Mathematische Annalen

Begründet 1868 durch *Alfred Clebsch* und *Carl Neumann*, früher herausgegeben von *Alfred Clebsch* (1869—1872), *Carl Neumann* (1869—1876), *Felix Klein* (1876—1924), *Adolph Mayer* (1876—1901), *Walther v. Dyck* (1888—1921), *David Hilbert* (1902—1939), *Otto Blumenthal* (1906—1938), *Albert Einstein* (1920—1928), *Constantin Carathéodory* (1925—1928), *Erich Hecke* (1929—1947), *Franz Rellich* (1947—1955), *Kurt Reidemeister* (1947—1963).

Band 1—80 Leipzig, B. G. Teubner, ab Band 81 (1920) Berlin, Springer.

An unsere Mitarbeiter!

Die Korrekturkosten sind bei den „Mathematischen Annalen" sehr hoch. Für ihre Verminderung muß unbedingt Sorge getragen werden. Wir richten deshalb an alle unsere Mitarbeiter die freundliche dringende Bitte, zu diesem Ziele an ihrem Teile mit beitragen zu wollen. Dazu ist nötig:

1. Das Manuskript muß *völlig druckfertig* und *gut leserlich* sein und für den Setzer *gemäß den „Hinweise für die Autoren"* (herausgegeben von Redaktion und Verlag) vorbereitet sein. — Diese Hinweise können von der Redaktion angefordert werden.

2. Veränderungen des Textes in der Korrektur sind auf die Fälle zu beschränken, wo sich nachträglich *wirkliche Irrtümer* herausstellen. Insbesondere sind rein stilistische Verbesserungen zu unterlassen. Sollte ein Irrtum bemerkt werden, noch bevor Korrektur eingetroffen ist, dann ist ein verbesserter Text sofort an die Redaktion zu schicken, die dafür Sorge tragen wird, daß das Manuskript noch vor dem Satz berichtigt wird.

Jede Änderung des schon gesetzten Textes muß zu Lasten der Autoren gehen.

Die Reihenfolge der Veröffentlichungen behält sich die Redaktion vor.

Die Redaktion der Mathematischen Annalen

Mathematische Annalen

Erscheinen zur Ermöglichung rascher Veröffentlichung zwanglos in Heften, die zu Bänden vereinigt werden. Sie sind durch jede Buchhandlung zu beziehen. Der Preis des Bandes beträgt DM 96,—.

Grundsätzlich dürfen nur Arbeiten eingereicht werden, die vorher weder im Inland noch im Ausland veröffentlicht worden sind. Der Autor verpflichtet sich, sie auch nachträglich nicht an anderer Stelle zu publizieren. Mit der Annahme des Manuskripts und seiner Veröffentlichung durch den Verlag geht das Verlagsrecht für alle Sprachen und Länder einschließlich des Rechts der fotomechanischen Wiedergabe oder einer sonstigen Vervielfältigung an den Verlag über. Jedoch wird gewerblichen Unternehmen für den innerbetrieblichen Gebrauch nach Maßgabe des zwischen dem Börsenverein des Deutschen Buchhandels e. V. und dem Bundesverband der Deutschen Industrie abgeschlossenen Rahmenabkommens die Anfertigung einer fotomechanischen Vervielfältigung gestattet. Wenn für diese Zeitschrift kein Pauschalabkommen mit dem Verlag vereinbart worden ist, ist eine Wertmarke im Betrage von DM 0,30 pro Seite zu verwenden. *Der Verlag läßt diese Beträge den Autorenverbänden zufließen.*

Die Mitarbeiter erhalten von ihrer Arbeit zusammen 75 Sonderdrucke unentgeltlich.

