THE FLABBY CLASS GROUP OF A FINITE CYCLIC GROUP

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Abstract. This is an expository paper on work of Endo and Miyata which leads to a computation of the flabby class group of a finite cyclic group.

1. Introduction

The aim of this paper is to give a proof of the calculation of the flabby class group of a finite cyclic group due to Endo and Miyata [2]. In the next section I will recall the definition and some basic facts about this group. In the final section I will give some examples to show that the invertibility conditions used by Endo and Miyata cannot be removed. I would like to thank M.-c. Kang for useful comments and for showing me his results on some related problems.

2. Some basic results

For convenience I will repeat here the results given in [6, §8] omitting the proofs. Let $\pi$ be a finite group and let $\mathcal{L}_\pi$ be the class of torsion free finitely generated $\mathbb{Z}\pi$–modules. As usual we refer to such modules as $\pi$–lattices. Let $\text{Perm}_\pi$ be the subclass of permutation modules, those having a $\mathbb{Z}$–base permuted by $\pi$. Let $\mathcal{P}_\pi$ be the class of invertible modules i.e. direct summands of permutation modules. Define $\mathcal{F}_\pi$ to be the class of $\pi$–lattices such that $\text{Ext}^1_{\mathbb{Z}\pi}(F, P) = 0$ for all permutation modules $P$ and therefore for all invertible modules $P$. We refer to the elements $F$ of $\mathcal{F}_\pi$ as flabby (or flasque) modules. Define $\mathcal{C}_\pi$ to be the class of $\pi$–lattices such that $\text{Ext}^1_{\mathbb{Z}_p\pi}(P, C) = 0$ for all permutation modules $P$ and therefore for all invertible modules $P$. We refer to the elements $C$ of $\mathcal{C}_\pi$ as coflabby (or coflasque) modules.

Lemma 2.1. A $\pi$–lattice $M$ lies in $\mathcal{F}_\pi$ if and only if $H^{-1}(\pi', M) = 0$ for all subgroups $\pi'$ of $\pi$ and it lies in $\mathcal{C}_\pi$ if and only if $H^1(\pi', M) = 0$ for all subgroups $\pi'$ of $\pi$.

Remark 2.2. Since $H^i(\pi', M) \to H^i(\pi'_p, M)$ is injective on $p$–torsion where $\pi'_p$ is a Sylow $p$–subgroup of $\pi'$, it is sufficient to assume $H^i(\pi', M) = 0$ for $p$–subgroups $\pi'$ in order that $H^i(\pi', M) = 0$ for all subgroups $\pi'$.

Lemma 2.3. $\mathcal{P}_\pi \subseteq \mathcal{C}_\pi \cap \mathcal{F}_\pi$.

Lemma 2.4. For any $\pi$–lattice $M$ there are short exact sequences $0 \to M \to P \to F \to 0$ and $0 \to C \to Q \to M \to 0$ where $P$ and $Q$ are permutation modules, $F \in \mathcal{F}_\pi$, and $C \in \mathcal{C}_\pi$.

Corollary 2.5. Let $M$ be a $\pi$–lattice. The following are equivalent:
(1) $M$ is invertible.
(2) $\text{Ext}^1_{\mathbb{Z}_\pi}(F, M) = 0$ for all $F$ in $\mathcal{F}_\pi$.
(3) $\text{Ext}^1_{\mathbb{Z}_\pi}(M, C) = 0$ for all $C$ in $\mathcal{C}_\pi$.

**Definition 2.6.** Define an equivalence relation on $\mathcal{F}_\pi$ by $F_1 \sim F_2$ if and only if we have $F_1 \oplus P_1 \approx F_2 \oplus P_2$ for permutation modules $P_1$ and $P_2$. Let $\mathcal{F}_\pi$ be the set of equivalence classes of $F \in \mathcal{F}_\pi$. It is a monoid under direct sum. I will refer to $\mathcal{F}_\pi$ as the flabby class monoid of $\pi$.

If $M$ is a $\pi$–lattice, choose an exact sequence $0 \to M \to P \to F \to 0$ as in Lemma 2.4 and let $\rho(M)$ be the class $[F]$ of $F$ in $\mathcal{F}_\pi$.

**Lemma 2.7.** $\rho(M)$ is well defined.

**Lemma 2.8.** Let $M$ and $N$ be $\pi$–lattices. Then $\rho(M) = \rho(N)$ if and only if there are exact sequences $0 \to M \to E \to P \to 0$ and $0 \to N \to E \to Q \to 0$ with $P$ and $Q$ permutation modules.

**Definition 2.9.** If $M$ is a $\pi$–module I will write $(M)_0$ for $M/t(M)$ where $t(M)$ is the torsion submodule of $M$.

Let $R$ be a Dedekind ring and let $\theta : \mathbb{Z}_\pi \to R$ be a ring homomorphism. Define $c_\theta : F_\pi \to C(R)$ by sending $[F]$ to $(R\otimes_{\mathbb{Z}_\pi} F)_0$. This is a well defined homomorphism of monoids. If $\pi$ is abelian and $A$ is the integral closure of $\mathbb{Z}_\pi$ in $\mathbb{Q}_\pi$ then $A$ is a product of Dedekind rings so we get a map $c : F_\pi \to C(A)$. Our object is to prove the following theorem.

**Theorem 2.10** (Endo and Miyata). If $\pi$ is a finite cyclic group then $c : F_\pi \to C(A)$ is an isomorphism.

Except for the examples in the final section, the results discussed in this paper are due to Endo and Miyata.

### 3. Some more useful facts

**Lemma 3.1.** Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of $\pi$–lattices with $M''$ invertible. Then $\rho(M) = \rho(M') + \rho(M'')$ in $\mathcal{F}_\pi$.

In section 7 I will show that the hypothesis that $M''$ is invertible cannot be omitted in general.

