Math 199 Introduction to Analysis and Linear Algebra, Section 43

## HOMEWORK # 6 , DUE FEBRUARY 14

**Problem 1**

Read Chapter 3.1, 3.2, 3.4 and 3.6 in *"Foundations of Mathematical Analysis"* by Paul J. Sally

**Problem 2**

Recall that we showed in class that the real numbers satisfies the *least upper bound property*, i.e. for every nonempty subset $A \subset \mathbb{R}$ which is bounded above, there exists a least upper bound $L \in \mathbb{R}$ for $A$. The least upper bound $L$ is often denoted by $\sup(A)$ and called the *supremum* of $A$. The greatest lower bound of a nonempty subset $B \subset \mathbb{R}$ is usually called the *infimum* of $B$ and is denoted by $\inf(B)$.

(1) Let $A, B \subset \mathbb{R}$ be nonempty subsets, which are both bounded above. Define the set $A + B := \{a + b \,|\, a \in A, b \in B\}$. Show that

$$\sup(A + B) = \sup(A) + \sup(B).$$

(2) Let $A, B \subset \mathbb{R}$ be nonempty subsets with the property that for every $a \in A$ and $b \in B$ we have $a < b$. Show that then $A$ is bounded above, $B$ is bounded below, and $\sup(A) \leq \inf(B)$. Give an example of two set $A, B \subset \mathbb{R}$ such that $\sup(A) = \inf(B)$.

(3) Find the least upper bound of the following subsets of $\mathbb{R}$:
$A = \{\sin(x) \,|\, x \in \mathbb{Q}\}$
$B = \{-2^n \,|\, n \in \mathbb{Z}\}$.

**Problem 3**

Recall that

- a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers is bounded if there exists $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.
- a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers is monotonic if $(a_n)$ is either monotonic increasing (i.e. $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$) or monotonic decreasing (i.e. $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$).

Prove the following statements, which we used in class to show that in $\mathbb{R}$ every Cauchy sequences converges.

(1) Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}$, then $(a_n)_{n\in\mathbb{N}}$ is bounded. (Remember what we did when we showed this for Cauchy sequences in $\mathbb{Q}$ in class.)

(2) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$, then $(a_n)_{n\in\mathbb{N}}$ has a monotonic subsequence. (Lemma 3.6.8 in the notes)

(3) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. If $(a_n)_{n\in\mathbb{N}}$ is bounded, then $(a_n)_{n\in\mathbb{N}}$ has a convergent subsequence. (Lemma 3.6.10. in the notes, use (1) and Lemma 3.6.9 in the notes which says that every bounded monotonic sequence has a convergent subsequence.)

**Problem 4**

Use the Archimedean property of $\mathbb{R}$ to prove the following statements.

(1) If $a, b \in \mathbb{R}$ with $a < b$, then there exists a rational number $r = \frac{p}{q} \in \mathbb{Q}$ such that $a < r < b$.

(2) We call $a \in \mathbb{R}$ *irrational* if $a \notin \mathbb{Q}$. Let $a \in \mathbb{R}$ be irrational and $r \in \mathbb{Q}$, $r \neq 0$. Show that $r \cdot a$ is irrational.

(3) (**Bonus - not required**) If $a, b \in \mathbb{R}$ with $a < b$, then there exists a irrational number $s \in \mathbb{R}$, $s \notin \mathbb{Q}$ such that $a < s < b$.

**Problem 5**

Find the accumulation points of the following sets (No proof!)

(1) $A = (0, 1)$
(2) $A = \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$
(3) $A = \mathbb{Z}$
(4) $A = \mathbb{Q}$.