Für die Mathematischen Annalen bestimmte Manuskripte können bei jedem der unten verzeichneten Redaktionsmitglieder eingereicht werden:

Inhaltsverzeichnis des Heftes auf Seite 4

MACKEY, G. W.
Math. Annalen 166, 187—207 (1966)

Ergodic Theory and Virtual Groups

To GOTTFRIED KÖTHE on his 60th Birthday

GEORGE W. MACKEY

1. Introduction

In a recent note [11] the author has given a highly condensed and largely unmotivated account of the basic notions in a theory whose aim is to bring to light and exploit certain apparently far reaching analogies between group theory and ergodic theory. Somewhat more motivation is contained on pages 652—654 of [10], but the brief account given there is embedded in a description of certain rather technical developments in the theory of infinite dimensional group representations. Moreover, some essential conceptual improvements in the theory were made between the times at which [10] and [11] were written. It is the purpose of this paper to give a leisurely account, with a minimum of mathematical technicalities, of the considerations which led to the formulation of the definitions and theorems in the first three sections of [11].

2. Subgroups and transitive actions

Let G be a group and let H be a subgroup. Let $S_H = G/H$ be the set of all right H cosets. For each $s = Hy$ in S_H and each $x \in G$ let $sx = Hyx$. Then the mapping taking s, y into sy is a mapping of $S_H \times G$ into S_H which defines an *action* of G on S_H in the sense that conditions (i) and (ii) below are satisfied.

(i) $(sx_1)x_2 = sx_1x_2$ for all x_1, x_2 in G and all s in S_H,

(ii) $se = s$ for all s in S_H where e is the identity of G.

This action is clearly *transitive* in the sense that for each pair s_1, s_2 of elements of S_H there exists an element x of G such that $s_1x = s_2$. Equivalently S_H has no invariant subsets except the empty set and its complement. (An invariant subset is a subset S^1 such that $sx \in S^1$ whenev $s \in S^1$ and $x \in G$.)

Conversely let $s, x \to sx$ define a transitive action of G on the set S and let H_s be the subgroup of all x with $sx = s$. Then the mapping $x \to sx$ defines a many-one mapping of G on S which is constant on each right coset. Thus it defines a one-to-one mapping θ of S_{H_s} on S. Evidently θ has the property that $\theta(sx) = \theta(s)x$ so that the actions of G on S and S_{H_s} are equivalent in an obvious sense. Thus every transitive G action is equivalent to one defined by a subgroup. One verifies easily that two subgroups define equivalent actions if and only if they are conjugate and hence that there is a natural one-to-one correspondence between the transitive actions of a group G and the conjugacy classes of its subgroups. These facts are of course elementary and well known. We review them in detail because they play such a central role in motivating what is to follow.

3. Measure theoretic actions and ergodicity

In the group actions that occur in analysis, the group G and the space S come equipped with topologies and the mapping $s, x \to sx$ is continuous. We shall be mainly interested in measure theoretic questions and accordingly find it convenient to replace the topologies of S and G by their underlying Borel structures. By a *Borel space* we shall mean a set together with a distinguished σ field of subsets called its Borel sets. By a Borel mapping of the Borel space S_1 into the Borel space S_2 we shall mean a mapping θ such that $\theta^{-1}(E)$ is a Borel subset of S_1 whenever E is a Borel subset of S_2. Two Borel spaces S_1 and S_2 will be said to be isomorphic if there exists a one-to-one Borel mapping θ of S_1 on S_2 such that θ^{-1} is also a Borel mapping. Let G be a separable locally compact group. By a Borel G space we shall mean a G space S which is also a Borel space and in such a way that $s, x \to sx$ is a Borel function from $S \times G$ to S. When H is a closed subgroup of G then the coset space S_H is a topological space in a natural way and hence is a Borel space. Since $s, x \to sx$ is continuous it is certainly a Borel function and S_H is a transitive Borel G space. Moreover S_H as a Borel space is "standard" in the sense that it is isomorphic as a Borel space to a Borel subset of a separable complete metric space. Conversely let S be any transitive Borel G space where S is standard and G is separable and locally compact. It is known ([8], p. 284) that for each $s \in S$ the subgroup H_s of all x with $sx = s$ is *closed* and that the mapping θ of § 2 is a Borel isomorphism between S_{H_s} and S. Thus when G is a separable locally compact group we have the following analogue of the result reviewed in § 2: The possible transitive actions of G on standard Borel spaces correspond one-to-one to the conjugacy classes of closed subgroups of G.