**Proof.** Choose a sequence $0 \to M \to P \to F \to 0$ with $P$ permutation and $F$ flabby. Factoring out $M'$ we get $0 \to M'' \to P/M' \to F \to 0$. This splits since $F$ is flabby and $M''$ is invertible. Therefore $P/M' \approx F \oplus M''$. It follows that $P/M'$ is flabby since $F$ and $M''$ are. The sequence $0 \to M' \to P \to P/M' \to 0$ shows that $\rho(M') = [P/M'] = [F] + [M']$. Since $M''$ is invertible we can write $M'' \oplus L = Q$ where $Q$ is permutation and the sequence $0 \to M'' \to Q \to L \to 0$ shows that $\rho(M'') = [L]$. Therefore $\rho(M) = [F] + [Q] = [F] + [M'] + [L] = \rho(M') + \rho(M'')$. □

**Lemma 3.2.** Let $0 \to M' \overset{i_1}{\to} M \overset{i_2}{\to} M'' \to 0$ be a short exact sequence of $\mathbb{Z}_\pi$–modules. If it splits over a Sylow $p$–subgroup $\pi_p$ for each prime $p$ then it splits over $\pi$.
Proof. Let \( f_p : M'' \to M \) be a \( \pi_p \)-homomorphism such that \( jf_p = 1 \). Let \( \pi = \cup \sigma \pi_p \) be a left coset decomposition and let \( g_p(x) = \sum \sigma v f_p(\sigma v^{-1} x) \). Then \( g_p \) is a \( \pi \)-homomorphism and \( jg_p = |\pi : \pi_p| \). Choose \( a_p \in \mathbb{Z} \) such that \( \sum a_p|\pi : \pi_p| = 1 \). Then \( h = \sum a_p g_p \) is the required splitting.

\[\]

**Lemma 3.3.** If \( M \) is a \( \pi \)-lattice which is invertible over each Sylow subgroup \( \pi_p \) then \( M \) is invertible.

Proof. Choose an exact sequence \( 0 \to M \to P \to F \to 0 \) as in Lemma 2.4 with \( F \) flabby over \( \pi \). By Lemma 2.1, \( F \) is flabby over all subgroups of \( \pi \). Therefore the sequence splits over all Sylow subgroups and therefore splits by Lemma 3.2.

4. Cyclic Groups

We now discuss some results which hold for cyclic groups and more generally for groups whose Sylow subgroups are cyclic.

**Lemma 4.1.** If all Sylow subgroups of \( \pi \) are cyclic then \( C_\pi = F_\pi \)

Proof. By Remark 2.2 it is enough to check this for cyclic groups since all \( p \)-subgroups are cyclic. Since the cohomology of a finite cyclic group is periodic with period 2 [1, Ch. XII], we have \( H^1(\pi', M) = 0 \) if and only if \( H^{-1}(\pi', M) = 0 \) for cyclic \( \pi' \).

**Lemma 4.2.** If \( f, g \in \mathbb{Z}[x] \) are non-zero then the sequence \( 0 \to \mathbb{Z}[x]/(f) \to \mathbb{Z}[x]/(g) \to 0 \) is exact.

**Lemma 4.3.** Let \( \pi \) be a finite cyclic \( p \)-group of order \( n \) with generator \( x \). If \( M \) is a finitely generated torsion free module over \( \mathbb{Z}\pi/\Phi_n(x) = \mathbb{Z}[\zeta_n] \) then \( \rho(M) \) is invertible.

Proof. Let \( n = pq \) and let \( \pi'' = \pi/ < x^q > \). The factorization \( x^n - 1 = \Phi_n(x)(x^q - 1) \) shows that the sequence \( 0 \to \mathbb{Z}[\zeta_n] \to \mathbb{Z}\pi' \to \mathbb{Z}\pi'' \to 0 \) is exact. It follows that \( \rho(\mathbb{Z}[\zeta_n]) = [\mathbb{Z}\pi''] = 0 \). Since \( M \) is projective over \( \mathbb{Z}[\zeta_n] \) we can write \( M \oplus N = \mathbb{Z}[\zeta_n]^r \) so \( \rho(M) + \rho(N) = r\rho(\mathbb{Z}[\zeta_n]) = 0 \) showing that \( \rho(M) \) is invertible.

It is clear that an element \( [F] \) of \( F_\pi \) has an inverse if and only if \( F \) is invertible. Therefore the following theorem characterizes the groups \( \pi \) for which \( F_\pi \) is a group.

**Theorem 4.4.** Let \( \pi \) be a finite group. Then \( P_\pi = F_\pi \) if and only if all Sylow subgroups of \( \pi \) are cyclic.

