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AUTOMORPHISMS AND NONINVARIANT PROPERTIES OF THE COMPUTABLY ENUMERABLE SETS

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To Bernard, Francine, and David Wald.

ABSTRACT

This thesis concerns automorphisms and noninvariant properties of the computably enumerable sets. We prove two results relating semilow sets and prompt degrees via automorphisms, one of which is complementary to a recent result of Downey and Harrington. We also show that the properties of quasicreativity, n-creativity, subcreativity, and effective simplicity are not invariant under automorphism, and that in fact every promptly simple set is automorphic to an effectively simple set. The techniques used in these proofs include a modification of the Harrington-Soare version of the method of Harrington-Soare and Cholak for constructing Δ^0_3 automorphisms; this modification takes advantage of a recent result of Soare on the extension of "restricted" automorphisms to full automorphisms.

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CHAPTER 1
SEMILOW SETS AND PROMPT DEGREES

1.1 Introduction

1.1.1 Background

A set $A \subseteq \omega$ is *computably enumerable (c.e.)* if its elements can be listed by an effective algorithm. The collection of computably enumerable sets can then be given two different kinds of structure, one based on their computational properties, and one based on their algebraic properties:

1. The c.e. Turing degrees form an upper semilattice $\mathcal{C}$ under $\leq_T$ (Turing reducibility), and

2. The c.e. sets form a lattice $\mathcal{E}$ under inclusion.

In 1944, Post [8] first looked at the connection between these structures; in the search for a noncomputable incomplete degree, he defined several new properties of c.e. sets (such as creativity, simplicity, and hyperhypersimplicity) that have turned out to be definable purely in terms of set inclusion. Ever since that time, there has been an ongoing program of examining the relationship between the structures of $\mathcal{C}$ and $\mathcal{E}$.

One fruitful area of research has been the study of automorphisms of $\mathcal{E}$. This began in the 1970s, starting with Soare [12], and being further developed by the work of Maass, Stob, Downey, Harrington, and others. This early work involved the construction of *effective* automorphisms of various sorts. Then in the mid 1990s, Harrington and Soare, and independently Cholak, introduced a powerful new method for constructing automophisms. This method, described in [6], combines the use of computable trees (as introduced by Lachlan) with previous automorphism methods to produce Δ^0_3 ($0''$-computable) automorphisms. This has allowed numerous results, which had not yielded to effective automorphism methods, to become attainable.

For any two c.e. sets A and B, if there an automorphism Φ of $\mathcal{E}$ such that $\Phi(A) = B$, we say that A *is automorphic to* B (denoted $A \simeq B$). Many past questions about automorphisms have concerned whether two c.e. sets of particular kinds may or may not be automorphic. For instance, Soare [12] showed that any two maximal c.e. sets A and B are automorphic, and more recently Downey and Stob [4] have shown that the same is true for hemimaximal sets. Numerous other results connect sets of different kinds; to take just one example, the above-mentioned method of Harrington-Soare and of Cholak was applied by them ([6], [2]) to show that every noncomputable c.e. set is automorphic to some high c.e. set.

1.1.2 Lowness and Promptness

In constructing an automorphism involving a given c.e. set A, we frequently wish to exert two forms of control regarding the enumeration of elements into A:

(1) We want to be able to guarantee that at least some "unwanted" elements will remain outside of A, and

(2) We want to be able to guarantee that at least some "wanted" elements eventually enter A.

Among the properties of c.e. sets that can particularly help us achieve (1) are the lowness properties:

Definition 1.1.1 A is *low* if $A' \in \mathbf{0}'$.

Definition 1.1.2 $\overline{A}$ is *semilow* if $\{e : W_e \cap \overline{A} \neq \emptyset\} \leq_T \mathbf{0}'$.

Among the properties that can help us with (2) are the promptness properties:

Definition 1.1.3 A is *promptly simple (p.s.)* if there are a computable function p and a computable enumeration $\{A_s\}_{s\in\omega}$ of A such that for all e,

$$W_e \text{ infinite} \implies (\exists s)(\exists x)[x \in W_{e,\text{ at } s} \cap A_{p(s)}].$$

Definition 1.1.4 A is *prompt* if $A \equiv_T B$ for some p.s. set B.

A classic result employing these properties is that of Maass [7]:

Theorem 1.1.5 (Maass, 1982) *If A and B are low (or even have semilow compliment) and are p.s. then $A \simeq B$.*

The first result of this paper deals with a similar case, but one with less information about our two sets:

Theorem 1.1.6 *For any c.e. set A which is low (or even has semilow complement) and for any promptly simple (p.s.) set C, there exists a c.e. set $B \leq_{\mathrm{T}} C$ with A automorphic to B (denoted by $A \simeq B$).*

The proof of this relies on a modification of the Δ^0_3-automorphism method described in [6]. A brief account of the machinery of this method will be given in §1.2.1. In §1.2.2, we will describe the major modification to the method that must be made; specifically, the construction will be *restricted* to the complements of the given set A and the constructed set B, and then the "New Extension Theorem" of Soare (Theorem 1.2.6) will be employed to show that this restricted construction implies the existence of a full automorphism between A and B. Then §1.2.3 will describe the other additions that are required, and §1.2.4 will describe the overall construction. The verification of this construction will be given in §1.3.

Theorem 1.1.6 gives rise to several corollaries. First of all, every c.e. degree contains a set with semilow compliment [14, p. 73]. Hence,

Corollary 1.1.7 $(\forall \mathbf{a} > \mathbf{0})(\forall$ *promptly simple* $C)(\exists A \in \mathbf{a})(\exists B \leq_{\mathrm{T}} C)[A \simeq B]$.

Also, as Downey and Cholak have observed, if we take two low p.s. degrees $\mathbf{d}$ and $\mathbf{c}$, then the lower cone of either one consists of sets that can fulfill the role of A above, and can therefore be automorphically mapped into the lower cone of the other. Hence,

Corollary 1.1.8 *There exist distinct noncomputable c.e. degrees* $\mathbf{d}$ *and* $\mathbf{c}$ *such that*

$$(\forall A \leq_T \mathbf{d})(\exists B \leq_T \mathbf{c})[A \simeq B] \;\;\&\;\; (\forall B \leq_T \mathbf{c})(\exists A \leq_T \mathbf{d})[A \simeq B].$$

1.1.3 Downey-Harrington and Theorem 1.1.11

A degree is said to be *prompt* if it contains a p.s. set, and *tardy* if it does not. Downey and Harrington have shown the following result on prompt low degrees and tardy degrees:

Theorem 1.1.9 (Downey-Harrington) *There are a prompt low degree* $\mathbf{c}$ *and tardy degree* $\mathbf{a}$ *such that*

$$(\forall B \in \mathbf{c})(\forall A \leq_T \mathbf{a})[A \not\simeq B].$$

Theorem 1.1.6 is complementary to this result, in that it shows that Theorem 1.1.9 cannot be extended to the cone below $\mathbf{c}$. That is, Theorem 1.1.6 has the following corollary:

Corollary 1.1.10 *For any prompt degree* $\mathbf{c}$ *(low or otherwise), and any degree* $\mathbf{a}$ *(tardy or otherwise), there exist some* $A \leq_T \mathbf{a}$ *(in fact, some* $A \in \mathbf{a}$*) and some* $B \leq_T \mathbf{c}$ *such that* $A \simeq B$*.*

Proof. Take A to be a set in $\mathbf{a}$ with semilow complement, and C to be a p.s. set in $\mathbf{c}$, and apply Theorem 1.1.6. ■

Theorem 1.1.9, in turn, places a restriction on how Theorem 1.1.6 may be extended. In the latter theorem, we show that for $\overline{A}$ semilow and C p.s., A is automorphic to some $B \leq_T C$. This raises the question of whether such a B can always be found so that B is not only computable from C, but actually Turing equivalent to C. Theorem 1.1.9 shows that this cannot always be done.

However, suppose we have an A that not only has semilow compliment, but is also promptly simple. Then we do get the following result:

Theorem 1.1.11 *If* A *is low (or has semilow complement) and p.s. and* C *is p.s., then* $(\exists B \equiv_T C)[A \simeq B]$*.*

We prove this in §1.4, with the aid of Maass's Theorem 1.1.5.

1.2 Proof Machinery

1.2.1 The Harrington-Soare Machinery

Since our work closely follows the methods of Harrington and Soare [6], it is necessary to review some of their machinery.

First of all, we consider the construction of our automorphism to be a game between two players, RED and BLUE. RED constructs two complete collections of the c.e. sets, $\{U_n\}_{n\in\omega}$ and $\{V_n\}_{n\in\omega}$, and BLUE in response constructs $\{\widehat{U}_n\}_{n\in\omega}$ and $\{\widehat{V}_n\}_{n\in\omega}$, so that our final automorphism will map each U_n to $\widehat{U}_n$ and each $\widehat{V}_n$ to V_n.

These constructions will be performed by enumerating elements into the appropriate sets. From one copy of the natural numbers (ω) RED enumerates elements into U_n and BLUE into $\widehat{V}_n$; from a second copy ($\widehat{\omega}$) RED enumerates elements into V_n and BLUE into $\widehat{U}_n$. To keep track of what sets a given x or $\hat{x}$ is in, we define the full e-state of x, as follows:

Definition 1.2.1 Given two sequences of r.e. sets $\{X_n\}_{n\in\omega}$ and $\{Y_n\}_{n\in\omega}$, define $\nu(e,x)$, the *full e-state* of x with respect to (w.r.t.) $\{X_n\}_{n\in\omega}$ and $\{Y_n\}_{n\in\omega}$ to be the triple $\langle e, \sigma(e,x), \tau(e,x)\rangle$, where

$$\sigma(e,x) = \{i : i \leq e \ \ \& \ \ x \in X_i\}, \text{ and}$$

$$\tau(e,x) = \{i : i \leq e \ \ \& \ \ x \in Y_i\}.$$

To guarantee that a construction of this type produces an automorphism, it suffices to show that the states on one side that are "well resided" (contain infinitely many elements) correspond exactly to the "well resided" states on the other side. That is, that

$$(\forall \nu)(\exists^\infty x \in \omega)[\nu(e,x) = \nu \ w.r.t. \ \{U_n\}_{n\in\omega} \text{ and } \{\widehat{V}_n\}_{n\in\omega}]$$
$$\Longleftrightarrow (\exists^\infty \hat{x} \in \hat{\omega})[\nu(e,\hat{x}) = \nu \ w.r.t. \ \{\widehat{U}_n\}_{n\in\omega} \text{ and } \{V_n\}_{n\in\omega}].$$

Second, this construction is performed on a tree T. Each node α has attached to it certain guesses regarding what states will be "well visited" (will have infinitely many elements enter them) and what states will nonetheless be "emptied" (will be well visited but not well resided). The nodes on the *true path* f will in the end be those that have guessed correctly. As the construction progresses, there are successive approximations f_s to f.

In addition, elements travel along the tree either downwards or leftwards (the latter generally when f_s shifts leftwards). We may thus describe for each node α, at each stage s, the following sets:

$$S_{\alpha,s} = \{x : \alpha(x,s) = \alpha\}$$

(the elements x "at" node α),

$$R_{\alpha,s} = \{x : \alpha(x,s) \supseteq \alpha\}$$

(the elements x at or below node α), and

$$Y_{\alpha,s} = \bigcup \{R_{\alpha,t} : t \leq s\}$$

(the elements x that have ever been at node α).

We also have the duals $\widehat{S}_{\alpha,s}$, $\widehat{R}_{\alpha,s}$, and $\widehat{Y}_{\alpha,s}$ of these, for elements $\hat{x}$.

For technical reasons, we will be dividing the elements that enter $S_{\alpha,s}$ into two sets, $S^0_{\alpha,s}$ and $S^1_{\alpha,s}$ (the former is used as a source of elements for coding and such purposes, while the latter provides the elements to be passed on down to successor nodes).

The elements of $R_{\alpha,s}$ ($\widehat{R}_{\alpha,s}$) may then be enumerated into sets U_α and $\widehat{V}_\alpha$ ($\widehat{U}_\alpha$ and V_α), which represent approximations to U_n and $\widehat{V}_n$ ($\widehat{U}_n$ and V_n), for $|\alpha| = 5n+1$ and $5n+2$ respectively. (That is, we will have $U_\alpha =^* U_n$ and $V_\alpha =^* V_n$ by construction, and $\widehat{U}_n = \widehat{U}_\alpha$ and $\widehat{V}_n = \widehat{V}_\alpha$ by definition, for any $\alpha \subset f$.)

For α in general, we let e_α ($\hat{e}_\alpha$) be the greatest n such that $5n+1$ ($5n+2$) $\leq |\alpha|$. (Thus, $e_\alpha > e_{\alpha^-}$ exactly when there exist sets U_α and $\widehat{U}_\alpha$, and $\hat{e}_\alpha > \hat{e}_{\alpha^-}$ exactly when there exist sets $\widehat{V}_\alpha$ and V_α.) Then to keep track of our α-sets we define *α-states* as follows:

Definition 1.2.2 (i) The *α-state of x at stage s*, $\nu(\alpha, x, s)$, is the triple $\langle \alpha, \sigma(\alpha, x, s), \tau(\alpha, x, s) \rangle$ where

$$\sigma(\alpha, x, s) = \{e_\beta : \beta \subseteq \alpha \;\;\&\;\; e_\beta > e_{\beta^-} \;\;\&\;\; x \in U_{\beta,s}\},$$

$$\tau(\alpha, x, s) = \{\hat{e}_\beta : \beta \subseteq \alpha \;\;\&\;\; \hat{e}_\beta > \hat{e}_{\beta^-} \;\;\&\;\; x \in \widehat{V}_{\beta,s}\}.$$

(ii) The *final α-state of x* is $\nu(\alpha, x) = \langle \alpha, \sigma(\alpha, x), \tau(\alpha, x) \rangle$ where $\sigma(\alpha, x) = \lim_s \sigma(\alpha, x, s)$ and $\tau(\alpha, x) = \lim_s \tau(\alpha, x, s)$.

For technical reasons, we will also sometimes need a modified version of $\nu(\alpha, x, s)$ or $\nu(\alpha, \hat{x}, s)$ that depends solely on α^-. Where $\beta = \alpha^-$, if $e_\alpha > e_\beta$ then we define $\nu^+(\alpha, x, s)$ to be the result of replacing $U_{\alpha,s}$ in the definition of $\nu(\alpha, x, s)$ with the similar set $Z_{e_\alpha,s}$ defined by

$$Z_{e_\alpha,s+1} =_{\text{dfn}} \{x : x \in U_{e_\alpha,s+1} \;\;\&\;\; x \in Y^1_{\beta,s}\}.$$

If $\hat{e}_\alpha > \hat{e}_\beta$ we similarly define $\nu^+(\alpha, \hat{x}, s)$ to be the result of replacing $\widehat{U}_{\alpha,s}$ in the definition of $\nu(\alpha, \hat{x}, s)$ with $\widehat{Z}_{\hat{e}_\alpha,s}$ defined by

$$\widehat{Z}_{\hat{e}_\alpha,s+1} =_{\text{dfn}} \{\hat{x} : \hat{x} \in V_{\hat{e}_\alpha,s+1} \;\;\&\;\; \hat{x} \in \widehat{Y}^1_{\beta,s}\}.$$

It is in terms of α-states that our guessing machinery for each node is phrased. For every node α, we have the following *lists* (sets of α-states):

1. $\mathcal{M}_\alpha$, α's guess about what α-states are well visited;

2. $\mathcal{B}_\alpha$, α's guess about what α-states are emptied by BLUE;

3. $\mathcal{R}_\alpha$, α's guess about what α-states are emptied by RED;

4. $\mathcal{N}_\alpha = \mathcal{B}_\alpha \cup \mathcal{R}_\alpha$, α's guess about what α-states are emptied overall;

and all of their duals. Our construction will guarantee that the guesses $\mathcal{M}_\alpha$, $\widehat{\mathcal{M}}_\alpha$, $\mathcal{N}_\alpha$, and $\widehat{\mathcal{N}}_\alpha$ are correct for all α on the true path. For any α our guess $\widehat{\mathcal{M}}_\alpha$ always equals $\{\hat{\nu} : \nu \in \mathcal{M}_\alpha\}$ and $\widehat{\mathcal{N}}_\alpha$ always equals $\{\hat{\nu} : \nu \in \mathcal{N}_\alpha\}$ (we will sometimes loosely state these correspondences as $\widehat{\mathcal{M}}_\alpha = \mathcal{M}_\alpha$ and $\widehat{\mathcal{N}}_\alpha = \mathcal{N}_\alpha$); we thus will guarantee that the sets of genuinely well resided states of both sides will likewise correspond. We will designate these sets of genuinely well resided states by $\mathcal{K}_\alpha$ and $\widehat{\mathcal{K}}_\alpha$. (In this we depart slightly from the notation of [6], in which $\mathcal{K}_\alpha$ and $\widehat{\mathcal{K}}_\alpha$ referred to the *complements* of these sets.)

In addition, we have the following machinery to aid in our construction:

$$\mathcal{E}_\alpha = \{\nu : (\exists^\infty x)(\exists s)[x \in S_{\alpha,s} - \bigcup \{S_{\alpha,t} : t < s\} \ \& \ \nu(\alpha, x, s) = \nu]\}$$

(the set of all α-states that are well visited by elements first coming into S_α), for $j = 1, 2$,

$$\mathcal{E}_\alpha^j = \{\nu : (\exists^\infty x)(\exists s)[x \in S_{\alpha,s}^j - \bigcup \{S_{\alpha,t} : t < s\} \ \& \ \nu(\alpha, x, s) = \nu]\}$$

(the set of all α-states that are well visited by elements first coming into S_α in S_α^j), and

$$\mathcal{F}_\alpha = \{\nu : (\exists^\infty x)(\exists s)[x \in R_{\alpha,s} \ \& \ \nu(\alpha, x, s) = \nu]\}$$

(the set of all α-states that are genuinely well visited by elements of R_α). For $\alpha \subset f$, we will have $\mathcal{E}_\alpha = \mathcal{E}_\alpha^0 = \mathcal{E}_\alpha^1 = \mathcal{F}_\alpha = \mathcal{M}_\alpha$.

Also, for technical reasons, we have:

1. A list $\mathcal{F}_\beta^+$ similar to $\mathcal{F}_\alpha$ that is attached to the node $\beta = \alpha^-$ and depends only on β. If $e_\alpha > e_\beta$ then we define

$$\mathcal{F}_\beta^+ = \{\nu : (\exists^\infty x)(\exists s)[x \in Y_{\beta,s}^1 \ \& \ \nu^+(\alpha, x, s) = \nu]\},$$

and $\widehat{\mathcal{F}}_\beta^+ = \{\hat{\nu} : \nu \in \mathcal{F}_\beta^+\}$; if $\hat{e}_\alpha > \hat{e}_\beta$ then we define $\widehat{\mathcal{F}}_\beta^+$ by the dual of the above expression and define $\mathcal{F}_\beta^+ = \{\nu : \hat{\nu} \in \widehat{\mathcal{F}}_\beta^+\}$.

2. A number k_α that is attached to the node α; k_α is a guess at an upper bound for the set of all x's and $\hat{x}$'s that are permanently in a non-well-resided α-state.

These are used in the course of defining f and f_s. Specifically, f and f_s are defined using a recursive collection of c.e. sets $\{C_\alpha\}$, with $\alpha \subset f$ iff $|C_\alpha| = \infty$ and $\alpha \subset f_{s+1}$ iff $\alpha^- \subset f_{s+1}$ and $|C_{\alpha,s+1}| > |C_{\alpha,s}|$. These $\{C_\alpha\}$ are defined so that if $|C_\alpha| = \infty$ then we automatically have that

1. $\mathcal{M}_\alpha = \mathcal{F}^+_{\alpha^-}$,
2. $\widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{F}}^+_{\alpha^-}$,
3. $\mathcal{R}_\alpha$ and $\widehat{\mathcal{R}}_\alpha$ are correct guesses, and
4. k_α is a correct guess.

Thus, to verify that our construction indeed yields an automorphism, we need only separately verify that f is infinite and that for all $\alpha \subset f$,

1. $\mathcal{F}^+_{\alpha^-} = \mathcal{F}_\alpha$,
2. $\widehat{\mathcal{F}}^+_{\alpha^-} = \widehat{\mathcal{F}}_\alpha$, and
3. $\mathcal{B}_\alpha$ and $\widehat{\mathcal{B}}_\alpha$ are correct guesses.

The final notions we must define are those of consistency; if a node α is inconsistent, then it is terminal (and we must ensure that it is not on the true path).

Definition 1.2.3 A node $\alpha \in T$ is $\mathcal{M}$-*inconsistent* if $e_\alpha > e_\beta$, where $\beta = \alpha^-$, and there are α-states $\nu_0 <_B \nu_1$ such that $\nu_0 \in \mathcal{M}_\alpha$ and $\nu_1 \restriction \beta \in \mathcal{M}_\beta$ but $\nu_1 \notin \mathcal{M}_\alpha$. Otherwise α is $\mathcal{M}$-*consistent.*

Definition 1.2.4 A node $\alpha \in T$ is $\mathcal{R}$-*consistent* if

$$(\forall \nu_0 \in \mathcal{R}_\alpha)(\exists \nu_1)[\nu_0 <_R \nu_1 \ \ \& \ \ \nu_1 \in \mathcal{M}_\alpha].$$

That is, α is $\mathcal{M}$-consistent if every state a BLUE move away from a putative well visited state is well visited, and $\mathcal{R}$-consistent if every state which RED is supposed to empty has a state where RED can put the elements it is supposed to be removing. We also have the corresponding dual notions of $\widehat{\mathcal{M}}$-consistency and $\widehat{\mathcal{R}}$-consistency. In our modified version of the Harrington-Soare construction, a node α is said to be *consistent* if it is $\mathcal{M}$-consistent, $\widehat{\mathcal{M}}$-consistent, $\mathcal{R}$-consistent, and $\widehat{\mathcal{R}}$-consistent. (In this, our definition of consistency differs from the definition in [6], which also includes a notion of $\mathcal{C}$-consistency or $\mathcal{D}$-consistency, which makes it possible to do certain kinds of coding.)

