu_B:=\mu_Q^{\{Q_\beta\}}$ and $\mu_\Gamma:=\mu_Q^{\{Q_\gamma\}}$. Denote by L_B, L_Γ their sources, and set $L_{B\Gamma}:=\otimes\lambda(Q_S\cap Q_T)^{(-1)^{(|S|+1)(|T|+1)}}$, the tensor product is indexed by all non-empty $S\subset B$, $T\subset\Gamma$. There is a natural morphism $\nu_B:L_{B\Gamma}\overset{\sim}{\to}L_B$ defined as the tensor product of morphisms $(\mu_{Q_S}^{\{Q_\gamma\cap Q_S\}})^{\otimes(-1)^{|S|+1}}$ for $\emptyset\neq S\subset B$, and a similarly defined morphism $\nu_\Gamma:L_{B\Gamma}\overset{\sim}{\to}L_\Gamma$. Now one has $\mu_B\nu_B=\mu_\Gamma\nu_\Gamma$; to prove this, one reduces the statement to the case when all Q_β and Q_γ coincide, where the statement is obvious.

(c) Let $\mathcal{I}\subset\mathcal{A}$ be a Boolean ideal, and $\mathcal{A}/\mathcal{I}$ the quotient Boolean algebra, so $\mathrm{Spec}\mathcal{A}/\mathcal{I}=:\mathcal{P}'$ is a closed subset in $\mathcal{P}$, and $\mathcal{I}$ consists of open compact subsets of $\mathcal{P}\smallsetminus\mathcal{P}'$. Let $\mathcal{M}eas(\mathcal{A},\mathcal{L})^{\mathcal{I}}$ be the Picard groupoid of pairs (λ,ι) where λ is an $\mathcal{L}$-valued measure on $\mathcal{A}$ and ι is a trivialization of the restriction $\lambda|_{\mathcal{I}}$ of λ to $\mathcal{I}$.[3] So ι is a datum of identifications $\iota(Q):\lambda(Q)\overset{\sim}{\to}1_{\mathcal{L}}$, $Q\in\mathcal{I}$, multiplicative with respect to disjoint union decompositions of Q's. The pull-back functor yields an equivalence of the Picard groupoids $\mathcal{M}eas(\mathcal{A}/\mathcal{I},\mathcal{L})\overset{\sim}{\to}\mathcal{M}eas(\mathcal{A},\mathcal{L})^{\mathcal{I}}$.

Take any (λ,ι) as above, and let Q, $\{Q_b\}$ be a datum as in (b) such that $Q,\{Q_b\}$ lie in $\mathcal{I}$. Since λ is trivial on $\mathcal{I}$, the lemma in (b) is applicable, so we have isomorphism (A2.1). Now ι trivializes both objects in (A2.1), and one has

$$\underset{S\subset B}{\otimes}\iota(Q_S)^{\otimes(-1)^{|S|+1}}=\iota(Q).\tag{A2.2}$$

[3] In the definition of $\mathcal{L}$-measure we need not assume that the Boolean algebra is unital, so one has the Picard groupoid $\mathcal{M}eas(\mathcal{I},\mathcal{L})$ of $\mathcal{L}$-measures on $\mathcal{P}\smallsetminus\mathcal{P}'$, and ι is an identification of $\lambda|_{\mathcal{I}}$ with the trivial object of this Picard groupoid.

(d) Let $\mathcal{I}$ be as in (c), and $\mathcal{D} \subset \mathcal{I}$ be a subset closed under $\cap$ such that every $Q \in \mathcal{I}$ can be represented as $\cup Q_\beta$ with $\{Q_\beta\} \subset \mathcal{D}$.

Lemma. *Let λ be any $\mathcal{L}$-measure on $\mathcal{A}$ and ι be any datum of trivializations $\iota(Q) : \lambda(Q) \xrightarrow{\sim} 1_{\mathcal{L}}$ defined for $Q \in \mathcal{D}$. If (A2.2) holds whenever $Q, \{Q_b\}$ are in $\mathcal{D}$, then ι extends in a unique manner to a trivialization ι of $\lambda|_{\mathcal{I}}$.*

