**Smooth Ergodic Theory II – Rigidity and Geometry. Homework Exercises.**

**Exercise 1.** *Let $M$ be a Riemannian manifold with metric $\langle\,,\,\rangle$ and Levi-Civita connection $\nabla$. Fix the standard identification of $TTM$ with $\otimes^3 TM$:*

$$[\beta] \to \big(\beta(0), (\pi\circ\beta)'(0), \nabla_{(\pi\circ\beta)'}\beta(0)\big) \in (T_{\pi\circ\beta(0)}M)^3.$$

*Prove the following.*

(1) *If $\theta$ is the canonical 1-form on the cotangent bundle $T^*M$, and $I : TM \to T^*M$ is the isomorphism $I(v) = \langle\cdot, v\rangle$, then $I^*\theta = \alpha$, where*

$$\alpha(u, v, w) = \langle u, v\rangle.$$

(2) *if $\xi_i = (u, v_i, w_i)$, $i = 1, 2$, then*

$$d\alpha(\xi_1, \xi_2) = \langle v_2, w_1\rangle - \langle v_1, w_2\rangle.$$

*(up to a sign).*

(3) *The vector field $\dot\varphi$ on $TM$ defined by*

$$\dot\varphi(u) = (u, u, 0)$$

*generates the geodesic flow.*

(4) *The restriction of the geodesic flow $\varphi$ to the unit tangent bundle $T^1M$ preserves the restriction of $\alpha$ to $T(T^1M)$.*

(5) *$\alpha$ is a contact 1-form on $T^1M$.*

**Exercise 2.** *Assume $M$ is compact without conjugate points, and for $v \in T^1\tilde M$, define the (positive and negative) Busemann functions*

$$b_v^+, b_v^- : \tilde M \to \mathbb{R}$$

*by*

$$b_v^+(p) = \lim_{t\to\infty} d(\gamma_v(t), p) - t.$$

*and*

$$b_v^-(v) = \lim_{t\to\infty} d(\gamma_v(-t), p) - t.$$

*Prove the following:*

(1) *The limits above exist, and for every $v \in T^1\tilde M$, the functions $b_v^\pm$ satisfy:*
 (a) *$b_v^\pm(\pi(v)) = 0$;*
 (b) *$|b_v^\pm(p) - b_v^\pm(q)| \le d(p, q)$, for all $p, q \in \tilde M$.*

(2) *$b^\pm_{\varphi_t(v)} = b_v^\pm - t$.*

**Exercise 3.** *The* Poincaré disk *(or hyperbolic disk) is the domain $\mathbb{D} = \{z : |z| < 1\}$ with the metric*

$$ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}.$$

*The group of orientation-preserving isometries of $\mathbb{D}$ is*

$$\{\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha}\end{pmatrix} : |\alpha|^2 - |\beta|^2 \neq 0\},$$

*which acts by Möbius transformations:*

$$\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} : z \mapsto \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}}.$$

*The hyperbolic disk is isometric via a Möbius transformation to the upper-half plane* $\mathbb{H} = Im(z) > 0$ *with the metric*

$$ds^2 = \frac{|dz|^2}{(Imz)^2}.$$

*The isometry group of* $\mathbb{H}$ *is*

$$\mathrm{PSL}(2,\mathbb{R}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1\}/\{\pm I\},$$

*also acting by Möbius transformations. The curvature of* $\mathbb{H}$ *is constant, equal to* $-1$*. We will refer to the* $\mathbb{D}$ *and* $\mathbb{H}$ *models interchangeably.*

*Hyperbolic geodesics in* $\mathbb{D}$ *are Euclidean circular arcs, perpendicular to* $\partial\mathbb{D} = \{|z| = 1\}$*. In* $\mathbb{H}$*, hyperbolic geodesics in* $\mathbb{H}$ *are Euclidean (semi) circular arcs, perpendicular to* $Im(z) = 0$ *(where lines are Euclidean circles with infinite radius).*