1.2.2 The New Extension Theorem

We now add to the techniques of [6] the notion of restriction to $\overline{A}$ and $\overline{B}$, which will enable us to use the New Extension Theorem. In [6], elements are enumerated into c.e. sets U_α, $\widehat{U}_\alpha$, V_α, and $\widehat{V}_\alpha$ (for $\alpha \in T$) by two players, RED and BLUE. In this construction, where $\rho = f \upharpoonright 1$, U_ρ and $\widehat{U}_\rho$ will, in the end, equal A (up to finite difference) and B respectively; $U_\rho =^* A$ because almost every element of A is eventually enumerated into U_ρ, and B is defined to equal $\widehat{U}_\rho$.

For technical reasons, we will find it advantageous to modify this construction slightly, so that

1. We know during the course of the construction exactly what ρ is,
2. At every stage s, $f_s \upharpoonright 1 = \rho$, and
3. We can therefore define enumerations $\{A_s\}$ and $\{B_s\}$ of A and B by setting $A_s = U_{\rho,s}$ and $B_s = \widehat{U}_{\rho,s}$.

We do this by modifying the definition of C_α for all α such that $\alpha = 1$. Let ρ_1 be the node such that $|\rho_1| = 1$, and

1. $\mathcal{M}_{\rho_1} = \{\langle \rho_1, \emptyset, \emptyset\rangle, \langle \rho_1, \{0\}, \emptyset\rangle\}$,
2. $\widehat{\mathcal{M}}_{\rho_1} = \{\langle \rho_1, \emptyset, \emptyset\rangle, \langle \rho_1, \{0\}, \emptyset\rangle\}$, and

3. $k_{\rho_1} = 0$.

That is, node ρ_1 guesses that

1. Both U_{ρ_1} and its complement, considered as ρ_1-states, will be well visited,

2. Both $\widehat{U}_{\rho_1}$ and its complement, considered as ρ_1-states, will be well visited, and

3. No element will remain in a non-well-visited ρ_1-state.

(Note that we need not specify $\mathcal{R}_{\rho_1}$, $\widehat{\mathcal{R}}_{\rho_1}$, $\mathcal{B}_{\rho_1}$, and $\widehat{\mathcal{B}}_{\rho_1}$, because these are the same for all α with $|\alpha| = 1$.) We then define $C_{\rho_1} = \omega$, and $C_\alpha = \emptyset$ for all other α with $|\alpha| = 1$.

This will indeed guarantee that $\rho = \rho_1$, and that $f_s \restriction 1 = \rho_1$ for all s; we will then define A_s and B_s as above. However, because we have redefined C_α for $|\alpha| = 1$, we now must verify the statements that the definition of C_α used to guarantee automatically; that is, we must separately verify the following *tree properties*:

1. $\mathcal{M}_\rho = \mathcal{F}^+_\lambda$,

2. $\widehat{\mathcal{M}}_\rho = \widehat{\mathcal{F}}^+_\lambda$, and

3. k_ρ is a correct guess.

(Again, we do not have to verify that $\mathcal{R}_\rho$ and $\widehat{\mathcal{R}}_\rho$ are correct guesses, because these are the same for all α with $|\alpha| = 1$.) We will do this as part of the verification of our construction, in §1.3.1 and §1.3.4.

Since we have now identified A with U_ρ and B with $\widehat{U}_\rho$, we may thus speak of A-states, $\overline{A}$-states, B-states, and $\overline{B}$-states, which will be those states including U_ρ, not including U_ρ, including $\widehat{U}_\rho$, and not including $\widehat{U}_\rho$, respectively. We can then speak of the *restriction* of a list of states to A, $\overline{A}$, B, or $\overline{B}$; for any list $\mathcal{S}$ ($\widehat{\mathcal{S}}$), $\mathcal{S}^A$ ($\widehat{\mathcal{S}}^B$) is the list of all A-states (B-states) in $\mathcal{S}$ ($\widehat{\mathcal{S}}$), and $\mathcal{S}^{\overline{A}}$ ($\widehat{\mathcal{S}}^{\overline{B}}$) is the list of all $\overline{A}$-states ($\overline{B}$-states) in $\mathcal{S}$ ($\widehat{\mathcal{S}}$).

So, for example, for any α we can consider $\mathcal{M}_\alpha$ as a disjoint union of its restriction to A, $\mathcal{M}^A_\alpha$, and its restriction to $\overline{A}$, $\mathcal{M}^{\overline{A}}_\alpha$, and we may treat these two sublists separately; similarly, $\widehat{\mathcal{M}}_\alpha$ can be considered as $\widehat{\mathcal{M}}^B_\alpha \sqcup \widehat{\mathcal{M}}^{\overline{B}}_\alpha$. (In particular, when we show that $\widehat{\mathcal{M}}_\rho = \widehat{\mathcal{F}}_\lambda^+$, we will do so by separately showing that $\widehat{\mathcal{M}}_\rho^{\overline{B}} = \widehat{\mathcal{F}}_\lambda^{+,\overline{B}}$ and that $\widehat{\mathcal{M}}_\rho^B = \widehat{\mathcal{F}}_\lambda^{+,B}$.)

Our designation of A-states, $\overline{A}$-states, and so forth also allows us to consider the moves in any RED/BLUE game to be divided into the following three types:

1. *"Exterior" move.* A enumeration into $U_{\rho,s+1} = A_{s+1}$ or $\widehat{U}_{\rho,s+1} = B_{s+1}$. (That is, a move from an A-state into an $\overline{A}$-state, or from a B-state into a $\overline{B}$-state.)

2. *$\overline{A}/\overline{B}$ move.* An enumeration into U_α or $\widehat{V}_\alpha$ ($|\alpha| > 1$) of some $x \in \overline{A}_s$ or an enumeration into $\widehat{U}_\alpha$ or V_α ($|\alpha| > 1$) of some $\hat{x} \in \overline{B}_s$. (That is, a movement between $\overline{A}$-states or between $\overline{B}$-states.)

3. *A/B move.* An enumeration into U_α or $\widehat{V}_\alpha$ ($|\alpha| > 1$) of some $x \in A_s$ or an enumeration into $\widehat{U}_\alpha$ or V_α ($|\alpha| > 1$) of some $\hat{x} \in B_s$. (That is, a movement between A-states or between B-states.)

However, suppose we only ever consider moves of type 1 and type 2, and completely ignore moves of type 3. This process can be considered a RED/BLUE game that is very much like the RED/BLUE game in [6], except that

1. It is entirely restricted to $\overline{A}$ and $\overline{B}$ (that is, every object in the machinery of [6] is replaced by its restriction to $\overline{A}$ or $\overline{B}$), and

2. Elements occasionally are removed entirely from the game by exterior moves.

Such a construction would only guarantee a correspondence between the $\overline{A}$-states and the $\overline{B}$-states. However, it turns out that if we also can guarantee a correspondence between the exterior moves on the two sides, then this is enough to guarantee the existence of an overall automorphism. The idea is that if we have elements entering A and B in "the same states," then a suitable set of type 3 moves, taking advantage of this supply of elements, could be added to guarantee a correspondence between A-states and B-states as well.

In order to keep track of this correspondence between exterior moves, we define for every α the sets $\mathcal{G}_\alpha^A$ and $\widehat{\mathcal{G}}_\alpha^B$:

$$\mathcal{G}_\alpha^A = \{\nu : (\exists^\infty x)(\exists s)[x \in A_{s+1} - A_s \ \ \& \ \ \nu = \nu(\alpha, x, s) \ \ \& \ \ x \in Y_{\alpha,s}]\},$$

and $\widehat{\mathcal{G}}_\alpha^B$ is defined correspondingly for B. That is, $\mathcal{G}_\alpha^A$ ($\widehat{\mathcal{G}}_\alpha^B$) is the set of all α-states ν ($\hat{\nu}$) from which infinitely many elements are enumerated directly into A (B).

To further formalize this idea, we must also introduce the notion of an *enumeration*:

Definition 1.2.5 Given a computable priority tree T with (infinite) true path $f \in [T]$, an *enumeration* $\mathbb{E}$ for T is a simultaneous computable enumeration of c.e. sets U_α, V_α, and $\widehat{U}_\alpha$, $\widehat{V}_\alpha$, for $\alpha \in T$, such that $\{U_\alpha\}_{\alpha\subset f}$ and $\{V_\alpha\}_{\alpha\subset f}$ are both complete collections of the c.e. sets up to finite differences.

The computable enumerations made while playing an $\overline{A}/\overline{B}$ game thus define some enumeration $\mathbb{E}$, and we want to show that if $\mathbb{E}$ guarantees a correspondence of $\overline{A}$-states and $\overline{B}$-states and of exterior moves on both sides, then we are guaranteed the existence of an enumeration giving rise to a complete automorphism. The New Extension Theorem (NET) states that this is indeed the case:

Theorem 1.2.6 (New Extension Theorem, Soare) *If an enumeration* $\mathbb{E}$ *satisfies:*

(T1) $\quad \mathcal{K}^{\overline{A}} = \widehat{\mathcal{K}}^{\overline{B}}$, *and*

(T2) $\quad \mathcal{G}^A = \widehat{\mathcal{G}}^B$,

then $A \cong B$ *by an automorphism of* $\mathcal{E}$ *which extends* $\mathbb{E}$.

The advantage of the NET, then, is that we automatically get an automorphism between A and B if we can play our restricted game, and guarantee that the true path f is infinite, and that on f

1. $\mathcal{M}^{\overline{A}}$, $\widehat{\mathcal{M}}^{\overline{B}}$, $\mathcal{N}^{\overline{A}}$, and $\widehat{\mathcal{N}}^{\overline{B}}$ are all correct guesses, and

2. $\mathcal{G}^A = \widehat{\mathcal{G}}^B$.

For the former, we will apply a form the proof used in [6] to show that the overall RED/BLUE game causes $\mathcal{M}$, $\widehat{\mathcal{M}}$, $\mathcal{N}$, and $\widehat{\mathcal{N}}$ all to be correct on the true path, modified slightly to take into account the occasional removal of elements by exterior moves. The latter we will ensure by an appropriate choice of exterior moves.

1.2.3 Additional Machinery

Our tree construction will employ the following machinery:

1. We have all of the machinery of Chapter 2 of [6], with the given modification to the definitions of "consistent" and of C_α, and restricted to $\overline{A}$ and $\overline{B}$. (Henceforth, when this will not cause confusion, we shall adopt the convention of omitting the superscript $\overline{A}$ and $\overline{B}$ that indicates this restriction; *e.g.*, we shall refer to $\mathcal{M}_\alpha^{\overline{A}}$ as $\mathcal{M}_\alpha$. This convention will be adopted in §1.2.4 and §1.3.2.)

2. We have a fixed enumeration $\{\widetilde{A}_s\}_{s\in\omega}$ of A.

3. In the course of our construction we will also be producing an enumeration $\{A_s\}_{s\in\omega}$ of our given A and defining B by an enumeration $\{B_s\}_{s\in\omega}$, as described in §1.2.2. A_s (B_s) will thus be equal to the set of all elements x ($\hat{x}$) that have been removed from our restricted construction at stage s by exterior moves.

4. We have a list $\mathcal{L}^{\mathcal{G}}$ of pairs of the form $\langle\alpha, \hat{\nu}\rangle$, used to satisfy the requirement that if $\nu \in \mathcal{G}_\alpha^A$ then $\hat{\nu} \in \widehat{\mathcal{G}}_\alpha^B$. Every time an element of some α-state ν_1 enters A, the pair $\langle\alpha, \hat{\nu}_1\rangle$ is added to $\mathcal{L}^{\mathcal{G}}$. We also guarantee that if $\langle\alpha, \hat{\nu}_1\rangle$ appears infinitely many times in $\mathcal{L}^{\mathcal{G}}$, then infinitely many elements from $\hat{\nu}_1$ are put into B.

5. For every node α, for every $\overline{A}$-state ν of α, we have the following infinite sets of markers and c.e. sets, which will be used, together with the semilowness of $\overline{A}$, to guarantee that the removal of elements from $\overline{A}$ does not interfere with our construction (by a technique introduced by Robinson [9], and used for example by Soare in [13]):

(a) For each $i \in \omega$, the marker $\Gamma^{1,\alpha}_{\nu,i}$ and c.e. set $Q^{1,\alpha}_{\nu,i}$. One marker of this type will be attached to any element x that enters S_α in α-state ν.

(b) For each $i \in \omega$, the marker $\Gamma^{2,\alpha}_{\nu,i}$ and c.e. set $Q^{2,\alpha}_{\nu,i}$. One marker of this type will be attached to any element x that enters α-state ν (whether it has just entered S_α or not).

(c) For each $i \in \omega$, the marker $\Gamma^{3,\alpha}_{\nu,i}$ and c.e. set $Q^{3,\alpha}_{\nu,i}$. One marker of this type will be attached to any element x in S_{α^-} if $\nu^+(x,\alpha)$ becomes equal to α-state ν.

By the Recursion Theorem, we will also be able to assume for this construction that we know indices $q^{1,\alpha}_{\nu,i}$, $q^{2,\alpha}_{\nu,i}$, and $q^{3,\alpha}_{\nu,i}$ uniformly for the sets $Q^{1,\alpha}_{\nu,i}$, $Q^{2,\alpha}_{\nu,i}$, and $Q^{3,\alpha}_{\nu,i}$ respectively.

6. For every node α, for every $\overline{B}$-state $\hat{\nu}$ of α, we will be constructing an infinity of c.e. sets $J^\alpha_{\nu,i}$. (These will help us find elements of C to permit necessary elements to enter B.) By the Recursion Theorem and the Slowdown Lemma [14, p. 284] we may assume that we know indices $j^\alpha_{\nu,i}$ uniformly for all these c.e. sets, such that any given element appears in $W_{j^\alpha_{\nu,i}}$ strictly later than we enumerate it into $J^\alpha_{\nu,i}$.

1.2.4 Construction

This construction is based on that in [6]. Throughout this construction, as mentioned in §1.2.3, we will adopt the convention of omitting the superscripts $\overline{A}$ and $\overline{B}$ that indicate restriction to a $\overline{A}/\overline{B}$ game.

Let $\hat{h}$ be the 0–1 valued function such that for all e, $W_e \cap \overline{A} \neq 0$ if and only if $\hat{h}(e) = 1$. By semilowness of $\overline{A}$, $\hat{h} \equiv_T \mathbf{0}'$, so by the Limit Lemma there must exist a computable function h such that $\lim_s h(e,s)$ exists and equals $\hat{h}(e)$ (see [14, p. 72]); we fix such a computable function h.

Because C is prompt, by the Promptly Simple Degree Theorem [14, p. 284] there exists a computable function p such that

$$W_e \text{ infinite } \implies (\exists x)(\exists s)[x \in W_{e,\,\text{at } s} \;\;\&\;\; C_s{\upharpoonright}x \neq C_{p(s)}{\upharpoonright}x].$$

We fix some such function p.

Every stage $s+1$ in our construction will be divided into steps as in [6]. Now, however, we must distinguish two different types of steps:

- *Interior steps.* These are the same as the corresponding numbered steps in [6], except that they are restricted to $\overline{A}$ and $\overline{B}$. In other words, a game played with the interior steps alone would be simply a RED/BLUE game played on the $\overline{A}$- and $\overline{B}$-states. Steps 1–5, $\widehat{1}$–$\widehat{5}$, and 11 of the present construction are of this type.

- *Exterior steps.* These are steps by which elements are moved into A_s or B_s. That is, if we consider just the RED/BLUE game played on $\overline{A}$- and $\overline{B}$-states, elements are allowed to disappear entirely from the game during exterior steps. In the current construction, the exterior steps are Step 0, which moves elements into A_s when necessary, and Step $\widehat{8}$, which moves elements into B_s in order to guarantee that $\mathcal{G}^A = \widehat{\mathcal{G}}^B$.

Our construction is then as follows:

Stage $s = 0$. For all $\alpha \in T$ define $U_{\alpha,0} = V_{\alpha,0} = \widehat{U}_{\alpha,0} = \widehat{V}_{\alpha,0} = \emptyset$, and define $m(\alpha, 0) = 0$. Define $Y_{\lambda,0} = \widehat{Y}_{\lambda,0} = \emptyset$, and $f_0 = \lambda$. Define every $Q^{\alpha}_{\nu,i,0} = \emptyset$ and every marker $\Gamma^{\alpha}_{\nu,i,0}$ to be unassigned. Define every $J^{\alpha}_{\nu,i} = \emptyset$. Define $A_0 = B_0 = \emptyset$.

Stage $s+1$. Find the least $n < 11$ such that Step n applies to some $x \in Y_{\alpha,s}$ and perform the intended action. If there is no such n, then find the least $n < 11$ such that Step $\hat{n}$ applies to some $\hat{x} \in \widehat{Y}_{\alpha,s}$, and perform the indicated action. If none of these steps applies, then apply Step 11, and go to stage $s+2$.

Step 0 (Moving elements into A).

Substep 0.1 (Enumerated elements). If $x \in (Y_{\lambda,s} \cap \widetilde{A}_{s+1}) - (Y_{\lambda,s-1} \cap \widetilde{A}_s)$, (0.1.1) Where x is in α-state ν, add to $\mathcal{L}^{\mathcal{G}}$ a pair $\langle \beta, \hat{\nu}\restriction\beta\rangle$ for every $\beta \subseteq \alpha$,

(0.1.2) Enumerate x into A_{s+1}, and

(0.1.3) Designate every Γ-marker attached to x as unassigned.

Substep 0.2 (Assigning a Γ-marker to an x believed not to go into A). In the following, to *challenge* x with regard to marker type j (= 1, 2, 3) and α-node ν means to do the following:

(i) Where i is the least number such that the marker $\Gamma^{j,\alpha}_{\nu,i}$ is currently unassigned, enumerate x into $Q^{j,\alpha}_{\nu,i}$.

(ii) Find the least t such that either

(a) $h(q^{j,\alpha}_{\nu,i}, t)\downarrow = 1$ or

(b) $x \in \widetilde{A}_t$.

In case (a), assign marker $\Gamma^{j,\alpha}_{\nu,i}$ to x. In case (b),

(iii) If $j = 1$ or 2, add the pair to $\mathcal{L}^{\mathcal{G}}$ a pair $\langle \beta, \hat{\nu}\restriction\beta\rangle$ for every $\beta \subseteq \alpha$; if $j = 3$, add a pair $\langle \beta, \hat{\nu}\restriction\beta\rangle$ for every $\beta \subseteq \alpha^-$;

(iv) Enumerate x into A_{s+1} immediately; and

(v) Designate every Γ-marker attached to x as unassigned.

Then Substep (0.2) consists of repeating the following three steps:

(0.2.1) If some element x is to be moved into some Y_α in $\overline{A}$-state ν by Step 1 or 2, then challenge x with regard to marker type 1 and α-state ν.

(0.2.2) If some element x is to be put into $\overline{A}$-state ν by one of Steps 1–5 or 11C, then challenge x with regard to marker type 2 and α-state ν.

(0.2.3) If there is some element x such that, as a result of x being enumerated into U_{e_α} and/or the action of Steps 1–5, 11C, or 11E, $\nu^+(x, \alpha)$ will become equal to $\overline{A}$-state ν, then challenge x with regard to marker type 3 and α-state ν.

until none of these three challenges described enters case (b) (that is, none of them causes an element to enter A_{s+1}).

Steps 1–5, $\widehat{\mathbf{1}}$–$\widehat{\mathbf{5}}$. As in [6], restricted to $\overline{A}$ and $\overline{B}$.

Step $\widehat{\mathbf{8}}$. (Moving elements into B.)

Find the first unmarked pair $\langle \alpha, \hat{\nu} \rangle$ in $\mathcal{L}^{\mathcal{G}}$ such that there exists some $\hat{x} \in \widehat{S}^0_{\alpha,s}$ in state $\hat{\nu}$. Then

($\widehat{8}$.1) Enumerate the least such $\hat{x}$ into $J^{\alpha}_{\nu,i}$.

($\widehat{8}$.2) Where $\hat{x} \in W_{j^{\alpha}_{\nu,i}, \text{at } t}$, if $C_t \upharpoonright x \neq C_{p(t)} \upharpoonright x$, then enumerate x into B_{s+1} and mark the current copy of $\langle \alpha, \hat{\nu} \rangle$.

Step 11. As in [6], restricted to $\overline{A}$ and $\overline{B}$.

1.3 Verification

1.3.1 Tree Properties (Restricted Version)

As mentioned in §1.2.2, there are certain properties of the tree T (the *tree properties*) that hold automatically in the construction of [6], but which in our construction must be proved to hold for ρ; namely

1. $\mathcal{M}_\rho = \mathcal{F}^+_\lambda$,
2. $\widehat{\mathcal{M}}_\rho = \widehat{\mathcal{F}}^+_\lambda$, and
3. k_ρ is a correct guess.