Proof. It is clear that $\lambda|_{\mathcal{I}}$ takes values in trivial objects of $\mathcal{I}$, so (A2.1) makes sense there. Take any $Q \in \mathcal{I}$ and represent it as $\cup Q_\beta$, $\beta \in B$, for $\{Q_\beta\} \subset \mathcal{D}$. Let $\iota^{\{Q_\beta\}}(Q)$ be a trivialization of $\lambda(Q)$ defined, using (A2.1), as $\underset{S \subset B}{\otimes} \iota(Q_S)^{\otimes(-1)^{|S|+1}}$. The comparison picture from Remark in (b) shows that $\iota^{\{Q_\beta\}}(Q)$ does not depend on the choice of particular $\{Q_\beta\}$. Set $\iota(Q) = \iota^{\{Q_\beta\}}(Q)$; it is immediate that ι is multiplicative, and we are done. □

A3. We return to the setting of A1. Below a *curve* is a subset $C \subset X$ whose closure $\bar{C}$ is a semi-analytic curve, $\bar{C} \smallsetminus C$ is finite, and $\bar{C} \cap Y = \emptyset$. A *stratification* of X is always assumed to be semi-analytic (its 1-strata are curves) and such that no 0 strata lie on Y (i.e., each Y_α lies in an open stratum). A *constructible set* in X is a union of strata of a stratification. Denote by $\mathcal{C}$ the Boolean algebra of constructible sets.

For every *locally closed* constructible subset $Q \subset X$, we have a perfect complex $R\Gamma_c(Q, F) := R\Gamma_c(Q, i_Q^* F)$, hence a graded super R-line $\det R\Gamma_c(Q, F)$ of the degree equal to the Euler characteristics $\chi(Q, F)$. If $Q_1 \hookrightarrow Q$ is an open embedding where Q_1 is also constructible, and $Q_2 := Q \smallsetminus Q_1$, then the exact triangle $R\Gamma_c(Q_1, F) \to R\Gamma_c(Q, F) \to R\Gamma_c(Q_2, F)$ yields an isomorphism

$$\text{(A3.1)} \qquad \det R\Gamma_c(Q_1, F) \otimes \det R\Gamma_c(Q_2, F) \xrightarrow{\sim} \det R\Gamma_c(Q, F).$$

Proposition. *There is an $\mathcal{L}_R$-measure λ_F on $\mathcal{C}$ together with identifications $\tau = \tau_Q : \lambda_F(Q) \xrightarrow{\sim} \det R\Gamma_c(Q, F)$ for each locally closed constructible $Q \subset X$, such that for every Q, Q_1, Q_2 as above, the τ identify the isomorphism from (A3.1) with the structure isomorphism $m : \lambda_F(Q_1) \otimes \lambda_F(Q_2) \xrightarrow{\sim} \lambda_F(Q)$. The datum (λ_F, τ) is unique up to a unique isomorphism.*

Proof. Use Remark in A2(a): Our α's run the set of all constructible stratifications with its usual ordering, $\mathcal{P}_\alpha$ is the set of strata of the stratification. Then τ specifies each line $\lambda_\alpha(b)$, $b \in \mathcal{P}_\alpha$, and (A3.1) defines the datum of $m_{\alpha\alpha'}$. The details are left to the reader. □

Remarks. (i) If Q is a constructible subset which is *not* locally closed, then Q is not locally compact, so $R\Gamma_c(Q, F)$ is not defined.

(ii) λ_F has local origin: For an open $U \subset X$, let $\mathcal{C}(U) \subset \mathcal{C}$ be the Boolean ideal of constructible Q's such that $\bar{Q} \subset U$. Then the restriction of λ_F to $\mathcal{C}(U)$ depends only on $F|_U$.

A4. Now let us switch in ν. We need an auxiliary datum of a *ν-cone* N, which is a continuous family of non-degenerate closed sectors $N_x \subset T_x X$, $x \in X \smallsetminus Y$, such that $\langle \nu_x, N_x \smallsetminus \{0\} \rangle < 0$.

A curve C is said to be N*-transversal* if for every $x \in \bar{C}$ each tangent line to $\bar{C}$ at x intersects $N_x \cup -N_x$ by $\{0\}$. A stratification is N*-transversal* if such are its 1-strata. A constructible set Q is said to be:

- N*-transversal,* if it is a union of strata in an N-constructible stratification;

- N*-special,* if it is N-transversal and a point $x \in \partial Q$ lies in Q if and only there are points $y \in \text{Int }(Q)$ close to x and such that $y - x \in N_x$ (in the evident sense);

- N*-lens,* if Q is N-special, locally closed, and $\bar{Q}$ is homeomorphic to a disc.