Proof. We know that \( P_\pi \subseteq F_\pi \) in any case by Lemma 2.3. Suppose all Sylow subgroups of \( \pi \) are cyclic. If \( M \) is flabby over \( \pi \) it is flabby over all subgroups of \( \pi \) by Lemma 2.1. By Lemma 3.3 it is enough to show \( M \) invertible over the Sylow subgroups so we can assume that \( \pi \) is a cyclic \( p \)-group. Write \( |\pi| = n \) and \( n = pq \) where \( n \) and \( q \) are powers of \( p \). Let \( x \) generate \( \pi \), let \( M' = \{ m \in M|\Phi_n(x)m = 0 \} \), and write \( 0 \to M' \to M \to M'' \to 0 \). Clearly \( M'' \) is torsion–free. Since \( x^n - 1 = \Phi_n(x)(x^q - 1) \), \( x^q - 1 \) annihilates \( M'' \) which is therefore a lattice over \( \pi'' = \pi/ < x^q > \). We can assume by induction that the theorem holds for \( \pi'' \), the case \( n = 1 \) being trivial. We claim that \( M'' \) is flabby over \( \pi \) i.e. \( H^{-1}(\pi', M'') = 0 \) for all subgroups \( \pi' \) of \( \pi \). This is clear for \( \pi' = 1 \). If \( \pi' \neq 1 \) then \( M''\pi' = 0 \) since it is annihilated by \( \Phi_n(x) \) and by \( x^q - 1 \) where \( x' = < x^q > \), \( d < n \), \( gcd(\Phi_n, x^d - 1) = 1 \) over \( \mathbb{Q} \) and \( M' \) is torsion free. It follows that \( \tilde{H}^0(\pi', M') = 0 \) and the exact
cohomology sequence gives $0 = H^{-1}(\pi, M) \rightarrow H^{-1}(\pi', M'') \rightarrow \hat{H}^0(\pi', M') = 0$ showing that $M''$ is flabby over $\pi$. It follows that $M''$ is also flabby over $\pi'' = \pi/ < x^q >$ since a sequence $0 \rightarrow P \rightarrow E \rightarrow M'' \rightarrow 0$ over $\pi''$ with $P$ permutation has the same properties over $\pi$ and therefore splits over $\pi$ and so over $\pi''$. Induction now shows that $M''$ is invertible over $\pi''$ and therefore also over $\pi$. By Lemma 3.1 we now have $\rho(M) = \rho(M') + \rho(M'')$. Now $\rho(M'') = -[M'']$ is invertible and so is $\rho(M')$ by Lemma 4.3. Therefore $\rho(M)$ is also invertible. Choose a sequence $0 \rightarrow M \rightarrow P \rightarrow F \rightarrow 0$ with $P$ permutation and $F$ flabby so that $\rho(M) = [F]$ and therefore $F$ is invertible. By Lemma 4.1, $M$ is coflabbly so the sequence splits giving $M \oplus F \approx P$ and showing that $M$ is invertible.

For the converse let $I$ be the augmentation ideal of $\mathbb{Z} \pi$. We will show that if $\rho(I^*)$ is invertible then all Sylow subgroups are cyclic. Let $n$ be the order of $\pi$. The cohomology sequence of $0 \rightarrow I \rightarrow \mathbb{Z} \pi \rightarrow \mathbb{Z} \rightarrow 0$ shows that $H^1(\pi, I) = \mathbb{Z}/n \mathbb{Z}$. Choose a sequence $0 \rightarrow I^* \rightarrow P \rightarrow F \rightarrow 0$ with $P$ permutation and $F$ flabby so that $\rho(I^*) = [F]$. Then $0 \rightarrow F^* \rightarrow P^* \rightarrow I^* \rightarrow 0$ gives $0 = H^1(\pi, P^*) \rightarrow H^1(\pi, I) \rightarrow H^2(\pi, F^*)$ showing that $H^2(\pi, F^*)$ has an element of order $n$. If $\rho(I^*)$ is invertible then so is $F$ and therefore so is $F^*$. Write $F^* \oplus L = Q$ where $Q$ is permutation. Then $H^2(\pi, Q)$ has an element of order $n$. Let $Q = \oplus \mathbb{Z} \pi/\pi_i$. Then $H^2(\pi, Q) = \oplus H^2(\pi, \mathbb{Z} \pi/\pi_i) = \oplus H^2(\pi, \mathbb{Z}, \mathbb{Z})$. Let $r = \text{ord}_p(n)$. Some $H^2(\pi, \mathbb{Z})$ must have an element of order $p^r$ so the next lemma shows that the Sylow subgroup of $\pi_i$ is cyclic of order $p^r$ and therefore the same is true of $\pi$.

\begin{lemma}
Let $\pi$ be a group of order $n$, let $p$ be a prime and let $q$ be the highest power of $p$ dividing $n$. If $H^2(\pi, \mathbb{Z})$ has an element of order divisible by $q$ then the Sylow $p$-subgroup of $\pi$ is cyclic of order $q$.
\end{lemma}

\begin{proof}
The sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \mathbb{Z} \rightarrow \mathbb{Z}/q \mathbb{Z} \rightarrow 0$ gives $0 = H^1(\pi, \mathbb{Z}) \rightarrow H^1(\pi, \mathbb{Z}/q \mathbb{Z}) \rightarrow H^2(\pi, \mathbb{Z}) \cong H^2(\pi, \mathbb{Z})$. It follows that $H^1(\pi, \mathbb{Z}/q \mathbb{Z}) = \text{Hom}(\pi, \mathbb{Z}/q \mathbb{Z})$ has an element of order $q$ so there is a map of $\pi$ onto $\mathbb{Z}/q \mathbb{Z}$. This clearly implies the result.
\end{proof}

5. Devissage

The aim of this section is to prove a devissage theorem which will be the main tool in the proof of the main theorem. The only property of $\rho: \mathcal{L}_\pi \rightarrow F_\pi$ which is needed is that of Lemma 3.1 so I will state the theorem more generally for a map $\phi: \mathcal{L}_\pi \rightarrow G$ assigning an element of an abelian group $G$ to each $\mathbb{Z}\pi$-lattice and satisfying the following property: If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is exact with $M''$ invertible then $\phi(M) = \phi(M') + \phi(M'')$. For any abelian group $M$ I will write $(M)_0$ for $M/t(M)$ where $t(M)$ is the torsion submodule of $M$.

\begin{theorem}
Let $\pi = \langle x : x^n = 1 \rangle$ be a cyclic group of order $N$ with generator $x$ and let $n | N$. Let $M$ be an invertible $\mathbb{Z}\pi$-lattice. Then
\[
\phi(M/(x^n - 1)M) = \sum_{d | n} \phi((M/\Phi_d(x)M)_0)
\]
\end{theorem}

In section 7 I will show that the hypothesis that $M$ is invertible cannot be omitted in general. By the Möbius inversion formula, this theorem is equivalent to the following result.
Corollary 5.2. Under the same conditions we have
\[ \phi(M/\Phi_n(x)M) = \sum_{d|n} \mu(\frac{n}{d}) \phi(M/(x^d - 1)M) \]
where \( \mu \) is the Möbius function.