In order to employ the NET, which holds for an enumeration $\mathbb{E}$ performed on a tree T defined as in [6], we need to show that these hold as given. However, our verfication of the correctness of $\mathcal{M}^{\overline{A}}$, $\widehat{\mathcal{M}}^{\overline{B}}$, $\mathcal{N}^{\overline{A}}$, and $\widehat{\mathcal{N}}^{\overline{B}}$ only requires the use of the *restrictions* to $\overline{A}$ and $\overline{B}$ of these tree properties. It will be most convenient, therefore, to verify the tree properties in separate stages:

1. In this section, we verify the restrictions of these tree properties to $\overline{A}$ and $\overline{B}$.
2. In §1.3.2 we use these tree properties to verify the correctness of $\mathcal{M}^{\overline{A}}$, etc.
3. In §1.3.3 we use the above verification in our verification that $\mathcal{G}^A = \widehat{\mathcal{G}}^B$.

4. In §1.3.4, we use the correspondence between $\mathcal{G}^A$ and $\widehat{\mathcal{G}}^B$ to show the full, unrestricted form of these tree properties.

We assume, first of all, that A is infinite and coinfinite, for otherwise A is recursive and Theorem 1.1.6 trivially holds. We now verify the following:

Lemma 1.3.1 $\mathcal{M}_\rho = \mathcal{F}_\lambda^+$

Proof. By Substep 11E, every element x of ω eventually enters Y_λ. Every element of the infinite set A eventually enters some A_s by Step 0, and no element of the infinite set $\overline{A}$ ever does. Thus, there are infinitely many x such that for some s, $x \in Y_{\lambda,s}$ and $x \in A_s$, and there are infinitely many x such that for some s, $x \in Y_{\lambda,s}$ and $x \notin A_s$, so $\mathcal{F}_\lambda^+ = \{\langle \rho_1, \emptyset, \emptyset\rangle, \langle \rho_1, \{0\}, \emptyset\rangle\} = \mathcal{M}_{\rho_1} = \mathcal{M}_\rho$. ■

(In particular, then, $\mathcal{M}_\rho^{\overline{A}} = \mathcal{F}_\lambda^{+,\overline{A}}$.)

Lemma 1.3.2 $\widehat{\mathcal{M}}_\rho^{\overline{B}} = \widehat{\mathcal{F}}_\lambda^{+,\overline{B}}$.

Proof. By Substep 11E, every element $\hat{x}$ of $\widehat{\omega}$ eventually enters Y_λ without first entering any B_s. Thus, $\langle \rho_1, \emptyset, \emptyset\rangle \in \widehat{\mathcal{F}}_\lambda^+$, so $\widehat{\mathcal{F}}_\lambda^{+,\overline{B}} = \{\langle \rho_1, \emptyset, \emptyset\rangle\} = \widehat{\mathcal{M}}_{\rho_1}^{\overline{B}} = \widehat{\mathcal{M}}_\rho^{\overline{B}}$. ■

Lemma 1.3.3 *No element of $\overline{A}$ or $\overline{B}$ remains permanently in a non-well-visited ρ-state.*

Proof. If $x \in \overline{A}$ ($\hat{x} \in \overline{B}$), then x ($\hat{x}$) is permanently in the ρ-state $\{\langle \rho_1, \emptyset, \emptyset\rangle\}$, which we have just seen is well-visited. ■

(In other words, the guess $k_\rho = 0$ is correct.)

We have thus verified the restricted versions of the tree properties for ρ. Since all C_α, $|\alpha| > 1$, are defined as in [6], these properties therefore hold automatically for all $\alpha \supseteq \rho$; that is, for all $\alpha \supseteq \rho$,

1. $\mathcal{M}_\alpha^{\overline{A}} = \mathcal{F}_\alpha^{+,\overline{A}}$,

2. $\widehat{\mathcal{M}}_\alpha^{\overline{B}} = \widehat{\mathcal{F}}_\alpha^{+,\overline{B}}$, and

3. k_α is the upper bound for all $x \in \overline{A}$ and $\hat{x} \in \overline{B}$ that remain in a non-well-visited α-state.

1.3.2 Correctness of $\mathcal{M}^{\overline{A}}$, $\widehat{\mathcal{M}}^{\overline{B}}$, $\mathcal{N}^{\overline{A}}$, and $\widehat{\mathcal{N}}^{\overline{B}}$ on f

Throughout this section, we will once again adopt the convention mentioned in §1.2.3 of omitting the superscripts $\overline{A}$ and $\overline{B}$ that indicate restriction to a $\overline{A}/\overline{B}$ game.

It suffices to verify that the versions of Lemmas 5.1 through 5.12 of [6] hold for the $\overline{A}/\overline{B}$ game as they did for the overall game in the Harrington-Soare construction. To aid us in this, we first prove the following additional lemma:

Lemma 5.0. (All markers eventually receive permanent assignments.)

(i) If α-state $\nu \in \mathcal{E}_\alpha$ then there exists an infinite set $\{x_i\}_{i\in\omega} \subseteq \overline{A}$ such that

$$(\forall i)[\lim_s \Gamma^{1,\alpha}_{\nu,i,s} = x_i \;\;\&\;\; (\exists s)[x_i \in S_{\alpha,s} - Y_{\alpha,s-1} \;\;\&\;\; \nu(\alpha, x_i, s) = \nu]].$$

(ii) If α-state $\nu \in \mathcal{F}_\alpha$ then there exists an infinite set $\{x_i\}_{i\in\omega} \subseteq \overline{A}$ such that

$$(\forall i)[\lim_s \Gamma^{2,\alpha}_{\nu,i,s} = x_i \;\;\&\;\; (\exists s)[x_i \in R_{\alpha,s} \;\;\&\;\; \nu(\alpha, x_i, s) = \nu]].$$

(iii) If α-state $\nu \in \mathcal{F}^+_\alpha$ then there exists an infinite set $\{x_i\}_{i\in\omega} \subseteq \overline{A}$ such that

$$(\forall i)[\lim_s \Gamma^{3,\alpha}_{\nu,i,s} = x_i \;\;\&\;\; (\exists s)[x \in R_{\alpha,s} \;\;\&\;\; \nu^+(\alpha, x_i, s) = \nu]].$$

Proof. All of these proofs are similar; we therefore give just the proof for (i), which serves with appropriate modifications for the other two:

(i) Assume by induction that this is true for all $i' < i$, and let t be a stage by which

(a) $\lim_s \Gamma^{1,\alpha}_{\nu,i',s} = \Gamma^{1,\alpha}_{\nu,i',t'}$ for all $t' \geq t$, $i' < i$; and

(b) $\lim_s h(q^{1,\alpha}_{\nu,i}, s) = h(q^{1,\alpha}_{\nu,i}, t')$ for all $t' \geq t$.

Then at some stage $t' \geq t$, $\Gamma^{1,\alpha}_{\nu,i}$ must be assigned to some x (either $\Gamma^{1,\alpha}_{\nu,i}$ is already assigned at stage t, or the next x to enter S_α in state ν must have $\Gamma^{1,\alpha}_{\nu,i}$ assigned to it by Step (0.2.1)). Thus, $\lim_s h(q^{1,\alpha}_{\nu,i}, s) = 1$ (for if it were 0, Step (0.2.1) could not assign a marker). Hence $Q^{1,\alpha}_{\nu,i} \cap \overline{A} \neq \emptyset$, so some y eventually goes into $Q^{1,\alpha}_{\nu,i}$ that does not eventually go into A. When this y goes into $Q^{1,\alpha}_{\nu,i}$, it has $\Gamma^{1,\alpha}_{\nu,i}$ assigned to it (since

y does not go into A), and since y never enters A this marker is permanently assigned to y.

Since only finitely many $\Gamma_\nu^{1,\alpha}$ markers may be attached to a given y, the set $\{x_i\}_{i\in\omega}$, where x_i is the y to which marker $\Gamma_{\nu,i}^{1,\alpha}$ is permanently assigned, must be infinite. ∎

We now examine Lemmas 5.1–5.12 individually.

The first three of these hold just as before:

Lemma 5.1. At stage $s+1$,

(i) if x enters R_α, $\alpha \neq \lambda$, then Step 1 or Step 2 applies to α and x;

(ii) if x moves from S_α to S_δ then one of the following steps must apply to x: Step 1_δ for $\delta <_{\mathrm{L}} \alpha$ or $\delta^- = \alpha$; Step 2_δ for δ such that $\delta^- = \alpha$; or Step 11_α Substep C applying to α, so $f_{s+1} <_{\mathrm{L}} \alpha$; and in the second two cases x enters S^1_δ;

(iii) if $x \in S_{\alpha,s}$ is enumerated in a red set U_α at stage $s+1$ then Step 1 or Step 4 must apply to x;

(iv) if $x \in S_{\alpha,s}$ is enumerated in a blue set $\widehat{V}_\alpha$ then Step 1, Step 3, Step 5, or Step n must apply to x.

Lemma 5.2 (True Path Lemma) $\quad f = \liminf_s f_s$.

Lemma 5.3 For all $\alpha \in T$,

(i) $f <_L \alpha \implies R_{\alpha,\infty} = \emptyset$,

(ii) $\alpha <_L f \implies Y_\alpha =^* \emptyset$,

(iii) $\alpha \subset f \implies Y_{<\alpha} =_{\mathrm{dfn}} \bigcup \{Y_\delta : \delta <_L \alpha\} =^* \emptyset$.

Lemma 5.4: (i) through (iv) hold as before. However, since it is now possible for an element x to disappear from the game by being enumerated (by an exterior move) into A (B in the dual lemma), we must slightly modify the statement of (v) by restricting x to elements of $\overline{A}$ ($\overline{B}$ in the dual), as shown:

Lemma 5.4. For every $\alpha \in T$ if $\alpha \neq \lambda$ and $\beta = \alpha^-$ then

(i) $Y_\alpha \setminus Y_\beta = \emptyset$ and $Y_\alpha \subseteq Y_\beta$,

(ii) $(\forall x)(\exists^{\leq 1} s)[x \in R_{\alpha,s+1} - R_{\alpha,s}]$,

(iii) $(\forall x)(\exists^{\leq 1} s)[x \in S^0_{\alpha,s+1} - S^0_{\alpha,s}]$,

(iv) $U_\alpha \setminus Y_\alpha = \widehat{V}_\alpha \setminus Y_\alpha = \emptyset$, and

(v) $\alpha \subset f \implies (\exists v_\alpha)(\forall x \in \overline{A})(\forall s \geq v_\alpha)[x \in R_{\alpha,s} \implies (\forall t \geq s)[x \in R_{\alpha,t}]]$ (and correspondingly with $\overline{B}$ in the dual).

Lemma 5.5 must be similarly restricted:

Lemma 5.5. For all $x \in \overline{A}$

(i) $\alpha(x) =_{\mathrm{dfn}} \lim_s \alpha(x, s)$ exists, and

(ii) x is enumerated in at most finitely many r.e. sets U_γ, $\widehat{V}_\gamma$, and hence for $\alpha = \alpha(x)$,

$$\nu(\alpha, x) =_{\mathrm{dfn}} \lim_s \nu(\alpha, x, s) \text{ exists.}$$

(And similarly with $\overline{B}$ in the dual.)

Lemma 5.6 is the same as before (though now "n" can include 0):

Lemma 5.6. (i) Step 11 applies infinitely often.

(ii) If the hypotheses of some Step 1–5, n (Step $\widehat{1}$–$\widehat{5}$, $\hat{n}$) remain satisfied then that step eventually applies.

The statement of Lemma 5.7 is as before:

Lemma 5.7. If $\alpha \subset f$, $\alpha \neq \lambda$, and $\beta = \alpha^-$ then

(i) $(\forall \gamma <_L f)[m(\gamma) =_{\mathrm{dfn}} \lim_s m(\gamma, s) < \infty]$,

(ii) $m(\alpha) =_{\mathrm{dfn}} \lim_s m(\alpha, s) = \infty$,

(iii) $\mathcal{E}_\alpha \supseteq \mathcal{M}_\alpha = \mathcal{F}^+_\beta$,

(iv) $\widehat{\mathcal{E}}_\alpha \supseteq \widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{F}}^+_\beta$, and

(v) $\mathcal{E}_\alpha = \mathcal{E}^0_\alpha = \mathcal{E}^1_\alpha$ and $\widehat{\mathcal{E}}_\alpha = \widehat{\mathcal{E}}^0_\alpha = \widehat{\mathcal{E}}^1_\alpha$.

However, the proof requires a slight modification. In the original, the proof of (ii) required the proof of an additional claim:

Claim 1. Every α-entry $\langle \alpha, \nu_1 \rangle$ on $\mathcal{L}$ ($\langle \alpha, \widehat{\nu}_1 \rangle$ on $\widehat{\mathcal{L}}$) is eventually marked.

The proof of this in the non-dual case must be modified slightly, since it is now possible for elements to leave the $\overline{A}/\overline{B}$ game before they can enter S_α. We use Lemma 5.0(iii) to guarantee a supply of elements ($\{x_i\}_{i \in \omega}$) that remain in $\overline{A}$ because their Γ^3-tags are never removed.

Its second paragraph now reads:

Now $\nu_1 \in \mathcal{M}_\alpha$ since $\langle \alpha, \nu_1 \rangle \in \mathcal{L}$. Also $\mathcal{M}_\alpha = \mathcal{F}_\beta^+$, since $\alpha \subset f$. Hence we have the infinite collection of elements $\{x_i\}_{i \in \omega} \subset \overline{A}$ described in Lemma 5.0(iii). By the choice of s_1 almost every such x_i also satisfies (1.1)–(1.7). Thus, some such x_i is moved to S_α under Step 1 at some stage $s + 1 > s_1$, and the entry $\langle \alpha, \nu_1 \rangle$ is then marked, contrary to hypothesis. This establishes the claim for $\mathcal{L}$.

(The proof in the dual case need not be altered, because we never enumerate any element of $\widehat{S}^1_{\beta,s}$ into B.)

Lemma 5.8: The statement and proof of this lemma must be modified:

Lemma 5.8. $\alpha \subset f \implies$

(i) $R_{\alpha,\infty} =^* Y_\alpha \cap \overline{A} =^* Y_\lambda \cap \overline{A} = \overline{A}$; and

(ii) Y_α is infinite. (And similarly for the dual lemma, with B for A.)

Proof. By Lemma 5.6(i) Step 11E must eventually put every element $x \in \omega$ into Y_λ. By induction we may assume that $R_{\beta,\infty} =^* Y_\beta \cap \overline{A} =^* \overline{A}$ and Y_β is infinite, for $\beta = \alpha^-$. By Lemma 5.7 $m(\alpha) = \infty$ and $m(\gamma) < \infty$ for all $\gamma <_L \alpha$ with $\gamma^- = \beta$.

By Lemma 5.3, $Y_{<\alpha} =^* 0$. For any x that is in S^0_β at some stage and is never moved into $Y_{<\alpha}$ by Step 11C, x is eventually moved either into A by Step 0, or into S^1_β by Step 11D. (For the dual case, this should read "$\hat{x}$ is eventually moved either into B by Step $\widehat{8}$, or into $\widehat{S}^1_\beta$ by Step 11D.") For any x that is in S^1_β at some stage and is never moved into $Y_{<\alpha}$ by Step 11C, x is eventually moved either into A by Step 0, or into S_α by Step 1 or Step 2. (For the dual case, this should read "$\hat{x}$ is eventually moved into $\widehat{S}_\alpha$ by Step $\widehat{1}$ or Step $\widehat{2}$, since there is no possibility of $\hat{x}$ being moved into B from $\widehat{S}^1_\beta$.)

Thus, almost every $x \in R_\beta$ not yet in R_α that never enters A will eventually enter S_α. By Lemma 5.4(v) almost every such x will remain in R_α forever.

To see that Y_α is infinite, observe that since Y_β is infinite, and there are only finitely many β-states, in particular some β-state ν_1 must be well-visited. Then by Lemma 5.0(ii), we have an infinite collection $\{x_i\}_{i \in \omega}$ of elements of ν_1 that never enter A. By the above reasoning, almost all of these must eventually enter S_α. ∎

The proof of the dual case is nearly the same, except that we make the indicated changes in the second paragraph, and the last paragraph is replaced by the following:

To see that $\widehat{Y}_\alpha$ is infinite, observe that since $\widehat{Y}_\beta$ is infinite, infinitely many elements must enter $\widehat{S}_\beta$ via Step $\widehat{1}$ or Step $\widehat{2}$. Elements entering $\widehat{S}_\beta$ in a given β-state $\hat{\nu}$ by Step $\widehat{1}$ alternate between going into $\widehat{S}^0_\beta$ and going into $\widehat{S}^1_\beta$, while all elements entering $\widehat{S}_\beta$ by Step $\widehat{2}$ go into $\widehat{S}^1_\beta$. Thus, infinitely many elements must enter $\widehat{S}^1_\beta$. By the above reasoning, almost all of these must eventually enter $\widehat{S}_\alpha$. ∎

The proof of Lemma 5.9 must be modified somewhat, and the modifications differ slightly in the two cases:

Lemma 5.9. $\alpha \subset f \implies \alpha$ is $\mathcal{M}$-consistent.

Proof. Let $\alpha \subset f$ and $\beta = \alpha^-$. Assume for a contradiction that α is not $\mathcal{M}$-consistent. Then $e_\alpha > e_\beta$ and there exist $\nu_0 \in \mathcal{M}_\alpha$, $\nu_1 \notin \mathcal{M}_\alpha$, $\nu_0 <_B \nu_1$ and $\nu_1 \restriction \beta \in \mathcal{M}_\beta$. By equation (42) of [6] α is a terminal node on T so $S_\alpha = R_\alpha$. Thus, by Lemma 5.4(v), for some υ_α, no $x \in S_{\alpha,s} \cap \overline{A}$ later leaves S_α.

By Lemma 5.7 $\nu_0 \in \mathcal{E}_\alpha$. Thus, by Lemma 5.0(i), we have an infinite set $\{x_i\}_{i\in\omega} \subseteq \overline{A}$ such that

$$(\forall i)(\exists s)[x_i \in S_{\alpha,s+1} - S_{\alpha,s} \ \ \& \ \ \nu(\alpha, x_i, s+1) = \nu_0].$$

Let x be any such x_i with the corresponding $s > \upsilon_\alpha$.

None of Steps 0–2 can move x at any stage $t > s$. Thus, Step 3_α must eventually apply to x at some stage $t+1 > s+1$, moving x from ν_0 either to ν_1, or to some other state ν_1' such that $\nu_0 <_B \nu_1'$ and $\nu_1' \restriction \beta \in \mathcal{M}_\beta$ and $\nu_1' \notin \mathcal{M}_\alpha$. Then α is provably incorrect at all stages $v \geq t+1$, so $\alpha \not\subset f$. ∎

The proof of the dual is the same, except that the second paragraph should read as follows:

By Lemma 5.7, $\hat{\nu}_0 \in \widehat{\mathcal{E}}^1_\alpha$. Take any $x \in \widehat{S}_{\alpha,s+1} - \widehat{S}_{\alpha,s}$ such that $\nu(\alpha, x, s+1) = \nu_0$ and $s > \upsilon_\alpha$.

Lemma 5.10 is just as in [6]:

Lemma 5.10. If $\alpha \subset f$ then

(i) $\widehat{\mathcal{M}}_\alpha = \{\hat{\nu} : \nu \in \mathcal{M}_\alpha\}$,

(ii) $\mathcal{M}_\alpha = \mathcal{F}_\alpha = \mathcal{E}_\alpha$, and

(iii) $\widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{F}}_\alpha = \widehat{\mathcal{E}}_\alpha$.

The proof of Lemma 5.11 must be modified much as the proof of Lemma 5.9 was.

Lemma 5.11. $\alpha \subset f \implies \alpha$ is $\mathcal{R}$-consistent.

Proof. Assume for a contradiction that $\alpha \subset f$ and α is not $\mathcal{R}$-consistent. Choose $\nu_1 \in \mathcal{R}_\alpha$ such that for all $\nu_2 \in \mathcal{M}_\alpha$, $\nu_1 \not<_R \nu_2$. By equation (42) of [6] α is a terminal node on T so $S_\alpha = R_\alpha$. Thus, by Lemma 5.4(v), for some υ_α, no $x \in S_{\alpha,s} \cap \overline{A}$ later leaves S_α.

Now $\nu_1 \in \mathcal{R}_\alpha \subseteq \mathcal{M}_\alpha = \mathcal{E}_\alpha$ by Lemma 5.10. Thus, by Lemma 5.0(i), we have an infinite set $\{x_i\}_{i\in\omega} \subseteq \overline{A}$ such that

$$(\forall i)(\exists s)[x_i \in S_{\alpha,s+1} - Y_{\alpha,s} \ \ \& \ \ \nu(\alpha, x_i, s+1) = \nu_1].$$

Let x be any such x_i with the corresponding $s > \upsilon_\alpha$. Note that Step 0 will not apply to x at any stage $t > s+1$ because $x \in \overline{A}$.