Let $\tilde{\mathcal{C}}^N \supset \mathcal{C}^N \supset \mathcal{C}^N_0$ be the subsets of $\mathcal{C}$ that consist of those Q that are, respectively, N-transversal, N-special, N-special and satisfisfy $\bar{Q} \cap Y = \emptyset$. Let $\mathcal{J}^N \subset \mathcal{C}$ be the subset of N-transversal Q's of dimension ≤ 1.

Proposition. *(i)* $\tilde{\mathcal{C}}^N$, $\mathcal{C}^N$ *are Boolean subalgebras of* $\mathcal{C}$, $\mathcal{J}^N$ *is a Boolean ideal in* $\mathcal{C}$, *and* $\mathcal{C}^N_0$ *is a Boolean ideal in* $\mathcal{C}^N$. *Every* N*-lens lies in* $\mathcal{C}^N_0$.

(ii) If Q *is* N*-special, then Int* (Q) *equals Int* $(\bar{Q})$, *and it is dense in* Q.

(iii) The composition $\mathcal{C}^N \hookrightarrow \tilde{\mathcal{C}}^N \twoheadrightarrow \tilde{\mathcal{C}}^N/\mathcal{J}^N$ *is a bijection.*

Denote the inverse Boolean algebras isomorphism $\tilde{\mathcal{C}}^N/\mathcal{J}^N \xrightarrow{\sim} \mathcal{C}^N$ *by* $V \mapsto V^+$.

(iv) Each point in $X \smallsetminus Y$ *admits a base of neighborhoods formed by* N*-lenses.*

(v) Any $Q \in \mathcal{C}^N_0$ *can be written as a union of finitely many* N*-lenses.*

(vi) If Q *is an* N*-lens, then* $\bar{Q} \smallsetminus Q$ *is a closed interval in the circle* $\partial\bar{Q}$.

(vii) Suppose that $P \in \mathcal{C}^N_0$ *is locally closed and is contained in an* N*-lens. Then each connected component of* P *is an* N*-lens. In particular, all the connected components of an intersection of finitely many* N*-lenses are* N*-lenses.*

Proof. (i), (ii), (iii) are straightforward.

(iv) Take any $a \in X \smallsetminus Y$; choose a real analytic local coordinate system (x, y) at a such that $a = (0, 0)$ and $\nu_a = dy$. For $R, \delta > 0$ let $U_{R\delta}$ be the intersection of two open discs of radius R centered at $(0, \pm(R - \delta))$. If R is very big and δ is very small, then $U^+_{R\delta}$ (which is the intersection of the open disc centered at $(0, R - \delta)$ and the closed one centered at $(0, \delta - R)$) is an N-lens in X. These N-lenses form a base of neighborhoods of a.

(v) We need a preliminary. Let a be a point in $X \smallsetminus Y$ and C be the germ at a of an N-transversal curve. Let (x, y) be coordinates as in (iv). By the implicit function theorem, one has two finite sets of functions: $\{y = g_1(x), \ldots, y = g_k(x)\}$ defined for $\epsilon > x \geq 0$, and $\{y = h_1(x), \ldots, y = h_r(x)\}$ defined for $-\epsilon < x \leq 0$, all vanishing at $x = 0$, such that C is the union of their graphs. Our C is semi-analytic, so for small enough ϵ all g_i, h_j are monotone and one can order them so that $g_1(x) < \ldots < g_2(x)$ for $\epsilon > x > 0$, and $h_1(x) < \ldots < h_r(x)$ for $-\epsilon < x < 0$. Choose R, δ as in (iv) such that $U^+_{R\delta}$ is an N-lens that lies in the interval $-\epsilon < x < \epsilon$. Suppose C meets both half-planes $x > 0$, $x < 0$, i.e., both sets $\{g_i\}$ and $\{h_j\}$ are non-empty. Then C cuts $U_{R\delta}$ into pieces $\{U_k\}$ such that each U^+_k is an N-lens, and $\cup U^+_k = U^+_{R\delta}$.

Let us return to the proof of (v). Our assertion is local: it suffices to find for any

$a \in \bar{Q}$ its neighborhood whose intersection with Q can be represented as a union of N-lenses. The case $a \in \operatorname{Int}(Q)$ is covered by (iv). For $a \in \partial Q$, choose (x, y) as above; let C be a curve defined as the germ of ∂Q at a if ∂Q intersects both half-planes $x > 0$, $x < 0$; if not, C is the union of ∂Q and the line $y = 0$. Then, by (iii), $Q \cap U^+_{R\delta}$ is the union of those of the above N-lenses U^+_k that meet Q, q.e.d.