Note that \( M/(x^n - 1)M \) will be torsion free by the following simple observation.

Lemma 5.3. If \( \pi \) acts on a set \( X \) and \( \pi' \) is a normal subgroup of \( \pi \) then \( \mathbb{Z}[\pi/\pi'] \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[X] = \mathbb{Z}[X/\pi'] \).

It follows that if \( M \) is a permutation lattice over \( \mathbb{Z} \pi \) then so is \( \mathbb{Z}[\pi/\pi'] \otimes_{\mathbb{Z}[\pi]} M \).

The same is therefore true for invertible lattices.

We also need the following fact which is easily checked.

Lemma 5.4. Let \( M' \rightarrow M \rightarrow M'' \rightarrow 0 \) be an exact sequence of abelian groups. Suppose that \( M'' \) is torsion free and the kernel of \( i \) is torsion. Then \( 0 \rightarrow (M')_0 \rightarrow (M)_0 \rightarrow M''_0 \rightarrow 0 \) is exact.

Lemma 5.5. Let \( \pi \) be a cyclic group of order \( N \) with generator \( x \). Let \( M \) be a \( \mathbb{Z} \pi \)-module. Let \( n|N \) and let \( f, g \in \mathbb{Z}[X] \) be such that \( fg \mathbb{Z}[x^n] = 1 \). If \( M/g(x)M \) is torsion-free then \( 0 \rightarrow (M/f(x)M)_0 \rightarrow (M/f(x)g(x)M)_0 \rightarrow M/g(x)M \rightarrow 0 \) is exact.

Proof. By Lemma 4.2 the sequence \( 0 \rightarrow \mathbb{Z}[X]/(f) \rightarrow \mathbb{Z}[X]/(fg) \rightarrow \mathbb{Z}[X]/(g) \rightarrow 0 \) is exact. Since \( fg \mathbb{Z}[x^n] = 1 \), this sequence is the same as \( 0 \rightarrow \mathbb{Z} \pi/(f(x)) \rightarrow \mathbb{Z} \pi/(f(x)g(x)) \rightarrow 0 \). Applying \( \otimes_{\mathbb{Z} \pi} M \) shows that \( \text{Tor}^1(\mathbb{Z} \pi/(g(x)) \otimes_{\mathbb{Z} \pi} M) \rightarrow M/f(x) \otimes_{\mathbb{Z} \pi} M \rightarrow M/f(x)g(x) \otimes_{\mathbb{Z} \pi} M \rightarrow M/g(x)M \rightarrow 0 \) is exact. Since the Tor term is torsion and \( M/g(x)M \) is torsion-free, Lemma 5.4 applies.

We now turn to the proof of Theorem 5.1. Let \( p_1, \ldots, p_r \) be the distinct prime divisors of \( n \). Construct a sequence \( d_0, d_1, \ldots, d_{2^r - 1} \) as follows. Let \( d_0 = 1 \) and \( d_1 = p_1 \). If \( d_0, d_1, \ldots, d_{2^r - 2} \) have been defined for \( s \geq 1 \) (using \( p_1, \ldots, p_{s-1} \)) let \( d_s = p_s(d_{s-1} - 1) \) for \( 2^s - 1 \leq \nu \leq 2^s - 1 \). Therefore \( d_2, \ldots, d_{2^r - 1} \) is \( p_r d_{2^{r-1} - 1}, \ldots, p_1 d_1 \), \( d_0 d_0 \). Note that \( d_s \) is squarefree since the \( p_i \) are distinct. Also the number of primes dividing \( d_s \) is congruent to \( \nu \) modulo 2. (If this holds for \( \nu \leq 2^s - 1 \) then for \( 2^s - 1 \leq \nu \leq 2^s - 1 \), the number of prime divisors of \( d_s \) is 1 more than that of \( d_{2^r - 2} \) and so is congruent to 1 + 2^s - \nu - 1 \equiv \nu \mod 2 \). It follows that \( \mu(d_\nu) = (-1)^\nu \).

Let \( e_\nu = n/d_\nu \) and define \( f_k(X) = \prod_{\nu=0}^k (X^\nu - 1)^{(-1)^\nu} \) for \( k = 0, \ldots, 2^r - 1 \). We also set \( f_{-1}(X) = 1 \)

Lemma 5.6.

1. \( f_k \) is monic and lies in \( \mathbb{Z}[X] \).
2. \( f_{2k}(X) = f_{2k-1}(X)(X^{e_{2k}} - 1) = f_{2k+1}(X)(X^{e_{2k+1}} - 1) \) for \( 0 \leq k \leq 2^r - 1 \).
3. \( f_k(X^n - 1) \), \( f_0 = X^n - 1 \), and \( f_{2^r - 1} = \Phi_n(X) \)