Because $x \in S_{\alpha,t}$ for all $t > s+1$, neither Step 1 nor Step 2 can apply to x at any stage $t > s+1$. Step 3 cannot apply to $x \in S_{\alpha,t}$ because α is $\mathcal{M}$-consistent by Lemma 5.9. Furthermore, Step 5 cannot apply to $x \in S_{\alpha,t}$ while $\nu(\alpha, x, t) = \nu_1$ because $\nu_1 \in \mathcal{R}_\alpha$ and $\mathcal{R}_\alpha \cap \mathcal{B}_\alpha = \emptyset$. But if $\nu(\alpha, x, t) = \nu_1$ for all $t \geq s$ then x witnesses that $F(\alpha^-, \nu_1)$ fails so $\nu_1 \in \mathcal{R}_\alpha$ contradicts $\alpha \subset f$. Hence, Step 4 applies to $x \in S_{\alpha,t}$ at some stage $t+1 > s+1$ such that $\nu_1 = \nu(\alpha, x, s) = \nu(\alpha, x, t)$, $\nu_2 = \nu(\alpha, x_i, t+1)$, and $\nu_1 <_R \nu_2$.

Choose ν_2 such that this happens for infinitely many $x \in \{x_i\}_{i\in\omega}$. Now $\nu_2 \in \mathcal{F}_\alpha$ so $\nu_2 \in \mathcal{M}_\alpha$ by Lemma 5.10. ∎

The proof of the dual requires that we change the second paragraph of this proof to the following:

Now $\hat{\nu}_1 \in \widehat{\mathcal{R}}_\alpha \subseteq \widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{E}}_\alpha = \widehat{\mathcal{E}}^1_\alpha$ by Lemmas 5.10 and 5.7(v), so

$$(\exists^\infty \hat{x})(\exists s > v_\alpha)[\hat{x} \in \widehat{S}^1_{\alpha,s+1} - \widehat{Y}_{\alpha,s} \;\;\&\;\; \hat{\nu}(\alpha, \hat{x}, s+1) = \hat{\nu}_1].$$

Take any such $\hat{x}$. Note that Step $\widehat{8}$ will not apply to $\hat{x}$ at any stage $t > s+1$ because $\hat{x} \in \widehat{S}^1_{\alpha,t}$.

Finally, we have Lemma 5.12, just as before:

Lemma 5.12. If $\alpha \subset f$ and $\nu_1 \in \mathcal{B}_\alpha$ then $\{x : x \in Y_\alpha \;\;\&\;\; \nu(\alpha, x) = \nu_1\} =^* \emptyset$.

We now reason very much as in the proof of Lemma 5.13 in [6]: Any $\alpha \subset f$ is consistent, by Lemmas 5.9, 5.11, and their duals; thus, f is infinite. Now, take any $\alpha \subset f$. Lemma 5.10 and its dual guarantee the correctness of the guesses $\mathcal{M}_\alpha$ and $\widehat{\mathcal{M}}_\alpha$. Lemma 5.12 and its dual guarantee the correctness of $\mathcal{B}_\alpha$ and $\widehat{\mathcal{B}}_\alpha$, and $\mathcal{R}_\alpha$ and $\widehat{\mathcal{R}}_\alpha$ are guaranteed to be correct because the (restricted) tree properties hold on our tree T below ρ, so $\mathcal{N}_\alpha$ and $\widehat{\mathcal{N}}_\alpha$ are also guaranteed to be correct.

Thus, just as we had that $\mathcal{M}$, $\widehat{\mathcal{M}}$, $\mathcal{N}$, and $\widehat{\mathcal{N}}$ were correct on f, and f was infinite, in [6], we now have (abandoning our superscript convention) that $\mathcal{M}^{\overline{A}}$, $\widehat{\mathcal{M}}^{\overline{B}}$, $\mathcal{N}^{\overline{A}}$, and $\widehat{\mathcal{N}}^{\overline{B}}$ are correct on f, and f is infinite.

1.3.3 Verifying that $\mathcal{G}^A = \widehat{\mathcal{G}}^B$

It suffices to verify two lemmas:

Lemma 1.3.4 *For any node α and α-state ν_1, $\mathcal{L}^{\mathcal{G}}$ contains infinitely many pairs $\langle \alpha, \hat{\nu}_1 \rangle$ if and only if $\nu_1 \in \mathcal{G}^A_\alpha$.*

Proof. Such a pair is added to $\mathcal{L}^{\mathcal{G}}$ exactly when Step 0 enumerates some $x \in \nu_1$ into A. Thus, $\mathcal{L}^{\mathcal{G}}$ contains infinitely many such pairs if and only if infinitely many $x \in \nu_1$ are enumerated into A; that is, if and only if $\nu_1 \in \mathcal{G}^A$.

Lemma 1.3.5 *For any node $\alpha \subset f$ and α-state ν_1, $\mathcal{L}^{\mathcal{G}}$ contains infinitely many pairs $\langle \alpha, \hat{\nu}_1 \rangle$ if and only if $\hat{\nu}_1 \in \widehat{\mathcal{G}}^B$.*

The proof of 1.3.5 will require the following additional lemma:

Lemma 1.3.6 *For any pair $\langle \alpha, \hat{\nu}_1 \rangle$ on list $\mathcal{L}^{\mathcal{G}}$, if $\langle \alpha, \hat{\nu}_1 \rangle$ is not eventually marked by Step $\widehat{8}$ then only finitely many elements x are enumerated into $J^{\alpha}_{\nu,i}$ by Step $\widehat{8}$.*

Proof. [1.3.6] Suppose that this is not the case for some $\langle \alpha, \hat{\nu} \rangle$. Then $J^{\alpha}_{\nu,i} = W_{j^{\alpha}_{\nu,i}}$ is an infinite c.e. set. Hence, by promptness of C, for some $\hat{x}$ and t we have that $\hat{x} \in W_{j^{\alpha}_{\nu,i}, \text{at } t}$ and $C_t \restriction \hat{x} \neq C_{p(t)} \restriction \hat{x}$. But then $\langle \alpha, \hat{\nu} \rangle$ must be marked by $(\widehat{8}.2)$ when $\hat{x}$ enters $J^{\alpha}_{\nu,i}$ at some stage $s < t$.

Proof. [1.3.5] To show the "if" part of this statement, we observe that

1. We do not move any element $\hat{x}$ in α-state $\hat{\nu}_1$ into B except when required to by some pair $\langle \gamma, \hat{\nu}'_1 \rangle$ in $\mathcal{L}^{\mathcal{G}}$ with $\alpha \subseteq \gamma$ and $\hat{\nu}_1 = \hat{\nu}'_1 \restriction \alpha$, and that

2. If there are infinitely many such pairs in $\mathcal{L}^{\mathcal{G}}$ then there are infinitely many pairs $\langle \alpha, \hat{\nu}_1 \rangle$ in $\mathcal{L}^{\mathcal{G}}$, since whenever we add one of the former we also add one of the latter.

To show the "only if" part, suppose that for our given α and ν_1, $\mathcal{L}^{\mathcal{G}}$ contains infinitely many pairs $\langle \alpha, \hat{\nu}_1 \rangle$. We will show that all of these pairs are eventually marked by $(\widehat{8}.2)$, and that therefore infinitely many elements $\hat{x}$ in state $\hat{\nu}$ are put by $(\widehat{8}.2)$ into B, *i.e.* that $\nu \in \widehat{\mathcal{G}}^B$.

Suppose that this is not the case, and take the first pair $\langle \alpha, \nu_1 \rangle$ that never gets marked. By some stage s,

1. Every pair in $\mathcal{L}^{\mathcal{G}}$ before that pair that will ever be marked already has been, and

2. For any $\langle \alpha', \hat{\nu}' \rangle$ before $\langle \alpha, \hat{\nu}_1 \rangle$ that will never be marked, all of the (by Lemma 1.3.6) finitely many $\hat{x}$ that will ever be enumerated into $J^{\alpha'}_{\nu',i'}$ by $(\widehat{8}.1)$ have been.

Now, since $\nu_1 \in \mathcal{G}^A \subseteq \mathcal{M}^{\overline{A}}$ and thus $\hat{\nu}_1 \in \widehat{\mathcal{M}^{\overline{B}}} = \widehat{\mathcal{E}}^{0,\overline{B}}$, we have that for infinitely many stages $t+1 > s$, some $\hat{x}$ is enumerated into $\widehat{S}^{0,\overline{B}}_{\alpha,t+1}$ by Step $\widehat{1}$. Neither Step $\widehat{1}$ nor Step $\widehat{2}$ can apply to $\hat{x}$ while it remains in $\widehat{S}^{0,\overline{B}}_{\alpha}$, and since α is $\widehat{\mathcal{M}}$-consistent (by Lemma 5.$\widehat{9}$) Step $\widehat{3}$ cannot apply to $\hat{x}$ either. Hence eventually each such element, still in $\hat{\nu}_1$, is enumerated into $J^{\alpha}_{\nu,i}$ by $\widehat{8}.2$. But then by Lemma 1.3.6 $\langle \alpha, \hat{\nu}_1 \rangle$ is eventually marked, and we have a contradiction.

1.3.4 Tree Properties (Unrestricted Version)

We proceed as in §1.3.1, showing that the tree properties of §1.2.2 hold for the $\alpha = \rho$ case, from which the rest follow automatically. The first was shown in §1.3.1:

Lemma 1.3.1. $\mathcal{M}_\rho = \mathcal{F}^+_\lambda$.

We now show the other two:

Lemma 1.3.7 $\widehat{\mathcal{M}}_\rho = \mathcal{F}^+_\lambda$.

Proof. Since A is infinite, $\mathcal{G}_\rho \neq \emptyset$, so $\widehat{\mathcal{G}}_\rho \neq \emptyset$, so B is infinite. Thus, since infinitely many elements $\hat{x}$ of $\widehat{Y}_\lambda$ enter $B = \widehat{U}_\rho$, $\langle \rho_1, \{0\}, \emptyset \rangle \in \widehat{\mathcal{F}}^+_\lambda$, so $\widehat{\mathcal{F}}^{+,B}_\lambda = \{\langle \rho_1, \{0\}, \emptyset \rangle\} = \widehat{\mathcal{M}}^B_{\rho_1} = \widehat{\mathcal{M}}^B_\rho$. By Lemma 1.3.2, we also have that $\widehat{\mathcal{M}}^{\overline{B}}_\rho = \widehat{\mathcal{F}}^{+,\overline{B}}_\rho$; together, these two results prove our lemma. ∎

Lemma 1.3.8 $k_\rho = 0$ *is a correct guess.*

Proof. Since both ρ-states are well-visited on both the ω and $\widehat{\omega}$ sides (by the above two lemmas), there are no elements of ω or $\widehat{\omega}$ remaining permanently in a non-well-resided ρ-state (there being no such state), so 0 is indeed the correct upper bound for the set of such elements. ∎

1.3.5 Verifying that $B \leq_T C$

For any $\hat{x}$, if $\hat{x} \in B$ then $\hat{x}$ must enter B_{s+1} at some stage $s+1$ by Step $\widehat{8}.2$. Then for some α, ν, and i, $\hat{x}$ enters $J^\alpha_{\nu,i}$ at stage $s+1$ and enters $W_{j^\alpha_{\nu,i}}$ at some stage $t+1 > s+1$ such that $C_{t+1} \upharpoonright \hat{x} \neq C_{p(t+1)} \upharpoonright \hat{x}$. Thus, given C, to determine if $\hat{x} \in B$ we need only find a stage $t+1$ such that $C_{t+1} \upharpoonright \hat{x} = C \upharpoonright \hat{x}$; we then have that $\hat{x} \in B$ if and only if $\hat{x} \in B_t$.

1.4 Proof of Theorem 1.1.11

We wish to prove the following:

Theorem 1.1.11. *If A is low (or has semilow complement) and p.s. and C is p.s., then* $(\exists B \equiv_T C)[A \simeq B]$.

One way to do this employs Maass's Theorem 1.1.5, and also the following easy fact, which we prove in this section (after proving it we discovered it to have been previously known to Cholak and perhaps others):

Theorem 1.4.1 *For any p.s. set C, there exists a p.s. set B with semilow complement such that $B \equiv_T C$.*

This gives us Theorem 1.1.11 as a corollary:

Proof. [Theorem 1.1.11.] Take any p.s. set A with semilow complement and any p.s. set C. By Theorem 1.4.1, we can find $B \equiv_T C$ such that B is p.s. and has semilow complement. Then by Theorem 1.1.5, $A \simeq B$. ∎

(A more direct proof of Theorem 1.1.11, not employing Theorem 1.1.5 or Theorem 1.4.1, might also be constructed. Briefly sketched, this involves combining the construction used in the proof of Theorem 1.1.6 with a step assigning coding markers permanently to *α-witnesses*, coding elements chosen from $\widehat{S}_\alpha^{0,\overline{B}}$ to be enumerated into B_{s+1} if $|\alpha| = n \in C_{s+1}$. The verifications that f is infinite and that $\mathcal{M}^{\overline{A}}$, $\widehat{\mathcal{M}}^{\overline{B}}$, $\mathcal{N}^{\overline{A}}$, and $\widehat{\mathcal{N}}^{\overline{B}}$ are correct guesses proceed approximately as before. The enumeration of α-witnesses into B might seem to raise a potential problem for the proof that $\mathcal{G}^A = \widehat{\mathcal{G}}^B$ by requiring states to exist in $\widehat{\mathcal{G}}^B$ that are not matched in $\mathcal{G}^A$; this turns out not to be the case, however, because the prompt simplicity of A requires $\mathcal{G}^A$ to equal all of $\mathcal{M}^{\overline{A}}$, for reasons that will be discussed in §2.5.4. We still have $B \leq_T C$ because every element entering B is either permitted by an element of C, as before, or is an α-witness coding an element of C. Finally, the coding guarantees that $C \leq_T B$.)

Theorem 1.4.1 has another corollary, incidentally, which was pointed out by Cholak:

Corollary 1.4.2 (Cholak) *For any two prompt degrees* $\mathbf{a}$ *and* $\mathbf{b}$*, there exist* $A \in \mathbf{a}$ *and* $B \in \mathbf{b}$ *such that* $A \simeq B$.

Proof. By Theorem 1.4.1 we can find $A \in \mathbf{a}$ and $B \in \mathbf{b}$ such that A and B are both p.s. with semilow complement. Then by Theorem 1.1.5, $A \simeq B$. ∎

We now prove Theorem 1.4.1:

Proof. [Theorem 1.4.1.]

Take any p.s. set C. Fix some enumeration $\{C_s\}$ of C. Then by the Promptly Simple Degree Theorem ([14], p. 284) there exists some computable function p such that for all s, $p(s) \geq s$, and for all e

$$|W_e| = \infty \implies (\exists x)(\exists s)[x \in W_{e,\,\mathrm{at}\ s} \ \ \& \ \ C_s \restriction x \neq C_{p(s)} \restriction x].$$

We now construct B as follows:

Write ω as the disjoint union of finite sets $H_0, H_1, H_2, ...$, where each H_n contains $2n+1$ elements. These will be our *coding sets*; if n enters C, we will encode this fact in B by enumerating at least one element of H_n into B. We will in this way guarantee that $C \leq_T B$.

We also have the following sets of requirements that our construction will fulfill:

$$N_e : |W_e \searrow B| = \infty \implies W_e \cap \overline{B} \neq \emptyset$$

(where by "$W_e \searrow B$" we mean the elements that either are in $W_e - B$ or are enumerated into W_e before being enumerated into B), and

$$P_e : |W_e| = \infty \implies (\exists x)(\exists s)[x \in W_{e,\,\mathrm{at}\ s} \cap B_s].$$

Fulfilling all the negative requirements N_e will guarantee that B is semilow; fulfilling all the positive requirements P_e will guarantee that B is p.s. (with its promptness function being the identity function). We order these requirements in a priority list $P_0, N_0, P_1, N_1, P_2, ...$; to fulfill one requirement we are only allowed to injure requirements of lower priority. (In this construction, in fact, only positive requirements are ever injured.)

We also define the sequence of markers $\Lambda_{e,s}$. In order to fulfill the requirement N_e, we may at stage $s+1$ put $\Lambda_{e,s+1}$ on some element x of $W_{e,s+1} - B_s - \bigcup_{i\leq e} E_i$, and attempt to prohibit x from ever entering B. This prohibition may be injured in order to fulfill P_i for some $i \leq e$, forcing Λ_e to be removed from x. However, if $W_e \setminus B$ is indeed infinite, then eventually some Λ_e-prohibition is permanent, and the requirement P_e is fulfilled.

Our construction will also be enumerating, for each e, a c.e. set E_e. By the Slowdown Lemma [14, p. 284] there exists a computable function h such that $E_e = W_{h(e)}$ for all e, and any element enumerated in E_e appears strictly later in $W_{h(e)}$. We may assume, by the Recursion Theorem, that we have this function available for our construction. We also define the speed-up function $g(e,s)$ such that for any e, for any stage s,

$$x \in E_{e,s+1} - E_{e,s} \iff x \in W_{h(e),g(e,s)+1} - W_{h(e),g(e,s)}.$$

(Thus $g(e,s) > s$ for all e, s.) These will be used, in conjunction with the p.s. property of C, to find elements of W_e that have permission from C to enter B, with which we can fulfill the requirement P_e while still keeping $B \leq_T C$.

Finally, in our construction we adopt the usual convention that at any stage s, there exists at most one e such that $W_{e,s+1} \neq W_{e,s}$, and in that case $|W_{e,s+1} - W_{e,s}| = 1$.

Construction.

Stage $s = 0$**.** Let $B_0 = \emptyset$. For all e, let $\Lambda_{e,0} = -1$, and $E_{e,0} = \emptyset$.

Stage $s+1$**.** (In the following, any item not otherwise specified remains the same at stage $s+1$ as at stage s.)

Step 1 (Fulfilling positive and negative requirements). If there exist x and e such that $x \in W_{e,s+1} - W_{e,s}$, $x \notin \bigcup_{j\leq e} H_j$, and $x \notin \{\Lambda_{j,s}\}_{j<e}$, then

Substep 1.1. If requirement P_e has not yet been acted on, then

(a) If $x \in B_s$, then mark requirement N_e as having been acted on, and end Step 1.

(b) Let $E_{e,s+1} = E_{e,s} \cup \{x\}$.

(c) If $C_{g(e,s)+1}\restriction x \neq C_{p(g(e,s)+1)}\restriction x$, then

(i) Enumerate x into B;

(ii) If $\Lambda_{j,s} = x$ for some $j \geq e$, then set $\Lambda_{j,s+1} = -1$;

(iii) Mark requirement P_e as having been acted on; and

(iv) End Step 1.

Substep 1.2. If $\Lambda_{e,s} = -1$, then set $\Lambda_{e,s+1} = x$.

Step 2 (Coding). If $e \in C_{s+1} - C_s$, enumerate into B every element of $H_e \cap \overline{B}_s$ with no attached Λ-marker.

(End of Construction.)

Verification. We must verify the following three lemmas:

Lemma 1.4.3 *B is promptly simple and semilow.*

Proof. It suffices to verify by induction that the following hold for all e:
(a) P_e is satisfied.
(b) $\Lambda_e =_{\text{dfn}} \lim_s \Lambda_{e,s}$ converges.
(c) N_e is satisfied.

(a) Suppose for a contradiction that P_e is not satisfied. Then W_e is infinite. Also, P_e is never marked as having been acted on, for it can only be so marked, by Substep 1.1(a) or (c), if there is some element x in both $W_{e,s+1} - W_{e,s}$ and B_{s+1}. Let s_0 be some stage such that for all $i < e$, $\Lambda_i = \Lambda_{i,s_0}$. Since $\{\Lambda_i\}_{i<e} \cup \bigcup_{i \leq e} H_i$ is finite, at infinitely many stages $t > s_0$ an element x of $W_{e,t+1} - W_{e,t}$ causes Substep 1.1 to be performed. No such x is in B_t, for this would satisfy P_e, so each such x is enumerated into E_e by Substep 1.1(b). Thus, $E_e = W_{h(e)}$ is infinite. But then for some x, s, $x \in W_{h(e), \text{at } s}$ and $C_s \restriction x \neq C_{p(s)} \restriction x$, and by definition of g this s must equal $g(t') + 1$ for some t' such that $x \in E_{t'+1} - E_{t'}$. But then x must enter B by Substep 1.1(c) at stage $t' + 1$, so P_e is satisfied.

(b) Let t be a stage at which every P_i with $i \leq e$ that will ever be acted on already has been. Then for any $t' \geq t$, if $\Lambda_{e,t'} \neq -1$ then $\Lambda_{e,t'+1} = \Lambda_{e,t'}$. Thus, either

(i) $(\forall t' \geq t)[\Lambda_{e,t'} = \Lambda_{e,t} \neq -1]$,

(ii) $(\forall t' \geq t)[\Lambda_{e,t'} = \Lambda_{e,t} = -1]$, or

(iii) $\Lambda_{e,t} = -1$ and $\Lambda_{e,t_0+1} \neq \Lambda_{e,t_0} = \Lambda_{e,t} = -1$ for some $t_0 \geq t$.

In cases (i) and (ii), $\Lambda_e = \Lambda_{e,t}$; in case (iii), $\Lambda_{e,t'} = \Lambda_{e,t_0+1}$ for all $t' > t_0$ and therefore $\Lambda_e = \Lambda_{e,t_0+1}$.

(c) Suppose that W_e is infinite, and take t as above. Take any $t' \geq t$ such that there exists an $x \in W_{e,t'+1} - W_{e,t'}$; then if $\Lambda_{e,t'} = -1$, $\Lambda_{e,t'+1} = x$ by Substep 1.2. Thus, case (ii) above cannot hold.