(vi) Set $I := \bar{Q} \setminus Q$. Since Q is locally closed, I is a union of finitely many closed intervals and points in the circle $\partial Q = \partial \bar{Q}$ (see (ii)).

Take any $a \in \partial Q$, and choose local coordinates (x, y) as in (v). The curve ∂Q has one branch at a, so, by (v), it is either the graph of a continuous function $y = k(x)$ defined for $-\epsilon < x < \epsilon$, or it lies in the half-plane $x \geq 0$, where it is the union pieces $y = g_1(x)$, $y = g_2(x)$, or it lies in the half-plane $x \leq 0$, where it is is the union of pieces $y = h_1(x)$, $y = h_2(x)$. In the first case Q equals either of the domains $y \leq k(x)$, or $y > k(x)$. In the second case, the assumption that Q is locally closed implies that Q equals the domain $g_1(x) < y \leq g_2(x)$; similarly, in the third case Q equals the domain $h_1(x) < y \leq h_2(x)$.

This shows, in particular, that I does not contain isolated points. In a moment we will define a continuous retraction $\pi : \bar{Q} \to I$. Its existence implies that I is connected and $\neq \partial Q$, hence it is a single interval, and we are done.

Choose a non-vanishing smooth vector field τ on a neighborhood of $\bar{Q}$ which takes values in N. For $x \in \bar{Q}$ follow the integral line $x(t)$, $x(0) = x$, of τ. Let us show that the trajectory $x(t)$ meets I at certain $t \geq 0$. If not, then $x \notin I$ and the trajectory $x(t)$, $t > 0$, stays in $\operatorname{Int}(Q)$. Our τ does not vanish, so, by the Poincaré-Bendixon theorem (see e.g. [KH]),[4] $\operatorname{Int}(Q)$ contains a periodic trajectory T of τ. Then T is the boundary of a disc D in $\operatorname{Int}(Q)$, and $\tau|_D$ is a non-vanishing vector field tangent to T, which does not exist; contradiction.

Take a smallest $t \geq 0$ such that $x(t) \in I$ (then $x((0, 1) \subset \operatorname{Int}(Q))$, and set $\pi(x) := x(t)$. The above picture of Q near the boundary shows that π is a continuous retraction onto I.

(vii) Every connected component of P is N-special, so we can assume that P is connected. We need to show that $\bar{P}$ is homeomorphic to a disc.

Let Q be an N-lens that contains P. Consider a retraction $\pi : \bar{Q} \to I$ from (vi). By construction, $\pi(Q)$ is the interior I° of I, and for every $t \in I^\circ$ the fiber $\bar{Q}_t$ of π is a closed interval. We orient it so that $\nu|_{\bar{Q}_t}$ is positive.

Set $K := \pi(\bar{P})$; this is a closed interval since $\bar{P}$ is connected and P is N-special. Let t be any interior point of K, so $\bar{P}_t := \bar{P} \cap \bar{Q}_t$ is a union of finitely many intervals and points in $\bar{Q}_t$. Let us show that

(∗) *The interior of $\bar{P}_t$ (in $\bar{Q}_t$) lies in the interior of P.*

If not, take any $y \in \operatorname{Int}(\bar{P}_t) \cap \partial P$. Since P is N-special, $\partial P = \partial \bar{P}$ does not contain isolated points and $\bar{P}_t \cap \partial P$ is finite. Thus a punctured neighborhood of y in $\bar{P}_t$ lies in $\operatorname{Int}(P)$, so $y \in P$ since P is N-special. Let $U \subset \operatorname{Int}(P)$ be any open subset with connected fibers such that y is the bottom point of the fiber U_t. For $t' \in I$, $t' \neq t$, let $s(t')$ be the bottom point of the connected component of $\bar{P}_{t'}$ that

[4] I am grateful to Benson Farb for comments and the reference.

contains $U_{t'}$. These points form a subset $S \subset \partial P$. Since P is N-special, one has $S \subset \partial P \setminus P$. From $y \in \partial P$ it follows easily that $y \in \bar{S}$. Since P is locally closed, this contradicts the fact that $y \in P$, and we are done.