Proof. The irreducible factors occurring in \( f_k \) are cyclotomic polynomials \( \Phi_h \) for certain \( h|n \). Fix such an \( h \). Since \( \text{ord}_{\Phi_h}(x^e - 1) = 1 \) if \( h|e \) and 0 if not, we see that \( \text{ord}_{\Phi_h} f_k = e_0 + \cdots + e_\nu \) where \( e_\nu = (-1)^\nu \) if \( h|e_\nu \) or, equivalently, \( d_\nu \neq \frac{n}{h} \) and \( e_\nu = 0 \) otherwise. Clearly \( e_0 = 1 \). We claim that the sequence \( e_0, \ldots, e_{2^r-1} \) is either 1, 0, 0, ..., 0 or consists of alternate +1’s and -1’s with 0’s between. Suppose this is true for \( e_0, \ldots, e_{2^r-2} \) where \( s \geq 1 \). If \( p_s \) does not divide \( \frac{n}{h} \) then \( d_\nu \) does not divide \( \frac{n}{h} \) for \( 2^s - 1 \leq \nu \leq 2^s - 1 \) so \( e_0, \ldots, e_{2^s-1} \) is \( e_0, \ldots, e_{2^s-2}, 0, 0, \ldots, 0 \) which
has the required form. If $p_s$ does divide $\frac{m}{k}$ then, for $2^{s-1} \leq \nu \leq 2^s - 1$, $d_\nu$ divides $\frac{m}{k}$ if and only if $d_{2^{s-1}-\nu}$ divides $\frac{m}{k}$. It follows that $\epsilon_\nu = -\epsilon_{2^{s-1}-\nu}$, so $\epsilon_0, \ldots, \epsilon_{2^s-1}$ is $\epsilon_0, \ldots, \epsilon_{2^{s-1}-1}, -\epsilon_{2^{s-1}-1}, \ldots, -\epsilon_1, -\epsilon_0$ which again clearly has the required form. It follows that $\text{ord}_{\Phi_n} f_k$ is either 0 or 1 showing that $f_k$ is a monic polynomial and divides $X^n - 1$. The preceding argument also shows that $\sum_{\nu=0}^{2^s-1} \epsilon_\nu = 0$ if $\sum_{\nu=0}^{2^{s-1}-1} \epsilon_\nu = 0$ or if $p_s$ divides $\frac{m}{k}$. Therefore $\sum_{\nu=0}^{2^s-1} \epsilon_\nu = 0$ unless no $p_s$ divides $\frac{m}{k}$ so that $h = n$. This shows that $f_{2^s-1} = \Phi_n(X)$. The remaining assertions are obvious.

Now let $\pi = < x : x^N = 1 >$ be a cyclic group of order $N$ with generator $x$ and let $\nu|N$. Let $M$ be an invertible $\mathbb{Z}^\pi$-lattice. Define $M_k = (M/f_k(x)M)_0$, set $M_{-1} = 0$, and let $Q_k = M/(x^k - 1)M$. Then $M_0 = M/(x^n - 1)M = Q_0$, $M_{2^n-1} = (M/\Phi_n(x)M)_0$ and $Q_k$ is invertible for all $k$ by Lemma 5.3. In particular, $Q_k$ is torsion-free. By Lemma 5.6 and Lemma 5.5 we get short exact sequences $0 \rightarrow M_{2k-1} \rightarrow M_{2k} \rightarrow Q_{2k} \rightarrow 0$ and $0 \rightarrow M_{2k+1} \rightarrow M_{2k} \rightarrow Q_{2k+1} \rightarrow 0$ for $0 \leq k \leq 2^{s-1} - 1$. By the additivity property assumed for our function $\phi$ we get $\phi(M_{2k}) = \phi(M_{2k-1}) + \phi(Q_{2k}) = \phi(M_{2k+1}) + \phi(Q_{2k+1})$. It follows that $\phi(M_{2k+1}) - \phi(M_{2k-1}) = \phi(Q_{2k}) - \phi(Q_{2k+1})$. Summing from $k = 0$ to $k = 2^{s-1} - 1$ and using $M_{-1} = 0$ we get $\phi(M_{2s-1}) = \sum_{\nu=0}^{2^s-1} (-1)^\nu \phi(Q_\nu)$. Since the $d_\nu$ run over all squarefree divisors of $n$, $\mu(d_\nu) = (-1)^{\nu}$ and $\mu(d) = 0$ if $d$ is not squarefree we can write this as $\phi(M/\Phi_n(x)M) = \sum_{d|n} \mu(d) \phi(M/(x^d - 1)M) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \phi(M/(x^d - 1)M)$. This proves Corollary 5.2 and Theorem 5.1 follows by the M"obius inversion formula.

6. PROOF OF THE MAIN THEOREM

Let $\pi = < x : x^N = 1 >$ be a cyclic group of order $N$ with generator $x$. Let $A$ be the integral closure of $\mathbb{Z}\pi$ in $A_\pi$. Then $A = \prod_{n|N} R_n$ where $R_n = \mathbb{Z}[(\zeta_n)]$ is the ring of integers of the cyclotomic field $\mathbb{Q}(\zeta_n)$ and $x$ maps to $(\zeta_n)$. The map $c : F \rightarrow C(A) = \prod_{n|N} C(R_n)$ sends $[F]$ to $(R_n \otimes_{\mathbb{Z}\pi} F)_0$. Therefore if $c([F]) = 0$ then all $(R_n \otimes_{\mathbb{Z}\pi} F)_0$ are free. Now, by Corollary 5.2, we have $\rho(R_n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \phi(M/(x^d - 1)) = 0$ since all $\mathbb{Z}\pi/(x^d - 1)$ are permutation modules. Therefore it follows that $\rho((R_n \otimes_{\mathbb{Z}\pi} F)_0) = 0$ since $(R_n \otimes_{\mathbb{Z}\pi} F)_0$ is free over $R_n$. By Theorem 5.1 we have $\rho(F) = \sum_{n|N} \rho((R_n \otimes_{\mathbb{Z}\pi} F)_0) = 0$. Since $F$ is invertible we can write $F \otimes G = P$ where $P$ is permutation and the sequence $0 \rightarrow F \rightarrow P \rightarrow G \rightarrow 0$ shows that $\rho(F) = [G] = -[F]$ and it follows that $[F] = 0$ showing that $c : F \rightarrow C(A)$ is injective.