Let $s = t_0$ if case (iii) holds, and if case (i) holds let $s < t'$ be the earliest stage s such that $\Lambda_{e,t'} = \Lambda_{e,s+1}$ for all $t' > s$. Then $\Lambda_{e,s} \neq \Lambda_{e,s+1} \neq -1$, so Substep 1.2 must act on some $x \in W_{e,s+1} - W_{e,s}$ at Stage $s+1$. Then $x \notin B_s$. Moreover, Substep 1.1 does not put x into any $B_{t'+1}$ with $t' \geq s$ (for then $\Lambda_{e,t'+1}$ would equal -1), and neither does Step 2 (since x has a permanently attached Λ-marker). Hence $x \notin B$, and thus x satisfies P_e. ∎

Lemma 1.4.4 *$B \leq_T C$.*

Proof. For any x, we can determine from C if $x \in B$ as follows:

1. Find t such that $C_t \restriction x = C \restriction x$. Then at any stage $t' \geq t$, for any i, $g(i,t') > t$, so $C_{g(i,t')} \restriction x = C \restriction x$ and therefore x cannot enter B by Step 1.1.

2. Find e such that $x \in H_e$. If $e \in C$, find s such that $e \in C_{s+1} - C_s$; then x can only enter B by Step 2 at stage $s+1$. If $e \notin C$, x can never enter B by Step 2; set $s = 0$. Thus either way x cannot enter B at any stage $t' > s+1$.

Then $x \in B$ if and only if $x \in B_{\max\{t,s+1\}}$. ∎

Lemma 1.4.5 *$C \leq_T B$.*

Proof. For any e, we can determine from B if $e \in C$ as follows: Find a stage s such that $B_s \cap H_e = B \cap H_e$. We claim that $e \in C$ if and only if $e \in C_s$. To show the "only if" part of this statement, suppose that $e \in C$, and let $t+1$ be the stage at which e enters C.

Substep 1.1 can only move an element of H_e into B in acting on requirement P_i for some $i < e$; thus, at most e elements can be moved in this way. The only Λ-markers that can be assigned to elements of H_e by Substep 1.2 are the e markers $\Lambda_{0,s}, ..., \Lambda_{e-1,s}$. Thus, at any stage at most $2e$ of the $2e+1$ elements of B_e are either marked or in B.

In particular, then, at stage $t+1$ Step 2 has at least one element of H_e to enumerate into B, so $B_{t+1} \cap H_e \neq B_t \cap H_e$. Thus, $t+1 \leq s$, so $e \in C_s$. ■

CHAPTER 2
NONINVARIANCE

2.1 Introduction

Another topic arising from the study of $\mathcal{E}$-automorphisms is the question of *invariance* of properties of the c.e. sets. That is, for any given property $P(A)$ of sets in $\mathcal{E}$, we may ask whether $P(A)$ is invariant under $\mathrm{Aut}(\mathcal{E})$. For instance, computability is an invariant property; this can be seen by observing that a c.e. set A in $\mathcal{E}$ is computable if and only if $\mathcal{E}$ also contains A's complement, so that the property of computability is definable purely in terms of the lattice $\mathcal{E}$. (Indeed, any property which is $\mathcal{E}$-definable must also be invariant, and many properties have been proved invariant by showing that they are $\mathcal{E}$-definable.) On the other hand, completeness turns out to be wildly noninvariant; Harrington and Soare have shown [6] that any c.e. set in any prompt degree is automorphic to some set in $\mathbf{0}'$.

In particular, we can consider the notion of creativity. A set A is said to be *creative* if there exists a computable function f such that

$$(\forall e)[W_e \subseteq \overline{A} \implies f(e) \in \overline{A} - W_e].$$

This notion was introduced by Post in 1944, as part of his program to find an incomplete noncomputable c.e. set; the creative sets, however, turned out all to be complete, and in fact the creative sets are exactly the m-complete sets. Ever since that time, creative sets have played an important role in computability theory.

In 1967 Rogers [10, p. 228] raised the question of whether creativity is invariant, and with the development in the 1970s of machinery for generating automorphisms, considerable effort was expended in the attempt to show that it is not. The question was finally settled in the mid-1980s in a surprising way, when Harrington [14, p. 339] make the remarkable discovery of an $\mathcal{E}$-definable property exactly characterizing creative sets, thus showing that creativity was indeed invariant.

This in turn raises the question of whether other, similar properties might be invariant. In 1957 Shoenfield [11] introduced the notion of quasicreativity (see also [14, p. 88]). A set A is said to be *quasicreative* (or *q-creative*) if there exists a computable function f such that

$$(\forall e)[W_e \subseteq \overline{A} \implies D_{f(e)} \subset \overline{A} \ \& \ D_{f(e)} \not\subseteq W_e]$$

(where $\{D_e\}_{e\in\omega}$ is an effective indexing of the finite sets). Quasicreativity is similar to creativity in many ways. For instance, quasicreativity guarantees completeness just as creativity does. Also, just as the creative sets are exactly the m-complete sets, the q-creative sets are exactly the q-complete sets (as defined by Shoenfield).

However, unlike creativity, q-creativity is not invariant. In §2.3 we prove the following theorem:

Theorem 2.1.1 *There exist a quasicreative set A and a nonquasicreative low set B such that $A \simeq B$. (Hence quasicreativity is non-invariant.)*

Another related set of properties arises from the consideration of the differences between creativity and q-creativity. If A is creative, then for any W_e in $\overline{A}$ we can effectively find a single element of $\overline{A}$ not contained in W_e. If A is quasicreative, then for any W_e in $\overline{A}$ we can effectively find a finite set (of some arbitrary size) of elements of $\overline{A}$ that is not entirely contained in W_e. Suppose, then, that we consider a notion in between, in which what we effectively find is a finite set with some particular bound on its size.

We thus introduce, for every n, the property of n-creativity. A set A is said to be *n-creative* if there exists a computable function f such that

$$(\forall e)[W_e \subseteq \overline{A} \implies (|D_{f(e)}| \leq n \ \& \ D_{f(e)} \subset \overline{A} \ \& \ D_{f(e)} \not\subseteq W_e)].$$

We have thereby introduced infinitely many different properties. In §2.2.2 we show that they are indeed all distinct, by proving the following theorem:

Theorem 2.1.2 *For any n, m, if $n < m$ then there exists a c.e. set A that is m-creative but not n-creative.*

We thus have a chain of proper implications

$$\text{Creative} \;=\; \text{1-creative} \underset{\not\Leftarrow}{\overset{\Rightarrow}{}} \text{2-creative} \underset{\not\Leftarrow}{\overset{\Rightarrow}{}} \text{3-creative} \underset{\not\Leftarrow}{\overset{\Rightarrow}{}} \text{4-creative} \ldots$$

In addition, while n-creativity implies q-creativity for any n, we will also show, in §2.2.3, that the q-creative sets are not just the union of the n-creative sets over all $n \in \omega$:

Theorem 2.1.3 *There exists a c.e. set A such that*

1. *A is quasicreative, and*
2. *A is not n-creative for any n.*

We may then ask whether n-creativity is invariant. The 1-creative sets are exactly the creative sets, so 1-creativity is certainly invariant. However, it turns out that this is not true for any other value of n. In §2.3 we prove:

Theorem 2.1.4 *For all $n \geq 2$, there exist an n-creative set A and a non-n-creative low set B such that $A \simeq B$. (Hence n-creativity is noninvariant for all $n \geq 2$.)*

Another notion related to creativity is subcreativity. A c.e. set A is said to be *subcreative* if there exists a computable function h such that

$$(\forall e)[|W_e \cap A| < \infty \implies A \subset W_{h(e)} \subseteq A \cup \overline{W}_e].$$

This notion was introduced by Blum in 1973 [1], and has applications to complexity theory. As with creative sets, q-creative sets, and n-creative sets, all subcreative sets are complete. And as with all but the first, we show (in §2.4) that subcreativity is noninvariant:

Theorem 2.1.5 *There exist a subcreative set B and a nonsubcreative low set A such that $A \simeq B$. (Hence subcreativity is noninvariant.)*

One final property we consider arises from the consideration of the orbits of simple sets. In 1996, Cholak [3] answered a question of Herrmann by showing that every simple set is automorphic to some hypersimple set, and indeed every simple set is automorphic to some dense simple set. It is reasonable to examine other notions of simplicity in this connection. For example, there is the notion of an *effectively* simple set, which arose (like the notion of a creative set) from Post's seminal 1944 paper [8]. We may ask whether every simple set is automorphic to some effectively simple set. This turns out not to be the case; Harrington and Soare [5] exhibited a simple set that was not automorphic to any complete set, and (as with creative sets) all effectively simple sets are complete.

Jockusch then raised the following question: If we replace simplicity with the stronger property of prompt simplicity, does this additional power enable us to get the desired result? We show in §2.5 that it does.

Theorem 2.1.6 *For any promptly simple set A there exists an effectively simple set B such that $A \simeq B$.*

This gives us, as a corollary, one final noninvariance result:

Corollary 2.1.7 *Effective simplicity is noninvariant.*

Proof. Take any low promptly simple set A, and construct B as in Theorem 2.1.6. Then $A \simeq B$, A is incomplete and therefore not effectively simple, and B is effectively simple. ∎

2.2 Distinctness of n-creativity Notions

2.2.1 Another Definition of n-creative

We shall find it useful for our exposition to have another, equivalent, definition of n-creativity:

Theorem 2.2.1 *The following are equivalent:*

$$\text{(a)}\quad (\exists \text{ comp } f)(\forall e)[W_e \subseteq \overline{A} \implies (|D_{f(e)}| \leq n \;\&\; D_{f(e)} \subset \overline{A} \;\&\; D_{f(e)} \nsubseteq W_e)]$$

(that is, A is n-creative).

$$\text{(b)}\quad (\exists \text{ comp } f)(\forall e)[W_e \subseteq \overline{A} \implies (|D_{f(e)}| = n \;\&\; D_{f(e)} \subset \overline{A} \;\&\; D_{f(e)} \nsubseteq W_e)].$$

Proof. Clearly, (b) implies (a). Now, suppose that (a) holds for some A. Since A is n-creative, and therefore noncomputable, $\overline{A}$ is infinite. Select n elements $\{y_1, y_2, ..., y_n\}$ in $\overline{A}$. Then for any i, define the computable function p by

$$p(i) = \begin{cases} (\mu k)[|D_i \cup \{y_j\}_{j=1}^{k}| = n] & \text{if } |D_i| \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Let f be a computable function by which A satisfies (a), and define the computable function f' so that for all i,

$$D_{f'(i)} = D_{f(i)} \cup \{y_j\}_{j=1}^{p(f(i))}.$$

Then A satisfies (b) by f'. ■

2.2.2 Proof of Theorem 2.1.2

We now prove

Theorem 2.1.2. *For any n, m, if $n < m$ then there exists a c.e. set A that is m-creative but not n-creative.*

Proof. Given $n < m$, we will be constructing a c.e. set A that is m-creative but not n-creative. To guarantee the former, our construction will enumerate elements into A in order to satisfy all of the positive requirements

$$P_e : W_e \subseteq \overline{A} \implies [D_{g(e)} \subset \overline{A} \,\&\, D_{g(e)} \nsubseteq W_e],$$

where g is a computable function such that

$$D_{g(e)} = \{me, me+1, me+2, ..., me+(m-1)\}$$

(so that $|D_{g(e)}| = m$ for all e).

To guarantee the latter, our construction will satisfy all of the negative requirements

$$\begin{array}{ll} N_e: & (\exists y)[W_y \subseteq \overline{A} \ \& \\ (0) & (\varphi_e(y)\uparrow \ \vee \ |D_{\varphi_e(y)}| \neq n \ \vee \ D_{\varphi_e(y)} \nsubseteq \overline{A} \ \vee \\ (1) & D_{\varphi_e(y)} \subseteq W_y)]. \end{array}$$

That is, we will attempt to show that for no e can φ_e serve as the computable function f in the alternative definition of n-creativity given in §2.2.1. In order to meet the requirement N_e, we will use the Recursion Theorem to find some y for which either clause (0) or clause (1) holds. If clause (0) holds, W_y will be the empty set, and no action need be taken to guarantee that $W_y \subseteq \overline{A}$. If it appears that clause (1) holds, then W_y will equal $D_{\varphi_e(y)}$, and we must attempt to restrain the elements of W_y from entering $\overline{A}$. A restraint may be injured finitely often, under the following circumstances:

1. When we are called upon to meet some positive requirement P_i by putting an element of $D_{g(i)}$ into A, it may be that every possible choice would violate some existing restraint. In this case, we preserve the restraint with highest priority, and may injure any or all of the others.

2. Whenever a new restraint added in order to meet some N_i, all existing restraints for N_e, $e > i$, are considered to be injured.

We need the following machinery for this construction:

1. We have two recursive functions $a(e)$ and $n(e)$ defined so that the negative requirement N_e, if satisfied, will guarantee that A is not $n(e)$-creative via function $\varphi_{a(e)}$. For the present construction, $a(e) = e$ and $n(e) = n$ for all e. In

the next section, in the proof of Theorem 2.1.3, the definition of our negative requirements will be slightly different, and $a(e)$ and $n(e)$ will be defined so that $e = \langle a(e), n(e) \rangle$ for all e.

2. At every stage s, we have the function p_s. For any e, $p_s(e)$ is the value of y that is being used at stage s to attempt to guarantee N_e.

3. At every stage s we have the binary function f_s. For any e, $f_s(e)$ equals 0 if we are attempting to satisfy clause (0) of N_e, 1 if we are attempting to satisfy clause (1). If $f_s(e) = 0$, then we take no action to restrain any element of $W_{p_s(e)}$ from entering A, for $W_{p_s(e)}$ currently appears to be $\emptyset$. If $f_s(e) = 1$, then $W_{p_s(e)}$ equals the n-element set $D_{\varphi_{a(e)}(p_s(e))}$, and we must attempt to restrain all n of those elements from entering A.

Each $p_s(e)$ and $f_s(e)$ eventually achieves a permanent value. We denote $\lim_s p_s$ by p and $\lim_s f_s$ by f.

We may now proceed to describe our construction. There will be two additional complexities in our exposition:

1. If an injury occurs at level e at some stage $s+1$, for every $i \geq e$ we set $f_{s+1}(i)$ to 0, but determining what value $p_{s+1}(i)$ will take requires a more involved calculation. For ease of explanation, then, we describe this p-initializing procedure (which is also used to start off our overall construction) separately from the main part of the construction.

2. When we are choosing a new value of $p_{s+1}(e)$ at some stage $s+1$, we must choose it so that if there is no future injury at level e, $W_{p(e)}$ will equal $\emptyset$ if clause (0) holds throughout the rest of our construction, and $D_{\varphi_{a(e)}(p(e))}$ otherwise. This requires that we be able to speak of a hypothetical continuation of our construction, whose "program code" is used as part of a calculation for $p_{s+1}(e)$ by the Recursion Theorem. In order to avoid confusing this continuation with the real construction, the names A', f', and p' are used for its machinery.

   This means that both the main part and the p-initalization procedure of our construction must be described in more generality than usual. We must be able to start at any given starting state $s_0 + 1$ (so that the same description can be used for a hypothetical continuation that starts at state $s_0 + 1$), and we must be able to perform a portion of the construction "with regard to" either A, f, and p or A', f', and p'.

Then to start everything off, we just initialize everything and perform "the main part of the construction at stage 1, with regard to A, f, and p," and the A thus constructed is the A we want.

Construction.

p-Initialization. To p-initialize from e at stage s with regard to A, f, and p, we proceed inductively on all $i \geq e$. For each i in turn:

1. Define the computable function h_i so that for every j, $h_i(j)$ is an index of the c.e. set U defined by the following procedure:
   (a) Let $A'_s = A_s$ and $f'_s = f_s$.
   (b) Define p'_s by setting $p'_s \upharpoonright i = p_s \upharpoonright i$ and $p'_s(i) = j$, and p-initializing from $i+1$ at stage s with regard to A', f', and p'.
   (c) Perform the main part of the construction at stage $s+1$ with regard to A', f', and p' until the first stage $t+1$ such that
      (a) N_i is injured at stage $t+1$, or
      (b) $f'_{t+1}(i) = 1$.
   (d) In case (a), let $U = \emptyset$. In case (b), where $t + 1 = 2q + 2$ we have that $\varphi_{a(e),q+1}(j) \downarrow =$ some k; let $U = D_k$. (And if no such $t+1$ is ever found, nothing is ever enumerated into U, so $U = \emptyset$.)
2. Set $p_{s+1}(i)$ equal to a fixed point of h_i (found by the Recursion Theorem).

Main Part of the Construction. To perform the main part of the construction at stage s_0 with regard to A, f, and p, perform the following procedure for all stages $s+1$ starting with s_0:

Stage $s+1 = 2q+1$ (Changing strategies for guaranteeing N_e). If $\varphi_{a(e),q}(p_s(e))\uparrow$, $\varphi_{a(e),q+1}(p_s(e))\downarrow = k$, $|D_k| = n(e)$, and $D_k \subseteq \overline{A}_s$, then

1. Set $f_{s+1}(e) = 1$,
2. For all $i > e$, mark N_i as injured at stage $s+1$.
3. For all $i > e$, set $f_{s+1}(i) = 0$.
4. p-Initialize from $e+1$ at stage $s+1$ with regard to A, f, and p.

Stage $s+1 = 2q+2$ (Guaranteeing P_e). If $D_{g(e)} \subseteq \overline{A}_s$, $D_{g(e)} \nsubseteq W_{e,q}$, and $D_{g(e)} \subseteq W_{e,q+1}$, then

1. If there exists some

   $$x \in D_{g(e)} - \bigcup\{D_{\varphi_{a(i),q+1}(p_s(i))} : \varphi_{a(i),q+1}(p_s(i))\downarrow \quad \& \quad f_{2q+1}(i) = 1\},$$

   then enumerate the least such x into A_{s+1}, and end Stage $2q+2$.
2. Otherwise, let

   $$j = (\mu i)[\varphi_{a(i),q+1}(p_s(i))\downarrow \quad \& \quad f_{2q+1}(i) = 1 \;\; \& \;\; D_{g(e)} \cap D_{\varphi_{a(i),q+1}(p_s(i))} \neq \emptyset].$$

3. If there is some x in $D_{g(e)} - D_{\varphi_{a(j),q+1}(p_s(j))}$, then
   (a) Enumerate the least such x into A_{s+1}.
   (b) For all $i > j$, mark N_i as injured at stage $s+1$.
   (c) For all $i > j$, set $f_{s+1}(i) = 0$.
   (d) p-Initialize from $j+1$ at stage $s+1$ with regard to A, f, and p.
   (e) End Stage $2q+2$.
4. Otherwise (Note: this never happens in the construction for Theorem 2.1.2, but it may in the construction for Theorem 2.1.3):

(a) Enumerate the least element of $D_{g(e)}$ into A_{s+1}

(b) For all $i \geq j$, mark N_i as injured at stage $s+1$.

(c) For all $i \geq j$, set $f_{s+1}(i) = 0$.

(d) p-Initialize from j at stage $s+1$ with regard to A, f, and p.

(e) End Stage $2q+2$.

Overall Construction of A.

1. Let $A_0 = \emptyset$.

2. For all e, let $f_0(e) = 0$.

3. p-initialize from 0 at stage 0 with regard to A, f, and p.

4. Perform the main part of the construction at stage 1 with regard to A, f, and p. The resulting A is the non-m-creative, n-creative set we are attempting to construct.

(*End of Construction.*)

Verification. We must verify the following two lemmas:

Lemma 2.2.2 *Every P_e is satisfied.*

Proof. Take any e. If $D_{g(e)} \not\subseteq W_e$, then no element of $D_{g(e)}$ is ever put into A at any stage $2q+2$, so $D_{g(e)} \subset \overline{A}$. If $D_{g(e)} \subseteq W_e$, then for the least q such that $D_{g(e)} \subseteq W_{e,q+1}$, an element of $D_{g(e)}$ is put into A at stage $2q+2$, so $W_e \not\subseteq \overline{A}$. So either way P_e is satisfied. ■

Lemma 2.2.3 *Every N_e is satisfied.*

Proof. We will prove by induction that the following hold for every e:

1. N_e is marked as injured only finitely many times.

2. $f(e) =_{\text{dfn}} \lim_s f_s(e)\downarrow$.

3. $p(e) =_{\text{dfn}} \lim_s p_s(e)\downarrow$.

4. N_e is satisfied.

Take any e and assume that the above hold true for all $i < e$. Let t be a state by which, for all $i < e$,

(a) N_i will never again be marked as injured,

(b) $f(i)$ and $p(i)$ have converged, and

(c) $\varphi_{a(i)}(p(i))$ has converged if it ever will.

Then by (b) neither $f_s(e)$ nor $p_s(e)$ can change, nor can N_e be marked as injured, at any stage $2q + 1 > t$.