Now let C_t be any connected component of $\bar{P}_t$ for a generic $t \in K$. This is an interval (since P is N-invariant). Let us move t, say, to the left. By $(*)$, our component changes continuously until it degenerates into a point. Same happens when we move t to the right. Notice that $C_t \setminus P$ is the bottom point of C_t, so the two points, in which C_t degenerate, do not lie in P as well, since P is locally closed. Thus $C_t \cap P$ sweep a connected component of P, so the whole P, and $\bar{P}$ is homeomorphic to a disc as stated. □

A5. If Q is an N-lens, then, by A4(vi), one has $R\Gamma_c(Q, F) = 0$. Denote by $\iota(Q)$ the corresponding trivialization of $\lambda_F(Q) = \det R\Gamma_c(Q, F)$.

Theorem. *The restriction of λ_F to $\mathcal{C}_0^N$ admits a unique trivialization ι^N such that for any N-lens Q one has $\iota^N(Q) = \iota(1_Q)$.*

Proof. The uniqueness of ι follows from A4(v)(vii). Let $\mathcal{D}$ be the subset of $\mathcal{C}_0^N$ whose elements are those Q that are locally closed and whose connected components are N-lenses. As above, for such a Q one has $R\Gamma_c(Q, F) = 0$, hence $\lambda_F(Q) = \det R\Gamma_c(Q, F)$ has a natural trivialization $\iota(Q)$. By A4(v)(vii), $\mathcal{D}$ satisfies the assumptions of A2(d) with $\mathcal{I} = \mathcal{C}_0^N$. Therefore the theorem follows from the lemma in A2(d) and the next statement:

Lemma. *Let Q be an N-lens and $\{Q_\beta\}$, $\beta \in B$, be a finite set of N-lenses such that $\cup Q_\beta = Q$. Then the trivializations $\iota(Q)$ and $\iota(Q_S)$, $\emptyset \neq S \subset B$, satisfy (A2.2).*

Proof. (a) To write (A2.1) explicitly, we choose an N-transversal stratification $\{K_r\}$ such that if $K_r \cap Q_\beta \neq \emptyset$ for some r, β, then $K_r \subset Q_\beta$. For each non-empty $S \subset B$ denote by $\lambda_F^{(S)}$ the tensor product $\otimes \det R\Gamma_c(K_r, F)$ with respect to all r such that $K_r \subset Q_\beta$ if and only if $\beta \in S$. Then $\det R\Gamma_c(Q, F) = \underset{S}{\otimes} \lambda_F^{(S)}$ and $\det R\Gamma_c(Q_S, F) = \underset{S' \supset S}{\otimes} \lambda_F^{(S')}$. Excluding $\lambda_F^{(S)}$'s from the equations, we get the isomorphism $\mu_Q^{\{Q_\beta\}} : \underset{S}{\otimes} \det R\Gamma_c(Q_S, F)^{\otimes(-1)^{|S|}} \xrightarrow{\sim} \det R\Gamma_c(Q, F)$.

(b) We want to check that $\mu_Q^{\{Q_\beta\}}$ is compatible with the trivializations $\iota(Q)$ and $\iota(Q_S)$. To do this, we will find a finite filtration $\emptyset = E_{-1} \subset E_0 \subset \ldots \subset E_n = Q$, where E_i are closed subsets of Q, such that for every i one has (here $P_i := E_i \setminus E_{i-1}$):

(i) For any N-transversal locally closed K the complex $R\Gamma_c(K \cap P_i, F)$ is perfect;

(ii) If $P_i \cap Q_\beta \neq \emptyset$, then $P_i \subset Q_\beta$;

(iii) One has $R\Gamma_c(P_i, F) = 0$.

Such a filtration yields usual factorizations $\det R\Gamma_c(Q, F) = \otimes \det R\Gamma_c(P_i, F)$, $\det R\Gamma_c(Q_S, F) = \otimes \det R\Gamma_c(Q_S \cap P_i, F)$. Now $R\Gamma_c(P_i, F)$, $R\Gamma_c(Q_S \cap P_i, F)$ are acyclic complexes by (iii) and (ii), so we have the corresponding trivializations $\iota(P_i)$, $\iota(Q_S \cap P_i)$ of their determinant lines; it is clear that $\iota(Q) = \otimes \iota(P_i)$ and $\iota(Q_S) = \otimes \iota(Q_S \cap P_i)$. Our $\mu_Q^{\{Q_\beta\}}$ equals the tensor product of the similarly defined isomorphisms $\mu_{P_i}^{\{Q_\beta\cap P_i\}}:\bigotimes_S \det R\Gamma_c(Q_S\cap P_i,F)^{\otimes(-1)^{|S|}} \xrightarrow{\sim} \det R\Gamma_c(P_i,F)$. Thus the compatibility of $\mu_Q^{\{Q_\beta\}}$ with the trivializations $\iota(Q)$ and $\iota(Q_S)$ follows from the compatibility of $\mu_{P_i}^{\{Q_\beta\cap P_i\}}$ with $\iota(P_i)$ and $\iota(Q_S\cap P_i)$. The latter is evident due to (ii), and we are done.