To see that $c$ is onto we consider the map $C(\mathbb{Z}\pi) \rightarrow F$ sending $[P] - [Q]$ in $C(\mathbb{Z}\pi)$ to $[P] - [Q]$ in $F$. Since $P$ is projective, $R_n \otimes_{\mathbb{Z}\pi} P$ is torsion free, being projective over $R_n$. Therefore the composition $C(\mathbb{Z}\pi) \rightarrow F \rightarrow C(A)$ is the canonical map sending $[P] - [Q]$ to $[A \otimes_{\mathbb{Z}\pi} P] - [A \otimes_{\mathbb{Z}\pi} Q]$. Since this map is onto by the next (well-known) lemma it follows that $F \rightarrow C(A)$ is onto and therefore an isomorphism. It also follows that $C(\mathbb{Z}\pi)/C(\mathbb{Z}\pi) \approx F_n$ where $C(\mathbb{Z}\pi)$ is the kernel of $C(\mathbb{Z}\pi) \rightarrow C(A)$.

Lemma 6.1. $C(\mathbb{Z}\pi) \rightarrow C(A)$ is onto.

Proof. Let $[P] - [Q]$ lie in $C(A)$. Find a sequence $0 \rightarrow P \rightarrow Q \rightarrow X \rightarrow 0$ where $X$ is finite of order prime to the index of $\mathbb{Z}\pi$ in $A$. Let $0 \rightarrow S \rightarrow F \rightarrow X \rightarrow 0$ be a resolution of $X$ over $\mathbb{Z}\pi$ with $F$ free. Then $S$ is projective and $[F] - [S]$ maps to $[P] - [Q]$ in $C(A)$. This follows by tensoring $A$ with $0 \rightarrow S \rightarrow F \rightarrow X \rightarrow 0$ over
\[\mathbb{Z}\pi\] and applying Schanuel’s lemma. Note that \(A \otimes_{\mathbb{Z}\pi} X \xrightarrow{\sim} X\) since locally either \(\mathbb{Z}\pi = A\) or \(X = 0\), and \(0 \rightarrow A \otimes_{\mathbb{Z}\pi} S \rightarrow A \otimes_{\mathbb{Z}\pi} F \rightarrow A \otimes_{\mathbb{Z}\pi} X \rightarrow 0\) is exact for the same reason. (This also follows from the fact that \(A \otimes_{\mathbb{Z}\pi} S\) is torsion free.) \(\Box\)

In the next section we will also need the following observation.

**Corollary 6.2.** The map \(\rho : C(A) \rightarrow F_\pi\) is an isomorphism.

**Proof.** Let \(M\) be a flabby \(\pi\)-lattice. Then \(M\) is invertible. By Theorem 5.1 (with \(n\) being the order of \(\pi\)) we see that \(\rho(M)\) lies in the image of \(\rho : C(A) \rightarrow F_\pi\). Since \(\rho(M) = -[M]\) in \(F_\pi\) this shows that \(\rho : C(A) \rightarrow F_\pi\) is onto. Since \(C(A) \approx F_\pi\) by Theorem 2.10 it follows that \(\rho : C(A) \rightarrow F_\pi\) is an isomorphism. \(\Box\)

### 7. Examples

In this section I will give examples to show that the invertibility conditions in Lemma 3.1 and Theorem 5.1 cannot be omitted.

As above let \(A\) be the integral closure of \(\mathbb{Z}\pi\) in \(\mathbb{Q}\pi\). If \(\pi\) has order \(n\) then \(A = \prod_{i=1}^{n} R_d\) where \(R_d = \mathbb{Z}[\zeta_d]\) with \(\zeta_d = e^{2\pi i d}\) and the generator \(\sigma\) of \(\pi\) maps to \(\zeta_d \in R_d\). The following lemma will be useful.

**Lemma 7.1.** Let \(R\) be a Dedekind ring and let \(A \supseteq R\) be a domain containing \(R\) and finite over \(R\). Let \(I\) be an ideal of \(A\) such that \(A = R + I\) i.e. \(R/J = A/I\) where \(J = R \cap I\). If \(I\) is principal then so is \(J\).

**Proof.** We can assume \(I \neq 0\). Since \(I\) is principal, we have an exact sequence \(0 \rightarrow A \rightarrow A \rightarrow A/I \rightarrow 0\). Regarding this as a sequence over \(R\) and noting that \(A\) is projective as an \(R\)-module we see that \([A/I] = [A] - [A] = 0\) in \(K_0(R)\). But \(A/I \approx R/J\) so \([R/J] = [R] - [J] = 0\). Since cancellation holds for finitely generated projective modules over a Dedekind ring we have \(J \approx R\). \(\Box\)

The following lemma is an extension of the example considered in [5].

**Lemma 7.2.** Let \(q\) be a prime such that \(R_q\) contains a non-principal prime ideal \(p\) whose norm is a prime \(p\). Let \(\mathfrak{P}\) be a prime ideal of \(R_{pq}\) extending \(p\). Then \(\mathfrak{P}\) is also non-principal and of norm \(p\) so that \(R_q/p \rightarrow R_{pq}/\mathfrak{P}\) is an isomorphism.

**Proof.** We have \(R_{pq} = R_q[\zeta_p] = R_q[x]/\Phi_p(x)\) so \(R_{pq}/pR_{pq} = \mathbb{F}_p[x]/\Phi_p(x) = \mathbb{F}_p[x]/(x - 1)^{p-1}\). This is local with residue field \(\mathbb{F}_p\) so there is a unique prime ideal \(\mathfrak{P}\) of \(R_{pq}\) lying over \(p\) and it has residue field \(\mathbb{F}_p\) and \(\mathfrak{P}\) is not principal otherwise Lemma 7.1 would imply that \(p\) was principal. \(\Box\)

**Remark 7.3.** Standard density theorems show that such an ideal \(p\) will exist whenever \(R_q\) has class number \(h \neq 1\). An explicit example used in [5] is given by \(q = 23\) and \(p = 47\). This is proved as follows: Since \(p = 1 \mod q\), \(\mathbb{F}_p\) contains a primitive \(q\)-th root of \(1\) so \(R_q\) has a prime ideal \(p\) with \(R_q/p \approx \mathbb{F}_p\). To see that \(p\) is non-principal let \(B = \mathbb{Z}[[\frac{1 + \sqrt{-23}}{2}]]\) which is a subring of \(R_q\). If \(p\) was principal then Lemma 7.1 would imply that \(p \cap B\) was principal but it is easy to check that there is no element of \(B\) with norm \(p\) so this is impossible. An similar argument is given in [5].