Let S be a standard Borel G space where G is separable and locally compact and let μ be a *measure* defined in S; that is let μ be a function from the Borel subsets of S to the positive real numbers augmented by $+\infty$ which is additive on countable unions of disjoint sets and which is σ *finite* in the sense that S is a countable union of subsets S_j such that $\mu(S_j) < \infty$. If $\mu(Ex) = \mu(E)$ for all $x \in G$ and all Borel subsets E of S we shall say that μ is *invariant*. If $\mu(Ex) = 0$ whenever $x \in G$ and $\mu(E) = 0$ we shall say that μ is *quasi invariant*. Two measures with the same sets of measure zero will be said to be in the same class, and a class of measures (= measure class) will be said to be invariant if $E \to \mu(Ex)$ is in the class whenever x is in G and μ is in the class. It is clear that the class of a quasi invariant measure is invariant and that every member of an invariant measure class is quasi invariant. It is known ([6], p. 106) that $S_H = G/H$ admits a unique invariant measure class whenever H is a closed subgroup of G but that S_H admits an invariant measure only for rather special choices of H. One is also lead naturally to invariant measure classes which do not need to contain invariant measures whenever S is a C_∞ manifold and the mappings $s \to sx$ preserve the differential structure. Thus, although classical ergodic theory is chief concerned with those G spaces S which have an invariant measure, it seems natural and is convenient to deal here with the more general situation in which our G spaces have only an invariant measure class.

Let G be as above and let C be an invariant measure class in the standard Borel G space S. Suppose that the action is not transitive and that E is an invariant subset. Then $S - E$ is also invariant and E and $S - E$ will be independent G spaces of which S is a direct sum in a natural way. If E is a Borel set they will both the standard and if E (resp. $S - E$) is not of measure zero we obtain a non trivial invariant measure class in it by taking any member μ of C and then taking the class of its restriction to E (resp. $S - E$). Thus if E is a Borel set and neither E nor $S - E$ has measure zero we obtain S, C as the direct sum of two invariant subsystems. However, even when G is the additive group of the integers or of the real line it is possible to choose S and C in such a way that

(1) Every measurable invariant subset of S is either of measure zero or the complement of a set of measure zero

(2) Every invariant subset of S on which G acts transitively is of measure zero.

For example if G is the additive group of the integers we may choose S to be the unit circle $|z| = 1$ in the complex plane, C to be the measure class of the length measure and the action of G on S to be such that $zn = ze^{in\alpha 2\pi}$ where α is an irrational number. Thus mere lack of transitivity does not ensure the possibility of a direct sum decomposition — even if we ignore invariant sets of measure zero. One is forced to replace transitivity as the basic notion by a more inclusive and sophisticated one which is sometimes called ergodicity and sometimes metric transitivity. We say that the action of G on S is *ergodic* (or metrically transitive) with respect to C if condition (1) above holds. This condition holds in particular when the action of G on S is transitive. It also holds when the action is *essentially transitive* in the sense that there exists an invariant subset of measure zero on whose complement G acts transitively. When conditions (1) and (2) above both hold, that is when the action is ergodic but not essentially transitive we shall say that the action is *strictly ergodic*.

4. The virtual subgroup point of view

Let G be a separable locally compact group. Since the ergodic actions of G constitute a natural generalization of the transitive actions (§ 3) and since the transitive actions correspond one-to-one in a natural way to the conjugacy classes of closed subgroups of G (§ 2) it is natural to ask the following question.

(a) Is there a generalization of the notion of closed subgroup such that the ergodic actions bear the same relationship to these generalized subgroups that the transitive ones do to actual subgroups? We shall show in the sequel that such a notion does exist and in further publications that a surprisingly large number of the notions and theorems of group theory have analogues which apply to it. We shall not only be able to define the notion of virtual subgroup of a genuine group G but also the notion of "virtual group" in the abstract. Before doing either of these things however we shall show that many of the developments which the notion of virtual group make possible may be carried out without actually defining virtual groups. The idea is this. Since a transitive action determines and is determined by a conjugacy class of closed subgroups we may hope