Math 199 Introdution to Analysis and Linear Algebra, Section 43

Homework # 2 , due January 17

Problem 1
Read Chapter 1.6 - 1.7 of *"Foundations of Mathematical Analysis"* by Paul J. Sally.
Problem 2
One can construct the rational number $\mathbb{Q}$ from the integers $\mathbb{Z}$ as follows. Consider the set

$$F := \{(a, b) \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}.$$

For $(a, b), (c, d) \in F$ we define $(a, b) \sim (c, d)$ if $ad = bc$.
Show that this relation $\sim$ is an equivalence relation on F.
The set of equivalence classes are the rational numbers, which we denote by $\mathbb{Q}$. A tuple (a, b) corresponds to the fraction $\frac{a}{b}$. We denote the equivalence class $C((a, b))$ by $\{(a, b)\}$.
Bonus-Problem - not required
We define the addition and multiplication on $\mathbb{Q}$ as follows

$$\{(a, b)\} + \{(c, d)\} = \{(ad + bc, bd)\}$$

and

$$\{(a, b)\} \circ \{(c, d)\} = \{(ac, bd)\}.$$

Show that addition and multiplication are well defined, that is you have to show that if $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, then $(ad + bc, bd) \sim (a'd' + b'c', b'd')$ (for the addition) and $(ac, bd) \sim (a'c', b'd')$ (for the multiplication).
Problem 3
Let A, B, C be nonempty sets and $f : A \to B$, $g : B \to C$ two surjective functions. Show that the composition function $g \circ f : A \to C$ is also surjective.
(The composition $g \circ f$ was defined by the following rule: if $x \in A$ then $(g \circ f)(x) = g(f(x))$.)
Problem 4
Let $f : A \to B$ a function. For $A_1 \subset A$ we defined the image of A_1 as

$$f(A_1) := \{b \in B \mid \exists a \in A_1 \text{ such that } f(a) = b\}.$$

For $B_1 \subset B$ we defined the preimage of B_1 as

$$f^{-1}(B_1) := \{a \in A \mid f(a) \in B_1\}.$$

(1) Prove that if $A_1, A_2 \subset A$, then

$$f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2).$$

(2) Give an example of sets A, B, a function $f : A \to B$ and subsets $A_1, A_2 \subset A$ such that $f(A_1 \cap A_2)$ is not equal to $f(A_1) \cap f(A_2)$.

(3) Prove that if $B_1, B_2 \subset B$, then
$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2).$$

Problem 5

Recall that a commutative ring R with identity is a nonempty set R with an internal law of composition $\circ : R \times R \to R$, written as $\circ((a,b)) = ab$, satisfying the axioms (A1)-(A5), (M1)-(M4) and (D) stated in the notes.

Let R be a commutative ring (with identity).

(1) Let $a \in R$. Show that a has a unique additive inverse. We denote this inverse by $-a$.

(2) Let $a, b \in R$. The difference $a-b$ is defined as $a-b := a+(-b)$, the square a^2 is defined as $a^2 := aa$. Show that
$$(a-b)(a+b) = (a^2 - b^2)$$
.

(3) Let $a, b, \in R$. Show that $(-a)(-b) = ab$.

Problem 6

Recall the equivalence relation "congruence modulo n" on the set of integers $\mathbb{Z}$, which was defined for any integer $n \geq 2$ in the first Homework assignment, Problem 6. For every $a \in \mathbb{Z}$ let us denote the congruence class $C(a)$ simply by $\overline{a}$. Let us denote the set of all congruence classes modulo n by $\mathbb{Z}_n$. We define the addition and multiplication in $\mathbb{Z}_n$ by the rules
$$\overline{a} + \overline{b} = \overline{a+b}$$
and
$$\overline{a} \circ \overline{b} = \overline{ab}.$$
We showed in the last homework that the addition and multiplication is well defined, i.e. the result is independent on which representative of the congruence class we choose.

Show that $\mathbb{Z}_n$ is a commutative ring with identity. (You have to show that it satifies all the axioms (A1)-(A5), (M1)-(M4) and (D).) Which of these properties of $\mathbb{Z}$ are inherited from the ring structure on $\mathbb{Z}$? What is the additive identity? What is the multiplicative identity?