We make \(R_{pq}/\mathfrak{P}\) into a \(\mathbb{Z}\pi\)-module by \(\mathbb{Z}\pi \rightarrow R_{pq} \rightarrow R_{pq}/\mathfrak{P}\) and make \(R_q/p\) into a \(\mathbb{Z}\pi\)-module by \(\mathbb{Z}\pi \rightarrow R_q \rightarrow R_q/p\). Our examples are based on the following observation.
Lemma 7.4. Let $p$ and $q$ be as in Lemma 7.2 and let $\pi$ be cyclic of order $pq$. Then the natural map $R_q/p \to R_{pq}/\mathfrak{P}$ is a $\mathbb{Z}\pi$–isomorphism.

Proof. We can identify $R_q/p$ and $R_{pq}/\mathfrak{P}$ with $\mathbb{F}_p$ so that $R_q \to \mathbb{F}_p$ is the restriction of $R_{pq} \to \mathbb{F}_p$. The map $\mathbb{Z}\pi \to R_{pq} \to R_{pq}/\mathfrak{P}$ sends $\sigma$ to the image $\xi$ of $\zeta_{pq}$ in $\mathbb{F}_p$, while the map $\mathbb{Z}\pi \to R_q \to R_q/p$ sends $\sigma$ to the image $\eta$ of $\zeta_q$ in $\mathbb{F}_p$. Since $\zeta_q = \zeta_{pq}$ this shows that $\eta = \xi^p$. Since $\mathbb{F}_p$ satisfies the identity $x^p = x$, we have $\eta = \xi$ so the two maps are the same. □

Now $K_0(A) = \prod_{d|n} K_0(R_d)$. Since $R_d$ is a Dedekind ring, $K_0(R_d) = \mathbb{Z} \oplus C(R_d)$ where the class $(a)$ in $C(R_d)$ corresponds to $[R_d] - [a]$ in $K_0(R_d)$. So $K_0(A) = C(A) \oplus F$ where $F = \prod_{d|n} \mathbb{Z}$ is free abelian and $C(A) = \prod_{d|n} C(R_d)$.

Let $G_0(\mathbb{Z}\pi)$ be the Grothendieck group having generators $[M]$ for all finitely generated $\pi$–modules $M$ with relations $[M] = [M'] + [M'']$ for all short exact sequences $0 \to M' \to M \to M'' \to 0$. Define $K_0(A) \to G_0(\mathbb{Z}\pi)$ by sending $[M]$ to $[M]$ with $M$ considered as a $\mathbb{Z}\pi$–module. This gives us a map $C(A) \to G_0(\mathbb{Z}\pi)$.

Theorem 7.5 ([4, Corollary 6.1]). Let $p$ and $q$ be as in Lemma 7.2 and let $\pi$ be cyclic of order $pq$. Then $C(A) \to G_0(\mathbb{Z}\pi)$ is not injective.

Proof. We have $C(A) = C(\mathbb{Z}) \times C(R_p) \times C(R_q) \times C(R_{pq})$. The class of $\mathfrak{P}$ lies in $C(R_{pq})$ while the class of $p$ lies in $C(R_q)$. The images of these elements in $G_0(\mathbb{Z}\pi)$ are $[R_{pq}] - [\mathfrak{P}] = [R_{pq}/\mathfrak{P}]$ and $[R_q] - [p] = [R_q/p]$. By Lemma 7.4, these elements are the same and the lemma follows. □

Corollary 7.6. Let $p$ and $q$ be as in Lemma 7.2 and let $\pi$ be cyclic of order $pq$. Then $\rho : \mathcal{L}_\pi \to F_\pi$ does not satisfy $\rho(M) = \rho(M') + \rho(M'')$ for all short exact sequences of $\pi$–lattices.

Proof. If this condition was satisfied then $\rho : \mathcal{L}_\pi \to F_\pi$ would factor through $G_0(\mathbb{Z}\pi)$ and therefore so would $\rho : C(A) \to F_\pi$. Since $C(A) \to G_0(\mathbb{Z}\pi)$ is not injective, this contradicts Corollary 6.2. □

If $f$ is an endomorphism of a module $M$ we write $fM$ for the set of elements of $M$ annihilated by $f$. Let $\pi$ be a finite cyclic group of order $n$ with generator $\sigma$. If $M$ is a $\mathbb{Z}\pi$–lattice then $\Phi_n(\sigma)M$ is the largest $R_n$–lattice contained in $M$ and $(M/\Phi_n(\sigma))_0$ is the largest $R_n$–lattice which is a quotient of $M$. We give an example to show that these two $R_n$–lattices need not be isomorphic.

Lemma 7.7. Let $\pi$ be a finite cyclic group of order $n$ with generator $\sigma$. Suppose $X^n - 1 = f(X)g(X)$ in $\mathbb{Z}[X]$ Then $f(\sigma)\mathbb{Z}\pi = g(\sigma)\mathbb{Z}\pi$

Proof. Clearly $g(\sigma)M \subseteq f(\sigma)M$ for any $\mathbb{Z}\pi$–module $M$. Suppose $f(\sigma)h(\sigma) = 0$ in $\mathbb{Z}\pi$. Then $f(X)h(X) = (X^n - 1)k(X) = f(X)g(X)k(X)$ in $\mathbb{Z}[X]$ so $h(X) = g(X)k(X)$.