Now, take r so that for every $i < e$ such that $\varphi_{a(i)}(p(i))$ converges, $D_{\varphi_{a(i)}(p(i))} \subseteq \bigcup_{z<r} D_{g(z)}$. Take $t' > t$ such that $A_t \cap \bigcup_{z<r} D_{g(z)} = A \cap \bigcup_{z<r} D_{g(z)}$. Suppose that at some stage $2q + 2 > t'$, an element x enters A from some $D_{g(z)}$. Then $z \geq r$, so $D_{g(z)} \cap D_{\varphi_i(p(i))} = \emptyset$ for all $i < e$, so if x enters A by clause 3 or 4 then $j \geq e$. Thus,

(i) If x enters A by clause 3, N_e is not marked as injured at stage 2q+2, $f_{2q+2}(e) = f_{2q+1}(e)$, and $p_{2q+2}(e) = p_{2q+1}(e)$.

(ii) If x enters A by clause 4, then $j \neq e$, for if $j = e$ then $|D_{\varphi_{a(e),q+1}(p_{2q+1}(e))\downarrow}| = n(e) = n$, and so $D_{g(z)} - D_{\varphi_{a(e),q+1}(p_{2q+1}(e))} \neq \emptyset$ because $|D_{g(z)}| = m > n$. Hence $j > e$, so again N_e is not marked as injured at stage 2q+2, $f_{2q+2}(e) = f_{2q+1}(e)$, and $p_{2q+2}(e) = p_{2q+1}(e)$.

(iii) If x enters A by clause 1, then no negative requirements are marked as injured and neither f nor p changes.

Hence by stage $t' > t$, N_e will never again be injured (at any stage $> t'$, whether of the form $2q+1$ or $2q+2$) and $f(e)$ and $p(e)$ have converged, so the first three parts of our statement are proved.

To show (d), let s be 0 if N_e is never injured, or the last stage at which N_e is injured otherwise. Then $f_s(e) = 0$, and $p_s(e)$ is some fixed point of g_e as described in the p-initialization procedure. Let $j = p_s(e)$; then $W_j = W_{h_e(j)}$. Now, $W_{h_e(j)}$ is determined by the results of running the main part of the construction at $s+1$ on $A'_s = A_s$, $f'_s = f_s$, and p'_s defined by setting $p'_s \upharpoonright e = p_s \upharpoonright e$, $p'_s(e) = j = p_s(e)$, and running the p-initialization procedure from $e+1$ at stage s with regard to p', f', and A'. Since $p_s(i)$ for all $i > e$ will in fact be determined by running the p-initialization procedure from $e+1$ at stage s with regard to p, f and A, and the real construction will then procede by running the main part of the construction at $s+1$ on A_s, f_s, and p_s, these results will be the same as those of the real construction.

In the real construction, N_e will never again be injured. Thus, $W_j = W_{h_e(j)} =$

(a) $D_{\varphi_{a(e),2q+1}(j)\Downarrow}$ if at some subsequent stage $2q+1$ $f_{2q+1}(e)$ becomes 1, and

(b) $W_j = W_{h_e(j)} = \emptyset$ otherwise.

In case (a), $D_{\varphi_{a(e)}(j)} \subseteq \overline{A}_{2q+1}$, and no element of $W_j = D_{\varphi_{a(e)}(j)}$ will ever enter A because N_e will never again be injured. Thus, $D_{\varphi_{a(e)}(j)} \subseteq W_j \subseteq \overline{A}$, and requirement N_e is indeed satisfied, via clause (1).

In case (b), $f_{2q+1}(e)$ never becomes 1 at any subsequent stage $2q+1$. Thus, either $\varphi_{a(e)}(j)\uparrow$, or $|D_{\varphi_{a(e)}(j)}| \neq n(e)$, or at every stage $2q'+1$ after $\varphi_{a(e)}(j)$ converges $D_{\varphi_{a(e)}(j)} \nsubseteq \overline{A}_{2q'+1}$ and therefore $D_{\varphi_{a(e)}(j)} \nsubseteq \overline{A}$. Thus, clause (0) of N_e must hold, and since $W_j = \emptyset \subseteq \overline{A}$, N_e is satisfied. ■

2.2.3 Proof of Theorem 2.1.3

The proof of

Theorem 2.1.3. *There exists a c.e. set A such that*

1. *A is quasicreative, and*

2. *A is not n-creative for any n.*

is very similar. The only changes that must be made are the following:

1. We will be constructing a set A that is q-creative, but not n-creative for any n. Thus, we must alter our positive requirements so that they no longer guarantee

n-creativity for some particular n, but still guarantee q-creativity. In fact, we will keep the wording of the requirements the same:

$$P_e : W_e \subseteq \overline{A} \implies [D_{g(e)} \subset \overline{A} \;\&\; D_{g(e)} \not\subseteq W_e],$$

but will change the definition of the function g employed. We instead take g to be a computable function such that:

(a) $(\forall e)[|D_{g(e)}| = e + 1]$, and

(b) $(\forall i, j)[i \neq j \implies D_{g(i)} \cap D_{g(j)} = \emptyset]$.

2. We need to show that A is not n-creative, not just for one particular n, but for all $n \geq 1$. We thus must expand our collection of negative requirements to cover every possible n-creativity function φ_a for every n. We therefore employ some suitable pairing function to indentify ω with $\omega \times (\omega - \{0\})$, and define our negative requirements as follows:

$$\begin{array}{lll} N_{\langle a,n \rangle} : & & (\exists y)[W_y \subseteq \overline{A} \;\& \\ & (0) & (\varphi_a(y)\uparrow \;\vee\; |D_{\varphi_a(y)}| \neq n \;\vee\; D_{\varphi_a(y)} \not\subseteq \overline{A} \;\vee \\ & (1) & D_{\varphi_a(y)} \subseteq W_y)]. \end{array}$$

That is, we will attempt to show that for no $e = \langle a, n \rangle$ can φ_a serve as the computable function f in the alternative definition of n-creativity given in §2.2.1.

3. Having changed our definition of N_e, we must change our defintion of the two recursive functions $a(e)$ and $n(e)$ so that the negative requirement N_e, if satisfied, still guarantees that A is not $n(e)$-creative via function $\varphi_{a(e)}$. We define these functions so that where $e = \langle a, n \rangle$, $a(e) = a$ and $n(e) = n$.

4. The description of the construction is now exactly as before.

   However, note that Clause 4 of Stage $2q + 2$ may now come into play; it is now possible that $|D_{g(e)}| = e + 1 \leq n(j) = |D_{\varphi_{a(j),q+1}(p_s(j))}|$ and it is therefore possible that $D_{g(e)} - D_{\varphi_{a(j),q+1}(p_s(j))} = \emptyset$. Fortunately, for any given j there are only finitely many e for which this can occur, so N_j will only be injured by Clause 4 finitely many times.

5. To reflect the fact that Clause 4 may now come into play, we must alter our verification slightly. The third paragraph of the proof of Lemma 2.2.3 should now read:

   Now, take $r \geq n(e)$ so that for every $i < e$ such that $\varphi_{a(i)}(p(i))$ converges, $D_{\varphi_{a(i)}(p(i))} \subseteq \bigcup_{z<r} D_{g(z)}$. Take $t' > t$ such that $A_t \cap \bigcup_{z<r} D_{g(z)} = A \cap \bigcup_{z<r} D_{g(z)}$. Suppose that at some stage $2q+2 > t'$, an element x enters A from some $D_{g(z)}$. Then $z \geq r$, so $D_{g(z)} \cap D_{\varphi_i(p(i))} = \emptyset$ for all $i < e$, so if x enters A by clause 3 or 4 then $j \geq e$. Thus,

   (a) If x enters A by clause 3, N_e is not marked as injured at stage 2q+2, $f_{2q+2}(e) = f_{2q+1}(e)$, and $p_{2q+2}(e) = p_{2q+1}(e)$.

   (b) If x enters A by clause 4, then $j \neq e$, for if $j = e$ then $|D_{\varphi_{a(e),q+1}(p_{2q+1}(e))\downarrow}| = n(e)$, and so $D_{g(z)} - D_{\varphi_{a(e),q+1}(p_{2q+1}(e))} \neq \emptyset$ because $|D_{g(z)}| = z+1 \geq r+1 > n(e)$. Hence $j > e$, so again N_e is not marked as injured at stage 2q+2, $f_{2q+2}(e) = f_{2q+1}(e)$, and $p_{2q+2}(e) = p_{2q+1}(e)$.

   (c) If x enters A by clause 1, then no negative requirements are marked as injured and neither f nor p changes.

The rest of the verification goes through exactly as in the proof of Theorem 2.1.2.

2.3 Noninvariance of Quasicreativity and n-creativity

Both

Theorem 2.1.1. *There exist a quasicreative set A and a nonquasicreative low set B such that $A \simeq B$. (Hence quasicreativity is noninvariant.)*

and

Theorem 2.1.4. *For all $n \geq 2$, there exist an n-creative set A and a non-n-creative low set B such that $A \simeq B$. (Hence n-creativity is noninvariant for all $n \geq 2$.)*

follow as corollaries of the following lemma:

Lemma 2.3.1 *There exists a coinfinite 2-creative set A with semilow complement.*

Proof. [Theorems 2.1.1 and 2.1.4.] Let A be as in Lemma 2.3.1, and take any p.s. low set C. Then by Theorem 1.1.6, there exists a c.e. set $B \leq_{\mathrm{T}} C$ with $A \simeq B$. Then A is quasicreative, and n-creative for all $n \geq 2$, while B is nonquasicreative and non-n-creative for all n (since B is low, and thus not complete).

Proof. [Lemma 2.3.1.]

Take a computable function f such that for all e, $|D_{f(e)}| = \{2e, 2e+1\}$. In order to make A 2-creative, will be meeting each of the requirements

$$P_e : W_e \subseteq \overline{A} \implies [D_{f(e)} \subset \overline{A} \,\&\, D_{f(e)} \not\subseteq W_e]$$

by putting an element of $D_{f(e)}$ into A whenever $W_e \subseteq \overline{A}$ has grown to include all of $D_{f(e)}$.

In order to make $\overline{A}$ semilow, we will also be meeting the negative requirements

$$N_e : |W_e \setminus A| = \infty \implies W_e \cap \overline{A} \neq \emptyset.$$

This will require that we keep elements out of A. As in the proof of Theorem 1.4.1, we will employ for each e the marker $\Lambda_{e,s}$, which we attach to an element of W_e that we wish to keep out of A. Our attempt to thus satisfy requirement N_e may be injured in the process of preserving the truth of N_i for any $i < e$, but this can only result in finitely many injuries, so eventually each marker must come to rest permanently on some element of $W_e \cap \overline{A}$.

We construct A in stages as follows:

Construction.

Stage $s = 0$**.** Let $A_0 = \emptyset$, and let $\Lambda_{e,0} = -1$ for all e.

Stage $s + 1$**.**

Step 1 (Guaranteeing P_e). If

1. $D_{f(e)} \nsubseteq W_{e,s}$,
2. $D_{f(e)} \subseteq W_{e,s+1}$, and
3. $W_{e,s+1} \subseteq \overline{A}$,

then

1. If at least one element of $D_{f(e)}$ has no attached Λ-marker, then let x be that element if there is only one such, or the lower of the two (*i.e.*, $2e$) if there are two. Let $A_{s+1} = A_s \cup \{x\}$.
2. If both elements of $D_{f(e)}$ bear one or more markers, then for the least i such that $\Lambda_{i,s}$ is attached to one element of $D_{f(e)}$, let x be the element of $D_{f(e)}$ that does *not* bear the marker $\Lambda_{i,s}$. Let $A_{s+1} = A_s \cup \{x\}$, and for all j such that $\Lambda_{j,s} = x$, let $\Lambda_{j,s+1} = -1$.

Step 2 (Guaranteeing N_e). If for some x

1. $x \in W_{e,s+1} - W_{e,s}$,
2. $x \in \overline{A}$, and
3. $\Lambda_{e,s} = -1$,

then let $\Lambda_{e,s+1} = x$.

(End of construction.)

Verification. We now demonstrate the following three lemmas:

Lemma 2.3.2 *A is coinfinite.*

Proof. All $D_{f(e)}$ are disjoint, and no more than one element of any $D_{f(e)}$ is put into A by Step 1, because if $D_{f(e)} \cap A_s \neq \emptyset$ then it cannot be the case that $D_{f(e)} \subseteq W_{e,s+1} \subseteq \overline{A}$. Thus, $\overline{A}$ contains at least one element from each $D_{f(e)}$. ∎

Lemma 2.3.3 *A is 2-creative.*

Proof. If $D_{f(e)} \subseteq W_e$, then at the first stage $s+1$ when $D_{f(e)} \subseteq W_{e,s+1}$ either $W_{e,s} \nsubseteq \overline{A}$ or we put an element of $D_{f(e)}$ into A, so in either case $W_{e,s+1} \nsubseteq \overline{A}$ and therefore $W_e \nsubseteq \overline{A}$. Also, since no element of $D_{f(e)}$ is put into A unless $D_{f(e)} \subseteq W_{e,s+1}$, if $D_{f(e)} \nsubseteq \overline{A}$ then $W_e \nsubseteq \overline{A}$. Thus, if $W_e \subseteq \overline{A}$, we must have that $D_{f(e)} \nsubseteq W_e$ and $D_{f(e)} \subset \overline{A}$. ■

Lemma 2.3.4 *$\overline{A}$ is semilow.*

Proof. It suffices to verify by induction that the following hold for all e:

1. $\Lambda_e =_{\text{dfn}} \lim_s \Lambda_{e,s}$ converges.

2. N_e is satisfied.

1. Suppose that Λ_e does not converge. Then Step 1 and Step 2 must each act on $\Lambda_{e,s}$ at infinitely many stages s. Let s_0 be some stage such that for all $i < e$, $\Lambda_i = \Lambda_{i,s_0}$. Let
$$S = \bigcup\{D_{f(j)} : (\exists i < e)[\Lambda_i \in D_{f(j)}]\}.$$
Then for some stage $s_1 > s_0$, $W_{e,s_1} \cap S = W_e \cap S$. Take $s_2 > s_1$ such that Step 1 acts on Λ_{e,s_2} at stage s_2+1; then $\Lambda_{e,s_2+1} = -1$. Let $t+1$ be the next stage after s_2+1 at which Step 2 acts on $\Lambda_{e,t}$. Then $\Lambda_{e,t+1} = x$ where $x \in W_{e,t+1} - W_{e,t}$. Then since $t+1 > s_0$, $x \notin S$. Let $t'+1$ be the next stage after $t+1$ at which Step 1 acts on $\Lambda_{e,t'}$. Since x is tagged by the marker $\Lambda_{e,t'+1}$, x can only be enumerated into A if it is in the same $D_{f(j)}$ as some $\Lambda_{i,t'+1}$, $i < e$. But then $x \in S$, and we get a contradiction.

2. By the above, we know that Λ_e converges. Suppose that $|W_e \setminus A| = \infty$. Then for any stage t such that $\Lambda_{e,t} = -1$, let $t'+1$ be the first stage $> t$ for which there exists $x \in W_{e,t'+1} - W_{e,t'} - A_{t'}$; then by Step 2, $\Lambda_{e,t'+1} = x \neq -1$. Thus, Λ_e does not converge to -1.

Let $s+1$ be the first stage such that $\Lambda_{e,s+1} = \Lambda_e$. Then $\Lambda_{e,s+1} =$ some $x \in W_{e,s+1} - W_{e,s} - A_s$, and x never enters A at any subsequent stage, so $x \in W_e \cap \overline{A}$. Thus if $|W_e \setminus A| = \infty$ then $W_e \cap \overline{A} \neq \emptyset$. ∎

2.4 Noninvariance of Subcreativity

2.4.1 Proof via Theorem 2.4.1

In order to prove

Theorem 2.1.5. *There exist a subcreative set B and a nonsubcreative low set A such that $A \simeq B$. (Hence subcreativity is noninvariant.)*

we will use the Prompt Coding Theorem of [6] to prove the following theorem, of which Theorem 2.1.5 is a corollary:

Theorem 2.4.1 *For any coinfinite prompt set A with semilow complement, there exists a subcreative set B such that $A \simeq B$.*

Proof. [Theorem 2.1.5.] Take A a low prompt set (so A is incomplete and thus nonsubcreative), and construct B as in Theorem 2.4.1. ∎

2.4.2 Intuition and Machinery

The proof employs similar techniques to the proof in [6] that every prompt c.e. set is automorphic to a complete set. That is, for every α on the true path we will have a α-state $\hat{\nu}$ containing an element which BLUE can, if desired, place immediately into B; we will use these elements to ensure the subcreativity of B. The promptness of A will guarantee that $\widehat{\mathcal{D}}_\alpha$, the collection of such states for a given α, is always nonempty when α is on the true path.

In order for $\hat{\nu}$ to be suitable, it must have the following properties:

1. BLUE must be allowed to move a desired element from $\hat{\nu}$ directly into B.

2. $\hat{\nu}$ must be RED-maximal, that is, no move by RED can take an element out of $\hat{\nu}$. Then BLUE can keep a chosen coding element in $\hat{\nu}$ until BLUE needs to move it into B.

3. $\hat{\nu}$ must be in $\widehat{\mathcal{K}}_\alpha$, that is, $\hat{\nu}$ must be well-resided, to guarantee a supply of coding elements.

In order for these conditions to be met, we need for the corresponding conditions to be true on the A side:

1. RED must be allowed to move elements of ν directly into A; that is, we need ν to be an element of the set $\mathcal{W}_\alpha^\#$ defined, as in [6], to consist of all $\nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle \in \mathcal{M}_\alpha$ such that

$$0 \notin \sigma_1 \ \ \& \ \ (\exists \nu_2 = \langle \alpha, \sigma_2, \tau_2 \rangle)[\nu_2 \in \mathcal{M}_\alpha \ \ \& \ \ \sigma_1 \subset \sigma_2 \ \ \& \ \ 0 \in \sigma_2].$$

2. ν must be BLUE-maximal.

3. ν must be in $\mathcal{K}_\alpha$.

We therefore define $\mathcal{D}_\alpha$ as in [6] to be the set of all α-states ν that appear to be thus suitable:

Definition 2.4.2 Let $\mathcal{D}_\alpha$ be the set of all $\nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle$ such that $\nu_1 \in \mathcal{M}_\alpha$ and

1. $\nu_1 \in \mathcal{W}_\alpha^\#$,

2. $\neg(\exists \nu_2 \in \mathcal{M}_\alpha)[\nu_1 <_B \nu_2]$, and

3. $\nu_1 \notin \mathcal{N}_\alpha$.

Then the promptness of A guarantees that $\mathcal{D}_\alpha \neq \emptyset$, for all $\alpha \subset f$. Thus, when we define $\widehat{\mathcal{D}}_\alpha$ to be $\{\hat{\nu} : \nu \in \mathcal{D}_\alpha\}$, we are guaranteed the existence of at least one state $\hat{\nu} \in \widehat{\mathcal{D}}_\alpha$ to give us a supply of suitable coding elements. The coding elements for a given α will be enumerated into a d.r.e. set $L_{\alpha,s}$, from which they will be removed when they either are used or need to be discarded for other reasons. They are also tagged in pairs with markers $\hat{y}_{\alpha,i,s}$ and $\hat{y}'_{\alpha,i,s}$, for all i up to some computable bound $g(\alpha,s)$ that depends on the particular construction being used. (In our construction, as in that of [6, §10], $g(\alpha,s)$ will always equal 1, so we will have at most one such pair of tagged elements for any given α at any time; in fact, unlike the construction in [6], our construction will only actually use the first element of any such pair.)

We then get the Prompt Coding Theorem (Theorem 10.8 of [6]), which, stated in full, is as follows:

Theorem 2.4.3 (Prompt Coding Theorem, Harrington-Soare) *Let $A = U_0$ be a given prompt set, and let $B = \widehat{U}_\rho$ where $\rho = f \restriction 1$. Let $g(\alpha,s)$ be a recursive function (to be used in Step $\widehat{7}$). Perform the basic coding construction consisting of Steps 1–6, 11, $\widehat{1}$–$\widehat{5}$, and $\widehat{7}$ of [6, §10], and possibly with additional Steps $\hat{n}$, $9 \leq n < 11$, defined later, which satisfy the conditions ($\widehat{P}1$)–($\widehat{P}3$):*

($\widehat{P}1$) If α is $\widehat{\mathcal{R}}$-inconsistent or $\widehat{\mathcal{M}}$-inconsistent then Step $\hat{n}$ does not apply to α. If α is $\widehat{\mathcal{D}}$-inconsistent then Step $\hat{n}$ applies to α only if $n = 6$.

($\widehat{P}2$) Step $\hat{n}$ cannot enumerate $\hat{x}$ in any red set V_α. If Step $\hat{n}$ at stage $s+1$ enumerates $\hat{x}$ in a blue set $\widehat{U}_\alpha$, then $\hat{x} \in \widehat{R}_{\alpha,s}$, and this enumeration must be α-legal, i.e., *$\hat{\nu}(\alpha, \hat{x}, s+1) \in \widehat{\mathcal{M}}_\alpha$.*

($\widehat{P}3$) Step $\hat{n}$ cannot move $\hat{x}$ from $\widehat{S}_\alpha$ to $\widehat{S}_\gamma$ for $\alpha \neq \gamma$, or from $\widehat{S}^1_\alpha$ to $\widehat{S}^0_\alpha$, but can only appoint some $\hat{x}$ already in $\widehat{S}^0_\alpha$ as an α-witness, and can later cancel $\hat{x}$ as an α-witness and simultaneously move $\hat{x}$ from $\widehat{S}^0_\alpha$ to $\widehat{S}^1_\alpha$.

Then

(i) $(\forall \gamma \subseteq f)[\liminf_s \, g(\gamma,s) < \infty] \implies A$ is Δ^0_3-automorphic to B.