(c) It remains to construct E_i's. Consider the projection $\pi : Q \to I^\circ$ from the proof of A4(vi)(vii). The fibers Q_t, $t\in I^\circ$, are semi-open intervals, hence $R\pi_! i_Q^*(F)=0$. By A4(vii), for any non-empty $S\subset B$ one has $R(\pi|_{Q_S})_! i_{Q_S}^* F = 0$.

Let $B\subset I^\circ$ be a finite subset such that over $I^\circ\smallsetminus B$ all the projections $Q_S\to I^\circ$ are locally trivial. Let I^γ be the partition of I° by the points in B and the open intervals between successive points in B; set $Q^\gamma := \pi^{-1}(I^\gamma)$. For any Q^γ every Q_β such that $Q_\beta\cap Q_\gamma\neq\emptyset$ yields two closed subspaces of Q_γ that consist of points lying below one or the other boundary components of $Q_\beta\cap Q_\gamma$. Let $\{E_j^\gamma\}$ be the set of all subspaces of Q^γ obtained in this manner, with Q^γ itself added; we order them by inclusion. This is a finite filtration of Q^γ by closed subspaces. Notice that each E_j^γ is a fibration over I^γ whose fibers are semi-open intervals, and $P_j^\gamma\cap Q_\beta\neq\emptyset$ implies $P_j^\gamma\subset Q_\beta$; here $P_j^\gamma := E_j^\gamma\smallsetminus E_{j-1}^\gamma$. Combining E_j^γ's for all γ's, we get the promised E_i's (with $\{P_i\}=\{P_j^\gamma\}$). □

Remarks. (i) By A4(v), ι^N has local origin: for an open $U\subset X$ the restriction of ι^N to $\mathcal{C}_0^N(U)$ depends only on $F|_U$ and $N|_U$.

(ii) If $N'\subset TX$ is another ν-cone, then $N+N'$ is also a ν-cone, $\mathcal{C}^{N+N'}\subset \mathcal{C}^N\cap\mathcal{C}^{N'}$, same for $\tilde{\mathcal{C}}^N$, $\mathcal{C}_0^N$, and $\iota^{N+N'}=\iota^N|_{\mathcal{C}_0^{N+N'}}=\iota^{N'}|_{\mathcal{C}_0^{N+N'}}$.

A6. Now we can make good the promise of A1.

Lemma. *For any component Y_α and an open U_α such that $U\cap Y=Y_\alpha$ there exists $Q_\alpha\in\mathcal{C}^N(U_\alpha):=\mathcal{C}^N\cap\mathcal{C}(U_\alpha)$ such that $Q_\alpha\supset Y_\alpha$.*

Proof. We can assume that $\partial U_\alpha\cap Y=\emptyset$. Using A4(iv), cover ∂U_α by a finite set of N-lenses $\{Q_i\}$. Then $V_\alpha := U_\alpha\smallsetminus\cup Q_i$ is N-transversal; set $Q_\alpha = V_\alpha^+$. □

Consider the quotient Boolean algebra $\mathcal{C}^N/\mathcal{C}_0^N$. We have a morphism of Boolean algebras $\mathcal{C}^N/\mathcal{C}_0^N\to 2^A$, $Q\mapsto Q\cap Y$; here 2^A is the Boolean algebra of all subsets of the set A of connected components of Y. By the lemma, this is an isomorphism of Boolean algebras; let $\kappa : 2^A \xrightarrow{\sim} \mathcal{C}^N/\mathcal{C}_0^N$ be the inverse isomorphism. The lemma also shows that for U_α from loc. cit. the map $\mathcal{C}^N(U_\alpha)/\mathcal{C}_0^N(U_\alpha)\to 2^{\{\alpha\}}$, $Q\mapsto Q\cap Y = Q\cap Y_\alpha$, is an isomorphism as well, so we have its inverse $\kappa_\alpha : 2^{\{\alpha\}}\xrightarrow{\sim}\mathcal{C}^N(U_\alpha)/\mathcal{C}_0^N(U_\alpha)$.