Theorem 7.8. Let $p$ and $q$ be as in Lemma 7.2 and let $\pi$ be cyclic of order $n = pq$. Then there is a $\mathbb{Z}\pi$–lattice $I$ such that $\Phi_n(\sigma)I$ is not isomorphic to $(I/\Phi_n(\sigma))_0$.

Proof. Let $I$ be the kernel of the map $\mathbb{Z}\pi \to R_n \to R_n/\mathfrak{P} = \mathbb{F}_p$. Then $(I/\Phi_n(\sigma))_0 \approx \mathfrak{P}$ since the sequence $0 \to I \to \mathbb{Z}\pi \to \mathbb{F}_p \to 0$ gives an exact sequence $I/\Phi_n(\sigma) \to R_n \to \mathbb{F}_p \to 0$ and the kernel of the left hand map is torsion since $\mathbb{Q}I = \mathbb{Q}\mathbb{Z}\pi$. On the other hand, $\Phi_n(\sigma)I \approx R_n$ since we have an exact sequence $0 \to I \to \mathbb{Z}\pi \to \mathbb{F}_p$ and, by Lemma 7.7, $\Phi_n(\sigma)\mathbb{Z}\pi = \Psi(\sigma)\mathbb{Z}\pi$ where $\Psi(X) = (X^n -$
1)/\Phi_n(X). By Lemma 7.7, we see that \(\Psi(\sigma) : \mathbb{Z}\pi \rightarrow \Psi(\sigma)\mathbb{Z}\pi\) has kernel \(\Phi_n(\sigma)\mathbb{Z}\pi\) so \(\Psi(\sigma)\mathbb{Z}\pi \approx R_n\) and \(\Psi(\sigma)\) maps to zero in \(\mathbb{F}_p\). The last statement follows from the fact that \(\Psi = \Phi_1\Phi_p\Phi_q = (X^q - 1)\Phi_p\) so \(\Psi(\zeta_n) = (\zeta_p - 1)\Phi_p(\zeta_n)\) but \(\zeta_p\) maps to 1 in \(\mathbb{F}_p\).

We now show that the hypothesis that \(M\) is invertible cannot be omitted from Theorem 5.1. We use the same \(I\) as in the proof of Theorem 7.8.

**Theorem 7.9.** Let \(I\) be the kernel of the map \(\mathbb{Z}\pi \rightarrow R_n \rightarrow \mathbb{F}_p\). Then
\[
\rho(I) \neq \sum_{d|n} \rho((I/\Phi_d(x)I)_{\mathfrak{p}})  
\]

*Proof.* Let \(\pi''\) be the quotient group of \(\pi\) of order \(q\). The map \(\mathbb{Z}\pi \rightarrow \mathbb{F}_p\) factors through \(\mathbb{Z}\pi''\). Let \(J\) be the kernel of the resulting map \(\mathbb{Z}\pi'' \rightarrow \mathbb{F}_p\). Chasing the diagram
\[
0 \longrightarrow I \longrightarrow \mathbb{Z}\pi \longrightarrow \mathbb{F}_p \longrightarrow 0  
\]

\[
\downarrow \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow  
\]

\[
0 \longrightarrow J \longrightarrow \mathbb{Z}\pi'' \longrightarrow \mathbb{F}_p \longrightarrow 0  
\]

gives the exact sequence \(0 \rightarrow I \rightarrow \mathbb{Z}\pi \oplus J \rightarrow \mathbb{Z}\pi'' \rightarrow 0\). By Lemma 3.1 we get \(\rho(I) + \rho(\mathbb{Z}\pi'') = \rho(\mathbb{Z}\pi) + \rho(J)\) so that \(\rho(I) = \rho(J)\). Now \(J\) is projective over \(\mathbb{Z}\pi''\) and therefore invertible so by Theorem 5.1 we have \(\rho(J) = \sum_{d|n} \rho((J/\Phi_d(\sigma)J)_{\mathfrak{p}})\). Now \(J/\Phi_d(\sigma)J\) is torsion for \(d = n\) and \(d = p\) and \((J/\Phi_d(\sigma)J)_{\mathfrak{p}}\) is free over \(R_1 = \mathbb{Z}\) so \(\rho(J) = \rho((J/\Phi_p(\sigma)J)_{\mathfrak{p}})\). The sequence \(0 \rightarrow J \rightarrow \mathbb{Z}\pi'' \rightarrow \mathbb{F}_p \rightarrow 0\) gives \(J/\Phi_p(\sigma)J \rightarrow R_q \rightarrow \mathbb{F}_p \rightarrow 0\) and the image of the left hand map is \((J/\Phi_p(\sigma)J)_{\mathfrak{p}} = \mathfrak{p}\) so we have \(\rho(I) = \rho(J) = \rho(\mathfrak{p})\). If Theorem 5.1 held for \(I\) we would have
\[
\rho(I) = \sum_{d|n} \rho((I/\Phi_d(\sigma)I)_{\mathfrak{p}}).  
\]

Since \(C(A) = \bigoplus_{d|n} C(R_d)\), Corollary 6.2 shows that \(F_\pi = \bigoplus_{d|n} \rho C(R_d)\) and the term \(\rho((I/\Phi_d(\sigma)I)_{\mathfrak{p}})\) lies in the summand \(\rho C(R_d)\). Since \(\rho(I) = \rho(\mathfrak{p})\) lies in \(\rho C(R_q)\), the other terms must be 0 so \(\rho((I/\Phi_n(\sigma)I)_{\mathfrak{p}}) = 0\). But in the previous section we showed that \((I/\Phi_n(\sigma)I)_{\mathfrak{p}} \approx \mathfrak{P}\). Since \(\mathfrak{P}\) is not principal and \(\rho : C(A) \rightarrow F_\pi\) is an isomorphism, this is a contradiction, proving the theorem. \(\square\)

**References**


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