In addition, if the Steps $\hat{n}$, $9 \leq n < 11$, satisfy the following conditions ($\widehat{Q}1$)–($\widehat{Q}4$) then conclusions (ii), (iii), (vii), and (viii) hold.

($\widehat{Q}1$) Step $\hat{n}$ may not put any element $\hat{x}$ into the witness set $\widehat{L}_\alpha$.

($\widehat{Q}2$) Step $\hat{n}$ may not remove any element $\hat{x}$ from the witness set $\widehat{L}_\alpha$ unless simultaneously $\hat{x}$ is enumerated in B.

($\widehat{Q}3$) If Step $\hat{n}$ puts $\hat{x}$ into $B_{s+1} - B_s$ then $\hat{x} \in \widehat{L}_{\alpha,s}$.

($\widehat{Q}4$) If $\hat{x} \in \widehat{S}^0_{\alpha,s}$ then Step $\hat{n}$ may not enumerate $\hat{x} \in \widehat{U}_{\alpha,s+1} - \widehat{U}_{\alpha,s}$ for any blue set $\widehat{U}_\alpha$ unless simultaneously $\hat{x}$ is enumerated in B.

Assume $\alpha \subset f$, $\alpha \neq \lambda$. Then for all $\hat{x}$ and s and all $i \geq 1$,

(ii) $\widehat{L}_{\alpha,s} \subseteq \widehat{S}^0_{\alpha,s}$, *and* $\widehat{L}_\alpha$ *is a d.r.e. set;*

(iii) $[\hat{x} \in (\widehat{S}^0_{\alpha,s} - B_s)\ \&\ \nu(\alpha, \hat{x}, s) = \hat{\nu}_1 \in \widehat{\mathcal{D}}_\alpha] \implies \nu(\alpha, \hat{x}, s+1) = \hat{\nu}_1$;

(vii) $i \leq \liminf_s g(\alpha, s) \implies (\exists^\infty s)[\hat{y}_{\alpha,i,s}{\downarrow}\ \ \&\ \nu(\alpha, \hat{y}_{\alpha,i,s}, s) \in \widehat{\mathcal{D}}_\alpha]$;

(viii) $[i \leq \liminf_s g(\alpha, s)\ \&\ (\exists^{<\infty} s)[\hat{y}_{\alpha,i,s} \in B_{s+1} - B_s]] \implies [\lim_s \hat{y}_{\alpha,i,s} < \infty]$.

In addition, if Steps $\hat{n}$, $9 \leq n < 11$, satisfy the following condition ($\widehat{Q}5$) then conclusion (ix) holds for α and i as above.

($\widehat{Q}5$) Step $\hat{n}$ may not put $\hat{y}'_{\alpha,i,s}$ into $B_{s+1} - B_s$, and may put $\hat{y}_{\alpha,i,s}$ into $B_{s+1} - B_s$ only if $\hat{y}'_{\alpha,i,s}$ is defined.

(ix) $i \leq \liminf_s g(\alpha, s) \implies [\ (a.e.\ s)[\hat{y}_{\alpha,i,s}{\downarrow}]\ \&\ (\exists^\infty s)[\hat{y}_{\alpha,i,s}{\downarrow}\ \ \&\ \ \hat{y}'_{\alpha,i,s}{\downarrow}]]$.

In [6, §10], the elements supplied by the states in $\widehat{\mathcal{D}}_\alpha$ were used for coding K into B. We will be using them to fulfill the following requirements:

$$P_e : |W_e \cap B| < \infty \implies [B \subset W_{h(e)} \subseteq B \cup \overline{W}_e].$$

for some h which will be defined by our construction.

We employ the following machinery:

1. We have all of the machinery of [6, §10].

2. In the course of our contruction, we will enumerate a collection of c.e. sets $\{E_e\}_{e \in \omega}$. We will thus have a computable function h such that $W_{h(e)} = E_e \cup B$ for all e; this is the h used in the statement of our requirements P_e.

3. Where we identify ω with $\omega \times \omega$ by a suitable pairing function, we have the function l defined by $l(\alpha) = e$ if $|\alpha| = \langle e, y \rangle$ for some y.

2.4.3 Construction

We first let $g(\alpha, s) = 1$ for all α, s. (That is, only one marker pair will be employed at any node α.) We then employ a modified version of the construction used in [6, §10] as follows:

Step 1–6, $\widehat{\mathbf{1}}$–$\widehat{\mathbf{5}}$. As in [6, §10].

Step $\widehat{\mathbf{7}}$. As in [6, §10], with an added accounting step that does not affect any other part of the construction, and therefore does not affect the applicability of the Prompt Coding Theorem:

($\widehat{7}$a) If we have just assigned the marker $\hat{y}_{\alpha,1,s+1}$ to some $\hat{x}$, then enumerate $\hat{x}$ into $E_{l(\alpha)}$.

Steps $\widehat{9}$ and $\widehat{10}$ are somewhat modified versions of the corresponding steps in [6, §10]. (Note that since Step $\widehat{7}$ only assigns a marker $\hat{y}_{\alpha,1}$ if α is $\widehat{\mathcal{M}}$-consistent, $\widehat{\mathcal{R}}$-consistent, and $\widehat{\mathcal{D}}$-consistent, Steps $\widehat{9}$ and $\widehat{10}$ also only apply to nodes that are consistent in all three senses.)

Step $\widehat{9}$ puts the elements of E_e into B when they go into W_e, in order to satisfy requirement P_e:

Step $\widehat{\mathbf{9}}$. Suppose $|\alpha| = 5i$, $\hat{y}_{\alpha,1,s}\downarrow = \hat{x}$, and $\nu(\alpha, \hat{x}, s) = \hat{\nu}_1$. If $\hat{x} \in W_e$ for some e, perform the action of Step $\widehat{8}$ of [6, §7.2] on $\hat{x}$ at stage s+1. (Namely, choose the least such pair $\langle \alpha, \hat{x} \rangle$. Let $\widehat{F}_\alpha(\hat{\nu}_1) = \hat{\nu}_2 = \langle \alpha, \widehat{\sigma}_2, \widehat{\tau}_2 \rangle$. Enumerate $\hat{x}$ in $\widehat{U}_{\delta,s+1}$ for all $\delta \subseteq \alpha$ such that $\hat{e}_\delta \in \widehat{\sigma}_2$. Since $\hat{x} \in B_{s+1} - B_s$ move $\hat{x}$ from $\widehat{S}^0_\alpha$ to $\widehat{S}^1_\alpha$, and let $\hat{x}$ be cancelled as an α-witness.)

Step $\widehat{10}$ puts witnesses that will be deactivated into B:

Step $\widehat{\mathbf{10}}$. Suppose $\hat{y}_{\alpha,1,s}\downarrow = \hat{x}$, and that at stage s+1 either:

(i) $\hat{x}$ will be removed from $\widehat{S}^0_\alpha$ (and from $\widehat{L}_\alpha$) because Step 11C applies to α (*i.e.*, $f_{s+1} <_L \alpha$); or

(ii) $\hat{x}$ will be pulled from $\widehat{S}^0_\alpha$ to some $\widehat{S}_\beta$, $\beta <_L \alpha$, under Step $\widehat{1}_\beta$.

Then perform the action of Step $\widehat{8}$ of [6, §7.2] on $\hat{x}$, as in Step $\widehat{9}$ above. (Note that $\hat{x}$ will be cancelled as an α-witness at stage s+1 under Step 11C or Step $\widehat{1}_\beta$ (1.12), as will whatever element $\hat{x}'$, if any, is tagged with $\hat{y}'_{\alpha,1,s}$.)

Substep 11A-F. This is the same as in [6].

2.4.4 *Verification*

We first note that steps $\widehat{9}$ and $\widehat{10}$ satisfy all the conditions ($\widehat{P}1$–$\widehat{P}3$) and ($\widehat{Q}1$–$\widehat{Q}4$). Thus, conclusions (i), (ii), (iii), (vii), and (viii) hold. In particular, since (i) holds, the A and B we construct in this way are automorphic. We thus need only verify that B is subcreative; that is, that all the requirements P_e are met.

Take any e such that $|W_e \cap B| < \infty$. Then Steps $\widehat{9}$ and $\widehat{10}$ put only finitely many elements of E_e into B. Take any α such that $l(\alpha) = e$. Then every element that ever bears the tag $\hat{y}_{\alpha,1}$ is in E_e, so only finitely many such elements are enumerated into B. Thus, by conclusion (viii) for $i = 1$, we have that $\lim_s \hat{y}_{\alpha,1,s} < \infty$. Also, by conclusion (vii) there are infinitely many s such that $\hat{y}_{\alpha,1,s}\downarrow$. Hence $\hat{y}_{\alpha,1}$ must eventually be attached permanently to some $\hat{x}$, which therefore never enters B. Since $\hat{x} \in E_e \subseteq W_{h(e)}$, and $B \subseteq W_{h(e)}$, then, we have that $B \subset W_{h(e)}$.

To show that $W_{h(e)} \subseteq \overline{W}_e$, we note that any element enumerated into E_e gets an attached $\hat{y}$-marker simultaneously (in Step $\widehat{7}$), any element of E_e that loses its $\hat{y}$-marker is enumerated into B (in Step $\widehat{9}$ or $\widehat{10}$), and any element with an attached $\hat{y}$-marker that is enumerated into W_e is enumerated into B. Thus any element of E_e that is in W_e must also be in B, so $E_e \subseteq B \cup \overline{W}_e$ and therefore $W_{h(e)} = B \cup E_e \subseteq B \cup \overline{W}_e$. Thus, P_e is indeed satisfied, and we are done.

2.5 Effective Simplicity

2.5.1 *Intuition and Machinery*

To prove

Theorem 2.1.6. *For any promptly simple set A there exists an effectively simple set B such that $A \simeq B$.*

we will again, as in the proof of Theorem 1.1.6, employ a version of the Harrington-Soare construction restricted to $\overline{A}$ and $\overline{B}$. However, the modifications we make to the construction of [6] will be different this time, in the following ways:

1. We no longer have that A is semilow, so we can no longer control the enumeration of A_s with the help of Γ-markers. Therefore, our construction will no longer have a step that challenges elements x before they change state in any way and assigns them such markers.

2. Because we need no longer challenge elements x before they change state, we can now put the exterior step that enumerates elements into A_s where it more naturally goes, as part of Step 4 (that is, with the rest of the moves forced by RED).

3. Also, because we no longer have control of the enumeration into A_s, we will not be able to guarantee the correspondence of $\mathcal{G}^A$ and $\widehat{\mathcal{G}}^B$ by the technique we were using previously; that is, using Step 0 to generate our list $\mathcal{L}^{\mathcal{G}}$ to tell us what elements to enumerate into B for covering purposes. However, it will turn out that the prompt simplicity of A guarantees that $\mathcal{G}^A = \mathcal{M}^{\overline{A}} = \widehat{\mathcal{M}}^{\overline{B}}$ on the true path. Therefore, we can generate a covering list $\mathcal{L}^{\mathcal{G}}$ purely from $\mathcal{M}^{\overline{B}}$, and on the true path it will be correct.

4. Finally, our lack of control over the enumeration into A_s means that we cannot guarantee $\mathcal{R}_\alpha^{\overline{A}} = \widehat{\mathcal{B}}_\alpha^{\overline{B}}$ by the technique of our proof of Theorem 1.1.6 (which was the restriction of the technique used in [6]; that is, ensuring that any $\alpha \subset f$ is $\mathcal{R}^{\overline{A}}$-consistent). However, it will turn out that the prompt simplicity of A gives us a much more direct solution: $\mathcal{G}^A = \mathcal{M}^{\overline{A}} = \widehat{\mathcal{M}}^{\overline{B}}$ on the true path, so there cannot be a problem with BLUE having to empty a state but having no state to send its elements to; it can simply enumerate them into B_s. (However, we will still be guaranteeing that $\mathcal{B}_\alpha^{\overline{A}} = \widehat{\mathcal{R}}_\alpha^{\overline{B}}$ by ensuring that any $\alpha \subset f$ is $\widehat{\mathcal{R}}^{\overline{B}}$-consistent.)

5. To guarantee that B is effectively simple, we will satisfy the requirement

$$P_e : |W_e| > e \implies W_e \not\subseteq \overline{B}$$

for every e by always enumerating at least one element of any given W_e into B if $|W_e| > e$. We can no longer rely on using the elements in some $\widehat{S}_\alpha^{0,\overline{B}}$ for this purpose, because the finite collection of elements we have to chose from may not include any elements in any $\widehat{S}_\alpha^{0,\overline{B}}$; we may instead have to enumerate elements from anywhere in our tree. We will therefore employ a system of $\widehat{\Gamma}$-markers to make sure that we choose elements whose enumeration will not interfere with our construction. Specifically, markers will be attached (in Step $\hat{0}$) to elements $\hat{x}$ entering a node α in any state in $\widehat{\mathcal{R}}_\alpha^{\overline{B}}$, to help us to choose elements that will not interfere with the demonstration that any node $\alpha \subset f$ must be $\widehat{\mathcal{R}}^{\overline{B}}$-consistent.

6. A version of Step $\widehat{8}$ is still used to enumerate elements of $\widehat{S}_\alpha^{0,\overline{B}}$ into B for covering purposes, but an element no longer needs permission from any set C to enter B. This step has also been moved, for technical reasons, to a position between Steps $\widehat{3}$ and $\widehat{4}$, and has correspondingly been renumbered "Step $\widehat{\pi}$."

7. We have a Step $\widehat{9}$ that enumerates elements into B to guarantee that B is effectively simple.

Our tree construction will employ the following machinery:

1. We have all of the machinery of Chapter 2 of [6], with the same redefinition of consistency as in §1.2.1, and again restricted to $\overline{A}$ and $\overline{B}$. (Note, however, that we shall make no use of the notion of $\mathcal{R}^{\overline{A}}$-consistency.) We will henceforth universally adopt the convention, used previously in §1.2.4 and §1.3.2, of omitting the superscript $\overline{A}$ and $\overline{B}$ that indicates this restriction.

2. We again fix an enumeration $\{\widetilde{A}_s\}_{s\in\omega}$ of A.

3. We will again be enumerating $\{A_s\}_{s\in\omega}$ and $\{B_s\}_{s\in\omega}$ through exterior moves, just as in the proof of Theorem 1.1.6.

4. Also as in that proof, we have a list $\mathcal{L}^{\mathcal{G}}$ of pairs of the form $\langle \alpha, \hat{\nu} \rangle$, used to satisfy the requirement that $\nu \in \mathcal{G}_\alpha^A$ implies $\hat{\nu} \in \widehat{\mathcal{G}}_\alpha^B$. In this construction, however, this list is periodically updated in accordance with $\widehat{\mathcal{M}}_\alpha$, rather than being updated when elements are enumerated into A_s.

5. From our construction we will define (in the proof of Lemma 2.5.1) a collection of c.e. sets $\{E^{\alpha}_{\nu,i}\}_{i\in\omega}$ for every node α and α-state ν. Since A is promptly simple, by [14, Proposition XIII.1.3] there exists a computable function q such that

$$|E^{\alpha}_{\nu,i}| = \infty \implies (\exists s)(\exists x)[x \in E^{\alpha}_{\nu,i,\text{ at } s} \cap \widetilde{A}_{q(s)}].$$

By the Recursion Theorem, we can assume that we know this function q in advance.

6. For every node α, for every state $\hat{\nu} \in \widehat{\mathcal{R}}_{\alpha}$, we have the infinite set of markers $\{\widehat{\Gamma}^{\alpha}_{\hat{\nu},i}\}_{i\in\omega}$. One marker of this type will be attached to any element $\hat{x}$ that enters $\widehat{S}_{\alpha}$ in α-state ν. This marker is removed only if $\hat{x}$ leaves state $\hat{\nu}$, either by
   (a) A RED move of some kind,
   (b) An exterior BLUE move, to help guarantee either the correspondence of $\mathcal{G}^A$ and $\widehat{\mathcal{G}}^B$ or the effective simplicity of B, or
   (c) Step $\widehat{1}$ or 11C.

   The Harrington-Soare construction guarantees that for any given α removals of type (c) occur only finitely often, and we will guarantee that for any given marker removals of type (b) occur only finitely often, so eventually each marker rests on an element suitable for use in the proof that an $\widehat{\mathcal{R}}$-inconsistent α cannot be on the true path.

7. We define an effective bijection $|\cdot|$ from the set of all possible $\widehat{\Gamma}$-markers to ω. (This can be done because we can regard the the tree T as a subset of ω^{ω} as in [6, §2.2], so the nodes of T can be given an effective numbering $a(\alpha)$, and for any α the finitely many α-states $\hat{\nu}$ can be given an effective numbering $\hat{\nu}$, so any tag $\widehat{\Gamma}^{\alpha}_{\hat{\nu},i}$ can be effectively identified with a triple $\langle a(\alpha), n(\hat{\nu}), i\rangle$.) We call $|\widehat{\Gamma}^{\alpha}_{\hat{\nu},i}|$ the *order* of $\widehat{\Gamma}^{\alpha}_{\hat{\nu},i}$.

2.5.2 Construction

Our construction is as follows:

Stage $s = 0$. For all $\alpha \in T$ define $U_{\alpha,0} = V_{\alpha,0} = \widehat{U}_{\alpha,0} = \widehat{V}_{\alpha,0} = \emptyset$, and define $m(\alpha, 0) = 0$. Define $Y_{\lambda,0} = \widehat{Y}_{\lambda,0} = \emptyset$, and $f_0 = \lambda$. Define $A_0 = B_0 = \emptyset$. Every marker $\widehat{\Gamma}^{\alpha}_{\nu,i,0}$ is unassigned.

Stage $s + 1$. Find the least $n < 11$ such that Step n applies to some $x \in Y_{\alpha,s}$ and perform the intended action. If there is no such n, then find the least $n < 11$ such that Step $\hat{n}$ applies to some $\hat{x} \in \widehat{Y}_{\alpha,s}$, and perform the indicated action. If none of these steps applies, then apply Step 11, and go to stage $s + 2$.

Steps 1, 2. In the non-dual case, the same as in [6] (restricted to $\overline{A}$ and $\overline{B}$). In the dual case, each has an added step at the end (($\widehat{1}$.13) or ($\widehat{2}$.7), respectively) to add a $\widehat{\Gamma}$-marker if appropriate:

($\widehat{1}$.13)/($\widehat{2}$.7) Where $\hat{\nu}_1 = \hat{\nu}(\alpha, \hat{x}, s+1)$, the α-state in which $\hat{x}$ has just entered $\widehat{Y}_\alpha$, if $\hat{\nu}_1 \in \widehat{\mathcal{R}}_\alpha$ then attach marker $\widehat{\Gamma}^{\alpha}_{\hat{\nu}_1,i,s+1}$ to $\hat{x}$, for the least i such that $\widehat{\Gamma}^{\alpha}_{\hat{\nu}_1,i,s}$ is not attached to any element.

Step 3. In the nondual case, the same as in [6] (restricted to $\overline{A}$ and $\overline{B}$). In the dual case, we need to add a step to remove any marker that is no longer applicable:

($\widehat{3}$.6) If any marker $\widehat{\Gamma}^{\alpha}_{\hat{\nu}_0,i,s}$ is attached to the $\hat{x}$ that we have just moved out of state ν_0, remove this marker.

Step $\hat{\pi}$. (Moving covering elements into B.)

Find the first unmarked pair $\langle \alpha, \hat{\nu} \rangle$ in $\mathcal{L}^{\mathcal{G}}$ such that for some $\hat{x} \in \widehat{S}^0_{\alpha,s}$,

($\hat{\pi}$.a) $\hat{x} \in \hat{\nu}$, and

($\hat{\pi}$.b) No $\widehat{\Gamma}$-marker on $\hat{x}$ has ever been detached by Step $\hat{\pi}$.

Then

($\hat{\pi}$.1) Enumerate the least such $\hat{x}$ into B,

($\hat{\pi}$.2) Mark the current copy of $\langle \alpha, \hat{\nu} \rangle$, and

($\hat{\pi}$.3) Detach all $\widehat{\Gamma}$-markers from $\hat{x}$.

(In other words, this is just Step $\widehat{8}$ from the proof of Theorem 1.1.6, modified to deal with $\widehat{\Gamma}$-markers, and moved into a position between Steps $\widehat{3}$ and $\widehat{4}$.)

We must alter Step 4 slightly to include the RED enumeration of elements into A_s:

Step 4. (Delayed RED enumeration into A_{s+1} and U_α.)

(4A) Suppose some $x \in Y_\lambda$ and $x \in \widetilde{A}_{q(s)}$. Enumerate the least such x into A_{s+1}, and end Step 4.

(4) Suppose $x \in R_{\alpha,s}$ and

(4.1) $e_\alpha > e_\beta$,

(4.2) $x \notin U_{\alpha,s}$, and

(4.3) $x \in Z_{e_\alpha,s} =_{\mathrm{dfn}} U_{e_\alpha,s} \cap Y_{\beta,s-1}$.

Action. Choose the least such pair $\langle \alpha, x \rangle$ and

(4.4) Enumerate x in $U_{\alpha,s+1}$.

Step $\widehat{4}$, however, is just the same as in [6] (restricted to $\overline{A}$ and $\overline{B}$), with an added action for removing obsolete markers:

($\widehat{4}$.5) For any marker $\widehat{\Gamma}^\alpha_{\hat{\nu}_1,i,s}$ attached to $\hat{x}$ such that we have just moved $\hat{x}$ out of state ν_1, remove that marker.