*The stabilizer of a point under this left action is the compact subgroup* $K = SO(2)/\{\pm I\}$*, which gives an identification of* $\mathbb{H}$ *with the coset space of* $K$*:*

$$\mathbb{H} = \mathrm{PSL}(2,\mathbb{R})/K.$$

*The derivative action of* $PSL(2,\mathbb{R})$ *on the unit tangent bundle* $T^1\mathbb{H}$ *is free and transitive, and gives an analytic identification between* $T^1\mathbb{H}$ *and* $PSL(2,\mathbb{R})$*. The action of* $PSL(2,\mathbb{R})$ *on* $T^1\mathbb{H}$ *by isometries corresponds to left multiplication in* $\mathrm{PSL}(2,\mathbb{R})$*.*

*If* $S$ *is a closed orientable surface with* $\tilde{S} = \mathbb{H}$*, then* $\pi_1(S)$ *acts by isometries on* $\mathbb{H}$ *and hence embeds as a discrete subgroup* $\Gamma < \mathrm{PSL}(2,\mathbb{R})$*. We thus obtain the following identifications:*

$$S = \Gamma\backslash\mathbb{H} = \Gamma\backslash \mathrm{PSL}(2,\mathbb{R})/K,$$

*and*

$$T^1S = \Gamma\backslash \mathrm{PSL}(2,\mathbb{R}).$$

*Endowing* $\mathrm{PSL}(2,\mathbb{R})$ *with a suitable left-invariant metric gives an isometry between* $\mathrm{PSL}(2,\mathbb{R})/K$ *and* $\mathbb{H}$*. This metric on* $\mathrm{PSL}(2,\mathbb{R})$ *also induces a metric on* $T^1\mathbb{H}$*, called the Sasaki metric (see the next section). In this metric, the lifts of geodesics in* $\mathbb{H}$ *via* $\gamma \mapsto \dot{\gamma}$ *gives Sasaki geodesics in* $T^1\mathbb{H}$ *(there are other Sasaki geodesics that do not project to geodesics in* $\mathbb{H}$ *but project to curves of constant geodesic curvature: for example, the orbits of the* $SO(2)$ *subgroup.)*

*If you have never done so before, verify these assertions about hyperbolic space. Useful fact: the curvature of a conformal metric* $ds^2 = h(z)^2|dz|^2$

*(where $h$ is real-valued and positive) on a planar domain is given by the formula:*

$$k = -\frac{\Delta \log h}{h^2},$$

*where $\Delta$ is the Euclidean Laplacian.*

*To verify the assertion about geodesics, it suffices to show that the curve $t \mapsto ie^t$ is a geodesic in $\mathbb{H}$ and then apply isometries. (note that this vertical ray in $\mathbb{H}$ is fixed pointwise by the (orientation-reversing) hyperbolic isometry $z \mapsto -\overline{z}$...). One can also find a formula for hyperbolic distance using this method.*

*To identify $T^1\mathbb{H}$ with $\mathrm{PSL}(2,\mathbb{R})$, start by identifying the unit vertical tangent vector based at $i$ with the identity matrix. It is helpful to understand the orbit of this vector under one-parameter subgroups that together generate $\mathrm{PSL}(2,\mathbb{R})$, for example, the groups in the Iwasawa ($KAN$) decomposition.*

**Exercise 4.** *An additional symmetry of the geodesic flow is* flip invariance*:*

$$\varphi_{-t}(-v) = -\varphi_t(v).$$

*Another way to state this is that $\varphi_t$ is conjugate to the reverse time flow $\varphi_{-t}$ via the involution on $I\colon T^1\tilde{S} \to T^1 S$ defined by:*

$$I(v) = -v.$$

*Verify this.*

**Exercise 5.** *Show that on $T^1\mathbb{H} = PSL(2,\mathbb{R})$, the geodesic flow is given by right multiplication by the 1-parameter subgroup:*

$$A = \left\{ a_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \right\}.$$

**Exercise 6.** *Verify that the geodesic flow is Anosov in the case of the hyperbolic plane. Recall from previous exercises that $T^1\mathbb{H} = \mathrm{PSL}(2,\mathbb{R})$, with the Sasaki/left-invariant metrics, and the geodesic flow acts by right multiplication by the one-parameter subgroup $A$.*

*Verify that the horocycle foliations $\mathcal{H}^{\pm}$ are the foliations by cosets of the horocyclic subgroups $P^{\pm}$ (fun fact: the leaves of these foliations are Sasaki geodesics in $T^1\mathbb{H}$). The commutators $[a_t, h_s^{\pm}]$ are relevant. If you prefer, work on the level of the Lie algebra.*