By A2(c), (λ_F,ι^N) can be considered as an $\mathcal{L}_R$-measure λ_F^N on $\mathcal{C}^N/\mathcal{C}_0^N$. Set $\mathcal{E}(F)_{\nu Y_\alpha} := \lambda_F^N\kappa_\alpha(\{\alpha\})$; this is our ε-factor. The image of X in $\mathcal{C}^N/\mathcal{C}_0^N$ equals $\kappa(A)=\sqcup\kappa_\alpha(\{\alpha\})$, so one has the canonical identifications $\otimes\mathcal{E}(F)_{\nu Y_\alpha} = \otimes\lambda_F^N\kappa_\alpha(\{\alpha\}) \xrightarrow{m} \lambda_F^N(\sqcup\kappa_\alpha(\{\alpha\})) = \lambda_F^N(\kappa(A)) = \lambda_F(X)\xrightarrow{a}\det R\Gamma(X,F)$. One defines (A1.1) as their composition.

Explicitly, $\mathcal{E}(F)_{\nu Y_\alpha}=\lambda_F(Q_\alpha)$, where Q_α is any constructible set as in the lemma. To define (A1.1), we choose neighborhoods U_α of Y_α such that different $\bar U_\alpha$ do not intersect, a ν-cone N, and a set of N-lenses Q_β such that $\cup Q_\beta\supset X\smallsetminus\cup U_\alpha$. Set $Q_\alpha := U_\alpha\smallsetminus\cup Q_\beta$. Then $\mathcal{E}(F)_{\nu Y_\alpha}=\lambda_F(Q_\alpha)$, and (A1.1) comes from the isomorphism $m : (\otimes\lambda_F(Q_\alpha)) \otimes \lambda_F(\cup Q_\beta) \xrightarrow{\sim} \lambda_F(X)$ and a trivialization of $\lambda_F(\cup Q_\beta)$ defined by the trivializations $\iota(Q_S)$ via identification (A2.1) (the latter described explicitly in part (a) of the proof of the lemma in A5).

The construction of $\mathcal{E}(F)_{\nu\alpha}$ and the ε-factorization does not depend on the auxiliary datum of U_α's and the ν-cone N: for U_α this is evident, for N use Remark (ii) in A5. Finally, the local origin of $\mathcal{E}(F)_{\nu\alpha}$ follows from Remark (i) in A5.

Exercise. The degree of the graded super R-line $\mathcal{E}(F)_{\nu Y_\alpha}$ equals $\chi(Y_\alpha, F) + rk(F)w_\alpha(\nu)$, where $w_\alpha(\nu) \in \mathbb{Z}$ is the winding number of ν around Y_α.

Examples. Suppose Y_α is a circle of radius r around $0 \in \mathbb{C}$ and U_α equals $\{z : r \le |z| < R\}$. Let us write Q_α from the corollary for some forms ν explicitly:

(a) For $\nu = \mathrm{Re}\, dz/z$ one can take $Q_\alpha = \bar{U}_\alpha$; for $\nu = -\mathrm{Re}\, dz/z$ take $Q_\alpha = U_\alpha$.

(b) $\nu = \mathrm{Re}\, z^{n-1}dz$, $n > 0$: Draw a cogwheel of a radius $> r$ centered at 0 with cogs at the arguments $\frac{\pi}{2n} + \frac{k\pi}{n}$, $k = 1, \dots, 2n$, pointing outside the circle. Our Q_α is the union of the interior of the cogwheel and the points with arguments $\frac{-\pi}{2n} + \frac{2j\pi}{n} < \theta < \frac{\pi}{2n} + \frac{2j\pi}{n}$, $j = 1, \dots n$, on its boundary.

(c) $\nu = \mathrm{Re}\, z^{n-1}dz, n < 0$: Draw a cogwheel with cogs in the same position as for $-n$, but pointing *inside* the circle. Our Q_α is the union of the interior of the cogwheel and the points with arguments on the boundary it is 0 for arguments $\frac{-\pi}{2n} + \frac{2j\pi}{n} \le \theta \le \frac{\pi}{2n} + \frac{2j\pi}{n}$ on its boundary.

6. The determinant of the cohomology

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