Step 5. This step, in the non-dual case, is the same as in [6] (restricted to $\overline{A}$ and $\overline{B}$). Its dual, however, must be changed to reflect the fact that we no longer have any consideration of $\mathcal{R}$-consistency:

Step $\widehat{5}$. (BLUE emptying of state $\hat{\nu} \in \widehat{\mathcal{B}}_\alpha$.) Suppose for $\alpha \in T$ there exists $\hat{x}$ such that

(5.1) $\hat{\nu}(\alpha, \hat{x}, s) = \hat{\nu}_0 \in \widehat{\mathcal{B}}_\alpha$, and

(5.2) $\hat{x} \in \widehat{S}_{\alpha,s}$.

Action. Choose the least such pair $\langle \alpha, \hat{x} \rangle$. Enumerate $\hat{x}$ into B_s. (No marker removal is necessary, because by definition of $\widehat{\mathcal{B}}$ and $\widehat{\mathcal{R}}$ there cannot exist any $\delta \subset \alpha$ such that $\hat{\nu} \in \widehat{\mathcal{B}}_\alpha$ and $\hat{\nu} \restriction \delta \in \widehat{\mathcal{R}}_\delta$.)

Step $\widehat{9}$. (Guaranteeing that B is effectively simple.) Take the least e such that $|W_{e,s} \cap \widehat{Y}_{\lambda,s}| > e$ and $W_{e,s} \subseteq \overline{B}_s$. Take the $\hat{x} \in |W_{e,s} \cap \widehat{Y}_{\lambda,s}|$ whose lowest-order marker is of the highest order. Enumerate $\hat{x}$ into B and remove all of its markers.

Step 11. This is the same as in [6], except that we redefine Substep 11B and 11C:

Substep 11B. (Defining $m(\alpha, s+1)$, $\mathcal{L}_{s+1}$, $\widehat{\mathcal{L}}_{s+1}$, and $\mathcal{L}^{\mathcal{G}}_{s+1}$.) For every $\alpha \subseteq f_{s+1}$ if every α-entry $\langle \alpha, \nu \rangle$ on $\mathcal{L}_s$, and α-entry $\langle \alpha, \hat{\nu} \rangle$ on $\widehat{\mathcal{L}}_s$, is marked we say that the lists are α-*marked* and we

(11.1) Define $m(\alpha, s+1) = m(\alpha, s) + 1$.

(11.2) Add to the bottom of list $\mathcal{L}_s$ ($\widehat{\mathcal{L}}_s$) a new (unmarked) α-entry $\langle \alpha, \nu \rangle$ ($\langle \alpha, \hat{\nu} \rangle$) for every such α and every $\nu \in \mathcal{M}_\alpha$. Let the resulting list be $\mathcal{L}_{s+1}(\widehat{\mathcal{L}}_{s+1})$.

(11.3) Add to the bottom of list $\mathcal{L}^{\mathcal{G}}_s$ a new (unmarked) α-entry $\langle \alpha, \hat{\nu} \rangle$ for every such α and every $\hat{\nu} \in \widehat{\mathcal{M}}_\alpha$. Let the resulting list be $\mathcal{L}^{\mathcal{G}}_{s+1}$.

If lists $\mathcal{L}_s$ and $\widehat{\mathcal{L}}_s$ are not both α-marked then let $m(\alpha, s+1) = m(\alpha, s)$, $\mathcal{L}_{s+1} = \mathcal{L}_s$, $\widehat{\mathcal{L}}_{s+1} = \widehat{\mathcal{L}}_s$, and $\mathcal{L}^{\mathcal{G}}_{s+1} = \mathcal{L}^{\mathcal{G}}_s$.

Substep 11C For every α such that $f_{s+1} <_L \alpha$, remove all markers from all elements of $\widehat{S}_{\alpha,s}$ and initialize α.

2.5.3 Verifying Correctness of $\mathcal{M}^{\overline{A}}$, $\widehat{\mathcal{M}}^{\overline{B}}$, $\mathcal{N}^{\overline{A}}$, and $\widehat{\mathcal{N}}^{\overline{B}}$ on f

Just as in §1.3.1, we may assume without loss of generality that A is infinite and coinfinite, and then we have that the tree properties of §1.2.2 hold for our restricted construction. Then it suffices to verify that the versions of Lemmas 5.1 through 5.12 for the $\overline{A}/\overline{B}$ game hold as they did for the overall game in the Harrington-Soare construction.

Lemmas 5.1–5.6 These are stated and proved exactly as in §1.3.2.

Lemma 5.7. This is stated exactly as in [6] and §1.3.2, and is proved exactly as in [6] (rather than as in §1.3.2).

Lemma 5.8. The statement of this lemma is as in §1.3.2. The proof must be altered in a few details, however:

Proof. By Lemma 5.6(i) Step 11E must eventually put every element $x \in \omega$ into Y_λ. By induction we may assume that $R_{\beta,\infty} =^* Y_\beta \cap \overline{A} =^* \overline{A}$ and Y_β is infinite, for $\beta = \alpha^-$. By Lemma 5.7 $m(\alpha) = \infty$ and $m(\gamma) < \infty$ for all $\gamma <_L \alpha$ with $\gamma^- = \beta$.

By Lemma 5.3, $Y_{<\alpha} =^* 0$. For any x that is in S^0_β at some stage and is never moved into $Y_{<\alpha}$ by Step 11C, x is eventually moved either into A by Step 4A, or into

S^1_β by Step 11D. (For the dual case, this should read "$\hat{x}$ is eventually moved either into B by Step $\widehat{5}$, $\widehat{\pi}$, or $\widehat{9}$, or into $\widehat{S}^1_\beta$ by Step 11D.") For any x that is in S^1_β at some stage and is never moved into $Y_{<\alpha}$ by Step 11C, x is eventually moved either into A by Step 4A, or into S_α by Step 1 or Step 2. (For the dual case, this should read "$\hat{x}$ is eventually moved either into B by Step $\widehat{5}$ or $\widehat{9}$, or into $\widehat{S}_\alpha$ by Step $\widehat{1}$ or Step $\widehat{2}$.")

Thus, almost every $x \in R_\beta$ not yet in R_α that never enters A will eventually enter S_α. By Lemma 5.4(v) almost every such x will remain in R_α forever. Thus, $R_{\alpha,\infty} =^* Y_\alpha \cap \overline{A} =^* Y_\beta \cap \overline{A} =^* \overline{A}$.

Also, since Y_β is infinite, $\mathcal{F}^+_\beta$ must contain at least one state ν. By Lemma 5.7(iii) ((iv) in the dual case), $\nu \in \mathcal{E}_\alpha$; that is, infinitely many elements enter Y_α in state ν. Thus, Y_α is infinite. ∎

Lemma 5.9 is stated exactly as in [6] and §1.3.2, but the proof is very slightly different from the proof in the former (because the statements of Lemmas 5.8 and 5.4(v) are slightly different) and significantly different from the proof in the latter (because now, as in [6], elements only move by RED in Step 4, so we don't have to worry about elements vanishing into A before Step 3 can act on them):

Lemma 5.9. $\alpha \subset f \implies \alpha$ is $\mathcal{M}$-consistent.

Proof. Let $\alpha \subset f$ and $\beta = \alpha^-$. Assume for a contradiction that α is not $\mathcal{M}$-consistent. Then $e_\alpha > e_\beta$ and there exist $\nu_0 \in \mathcal{M}_\alpha$, $\nu_1 \notin \mathcal{M}_\alpha$, $\nu_0 <_B \nu_1$ and $\nu_1 \restriction \beta \in \mathcal{M}_\beta$. Then α is a terminal mode on T so $S_\alpha = R_\alpha$. By Lemmas 5.8 and 5.4(v), $S_{\alpha,\infty} \cap \overline{A}$ is infinite and no $x \in S_{\alpha,s}$, $s > v_\alpha$, later leaves S_α except possibly to enter A. By Lemma 5.7, $\mathcal{E}_\alpha \supseteq \mathcal{M}_\alpha$ so

$$(\exists^\infty x)(\exists s)[x \in S_{\alpha,s+1} - S_{\alpha,s} \;\;\&\;\; \nu(\alpha, x, s+1) = \nu_0].$$

Choose any such x and $s > v_\alpha$. Now neither Step 1_γ nor Step 2_γ can apply to x at any stage $t > s$. Hence, by the ordering of the steps, Step 3_α must apply to some such x' at some stage $t+1 > s+1$ with $\nu(\alpha, x', t) = \nu_0$ and must cause $\nu(\alpha, x', t+1) = \nu_1$. Thus, α is provably incorrect at all stages $v \geq t+1$, so $\alpha \not\subset f$. ∎

Lemma 5.10. Just as in [6] and §1.3.2.

We have no Lemma 5.11 in the nondual case, because we have no consideration of $\mathcal{R}$-consistency. Lemma 5.$\widehat{11}$, on the other hand, is stated exactly as in [6] and §1.3.2. The proof is somewhat more complicated, because we have to make sure that the disappearance of elements by Steps $\widehat{\pi}$ and $\widehat{9}$ does not cause problems.

Lemma 5.$\widehat{11}$. $\alpha \subset f \implies \alpha$ is $\widehat{\mathcal{R}}$-consistent.

Proof. Assume for a contradiction that $\alpha \subset f$ and α is not $\widehat{\mathcal{R}}$-consistent. Choose $\hat{\nu}_1 \in \widehat{\mathcal{R}}_\alpha$ such that for all $\hat{\nu}_2 \in \widehat{\mathcal{M}}_\alpha$, $\hat{\nu}_1 \not<_R \hat{\nu}_2$. By equation (42) of [6] α is a terminal node on T so $\widehat{S}_\alpha = \widehat{R}_\alpha$. Thus, by Lemma 5.4(v), for some υ_α, no $\hat{x} \in \widehat{S}_{\alpha,s} \cap \overline{B}$ later leaves $\widehat{S}_\alpha$.

Assume for a contradiction that there are only finitely many elements $\hat{x}$ such that Step $\widehat{4}$ applies to $\hat{x}$ at some stage s when $\hat{\nu}(\alpha, \hat{x}, s) = \hat{\nu}_1$. Then we can prove the following claim:

Claim 1. For every $i \in \omega$, there is an element $\hat{x}_i$ and a stage s_i such that

1. $\hat{x}_i \in \widehat{Y}_{\alpha,s_i+1} - \widehat{Y}_{\alpha,s_i}$, and

2. $\widehat{\Gamma}^\alpha_{\hat{\nu}_1,i,t} = \hat{x}_i$ for all $t \geq s_i + 1$.

Proof. [Claim 1] We proceed by induction on i. Suppose we have our statement for all $i' < i$. Take $s = \max\{s_{i'} : i' < i\}$.

Now, any collection of more than e elements must include one whose lowest order marker is of order $\geq e$. Thus, for any $e > |\widehat{\Gamma}^\alpha_{\hat{\nu},i}|$, if Step $\widehat{9}$ acts on W_e by putting an element of $W_{e,s} \cap \widehat{Y}_s$ into B_{s+1} then that element will not be $\widehat{\Gamma}^\alpha_{\hat{\nu},i}$, because there are $|W_{e,s} \cap \widehat{Y}_s| > e$ elements to choose from, and one of them will have a higher-order lowest-order marker. Thus, Step $\widehat{9}$ can remove the marker $\widehat{\Gamma}^\alpha_{\hat{\nu},i}$ at most $|\widehat{\Gamma}^\alpha_{\hat{\nu},i}| + 1$ times (in acting on W_e for $e \leq |\widehat{\Gamma}^\alpha_{\hat{\nu},i}| + 1$).

Then take any stage $t > \max\{\upsilon_\alpha, s\}$ by which

1. Step $\widehat{9}$ has removed marker $\widehat{\Gamma}^\alpha_{\hat{\nu}_1,i}$ all the times it ever will.

2. Step $\widehat{\pi}$ has removed marker $\widehat{\Gamma}^\alpha_{\hat{\nu}_1,i}$ if it ever will. (Note that Step $\widehat{\pi}$ can remove a given marker at most once.)

3. Step $\widehat{4}$ will never again apply to any $\hat{x}$ in state $\hat{\nu}_1$.

Then after stage t, $\widehat{\Gamma}^{\alpha}_{\hat{\nu}_1,i}$ can never be removed from an element by Step $\widehat{1}$ or 11C (since $t > \upsilon$), or by Step $\widehat{\pi}$, $\widehat{4}$, or $\widehat{9}$, or by Step $\widehat{3}$ (since by Lemma 5.$\widehat{9}$, α is $\widehat{\mathcal{M}}$-consistent) or by Step $\widehat{5}$ (since there cannot exist any $\delta \subset \alpha$ such that $\hat{\nu} \in \widehat{\mathcal{B}}_\alpha$ and $\hat{\nu} \restriction \delta \in \widehat{\mathcal{R}}_\delta$); that is, if $\widehat{\Gamma}^{\alpha}_{\hat{\nu}_1,i}$ is ever attached to an element at some stage $> t$, it remains attached permanently. Also, since $t > s$, if any element $\hat{x}$ ever enters $\widehat{Y}_\alpha$ in state $\hat{\nu}$ at any stage $> t$, it will be marker $\widehat{\Gamma}^{\alpha}_{\hat{\nu}_1,i}$ that will be attached to $\hat{x}$ unless $\widehat{\Gamma}^{\alpha}_{\hat{\nu}_1,i}$ is already attached to some element. Thus, since $\hat{\nu}_1 \in \widehat{\mathcal{R}}_\alpha \subseteq \widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{E}}_\alpha$, by some stage later than t $\widehat{\Gamma}^{\alpha}_{\hat{\nu}_1,i}$ must be permanently attached to some element $\hat{x}$, which must have received the marker when entering $\widehat{Y}_\alpha$ in state $\hat{\nu}$ at some stage $s_i + 1$. ■

But if, say, $\hat{\nu}(\alpha, \hat{x}_0, t) = \hat{\nu}_1$ for all $t > s_0$ then $\hat{x}_0$ witnesses that $\widehat{F}(\alpha^-, \hat{\nu}_1)$ fails so $\hat{\nu}_1 \in \widehat{\mathcal{R}}_\alpha$ contradicts $\alpha \subset f$. Hence, by contradiction we know that Step $\widehat{4}$ applies infinitely often to elements $\hat{x} \in \widehat{S}_\alpha$ in state $\hat{\nu}_1$. Then there must be some state $\hat{\nu}_2$ to which infinitely many of these elements are moved. Now $\hat{\nu}_2 \in \widehat{\mathcal{F}}_\alpha$, so $\hat{\nu}_2 \in \widehat{\mathcal{M}}_\alpha$ by Lemma 5.10. ■

The statement of Lemma 5.12 is the same as in [6] and §1.3.2, and so is the proof in the dual case, but in the nondual case it is somewhat different because there is no longer any consideration of $\mathcal{R}$-consistency.

Lemma 5.12. If $\alpha \subset f$ and $\nu_1 \in \mathcal{B}_\alpha$ then $\{x : x \in Y_\alpha \ \& \ \nu(\alpha, x) = \nu_1\} =^* \emptyset$.

Proof. Fix $\alpha \subset f$ and $\nu_1 \in \mathcal{B}_\alpha$. Let v_α be as in Lemma 5.4(v). Assume for a contradiction that $x \in R_{\alpha,s}$ for some $s > v_\alpha$ and that for all $t \geq s$, $\gamma = \alpha(x,t)$, and $\nu_1 = \nu(\alpha, x, t)$. Now $\gamma \supseteq \alpha$ and $\alpha \in T$ so $\nu_1' \in \mathcal{B}_\gamma$ for all $\nu_1' \in \mathcal{M}_\gamma$ such that $\nu_1' \restriction \alpha = \nu_1$. Then Step 5 applies to x and γ at some stage $t + 1 > s$ so x is enumerated into B_{t+1}.
■

Thus, just as in §1.3.2, we now have that $\mathcal{M}^{\overline{A}}$, $\widehat{\mathcal{M}}^{\overline{B}}$, $\mathcal{N}^{\overline{A}}$, and $\widehat{\mathcal{N}}^{\overline{B}}$ are all correct on f, and f is infinite.

2.5.4 Verifying that $\mathcal{G}^A = \widehat{\mathcal{G}}^B$.

Since $\mathcal{M}_\alpha = \widehat{\mathcal{M}}_\alpha$ for any α, it suffices to verify two lemmas:

Lemma 2.5.1 *For any $\alpha \subset f$, $\mathcal{G}^A_\alpha = \mathcal{M}_\alpha$.*

Lemma 2.5.2 *For any $\alpha \subset f$, $\widehat{\mathcal{G}}^B_\alpha = \widehat{\mathcal{M}}_\alpha$.*

Proof. [Lemma 2.5.1.] For any node α, α-state ν, and $i \in \omega$, we define the c.e. set $E^\alpha_{\nu,i}$ to be the set of all elements entering S^0_α in state ν after stage i (enumerated by enumerating each such element into $E^\alpha_{\nu,i}$ in the same stage in which it enters S^0_α). Then take any $\alpha \subset f$. Since $\mathcal{G}^A_\alpha \subseteq \mathcal{M}_\alpha$, we need only show that $\mathcal{M}_\alpha \subseteq \mathcal{G}^A_\alpha$. Take any $\nu \in \mathcal{M}_\alpha$.

Since $\nu \in \mathcal{M}_\alpha = \mathcal{E}^0_\alpha$, $E^\alpha_{\nu,i}$ is infinite for every i, so for each i there is some $x_i \in E^\alpha_{\nu,i}$ and some stage s_i such that $x_i \in E^\alpha_{\nu,i \text{ at } s_i} \cap \widetilde{A}_{q(s_i)}$. Note that such an s_i must be greater than i.

We define the sequence $i_0 < i_1 < i_2 < \ldots$ inductively as follows:

1. Let $i_{-1} = 0$.

2. Given any i_n, we can find x_{i_n} and s_{i_n} as above. Let $i_{n+1} = s_{i_n}$.

Then x_{i_n} enters S^0_α in state ν at stage i_{n+1}, by Step 1 or 2. Neither Step 1 nor Step 2 can act on any x_{i_n} while x_{i_n} is in S^0_α, and Step 3 cannot either because α is $\mathcal{M}$-consistent (by Lemma 5.9). Thus, since each $x_{i_n} \in \widetilde{A}_{q(s_i)}$, eventually each x_{i_n} is enumerated by Step 4A into A_s from state ν. Thus, $\nu \in \mathcal{G}^A_\alpha$. ■

Proof. [Lemma 2.5.2.] Take any $\alpha \subset f$. Again, we need only show that $\widehat{\mathcal{M}}_\alpha \subseteq \widehat{\mathcal{G}}^B_\alpha$. Take any $\hat{\nu} \in \widehat{\mathcal{M}}_\alpha$. Then since $\lim_s m(\alpha, s) = \infty$, infinitely many copies of the pair $\langle \alpha, \hat{\nu} \rangle$ appear in $\mathcal{L}^{\mathcal{G}}$. Take any such copy, and let s be a stage such that every element ahead of it in $\mathcal{L}^{\mathcal{G}}$ that will ever be marked has already been. By Lemma 5.7, $\hat{\nu} \in \widehat{\mathcal{E}}^0_\alpha$, so at some stage $t > s$ an element $\hat{x}$ enters S^0_α in state $\hat{\nu}$. Then neither Step $\widehat{1}$ nor Step $\widehat{2}$ can act on $\hat{x}$ as long as $\hat{x}$ remains in S^0_α, and Step $\widehat{3}$ cannot either because α is $\widehat{\mathcal{M}}$-consistent (by Lemma 5.$\widehat{9}$). Thus, the given copy of $\langle \alpha, \hat{\nu} \rangle$ must be marked at some stage $t' > t$ by Step $\widehat{\pi}$ acting on $\hat{x}$, unless it is marked by some other action of Step $\widehat{\pi}$ on an element of state $\hat{\nu}$.

Hence, every copy of $\langle \alpha, \hat{\nu} \rangle$ is eventually marked, so infinitely many elements enter B from state $\hat{\nu}$, so $\hat{\nu} \in \widehat{\mathcal{G}}^B_\alpha$. ■

Having shown that $\mathcal{G}^A = \widehat{\mathcal{G}}^B$, we also have that the tree properties of §1.2.2 hold for the overall construction, just as in §1.3.4.

2.5.5 Verifying that B is Effectively Simple.

We wish to verify that every requirement P_e is satisfied. Take any e such that $|W_e| > e$. Take s such that $|W_{e,s}| > e$. Let $t > s$ be a stage by which

1. Every $i < e$ that will ever be acted on by Step $\widehat{9}$ already has been, and

2. $W_{e,s} \subseteq \widehat{Y}_{\lambda,t}$.

Then at the next stage $t'+1$ at which Step $\widehat{9}$ is reached, we will have $W_{e,s} \subseteq W_{e,t'+1} \cap \widehat{Y}_{\lambda,t'}$ and therefore $|W_{e,t'+1} \cap \widehat{Y}_{\lambda,t'}| > e$, so if $W_{e,t'+1} \subseteq \overline{B}_{t'}$ then Step $\widehat{9}$ will enumerate an element of $W_{e,t'+1}$ into B. Thus, $W_{e,t'+1} \nsubseteq \overline{B}_{t'+1}$ and therefore $W_e \nsubseteq \overline{B}$. ■

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